1. (a) Show that any function \( f : \mathbb{R} \rightarrow \mathbb{R} \) can be decomposed as a sum of an even function \( f_e \) and an odd function \( f_o \). Hint: Construct explicit expressions for \( f_e \) and \( f_o \) in terms of \( f(x) \) and \( f(-x) \) and show that they are even and odd functions, respectively.

Let

\[
\begin{align*}
  f_e(x) &= \frac{f(x) + f(-x)}{2}, \\
  f_o(x) &= \frac{f(x) - f(-x)}{2},
\end{align*}
\]

Then \( f(x) = f_e(x) + f_o(x) \).

(b) Show using Corollary 5.2.1 and an appropriate substitution that if \( f_e \) is an even integrable function on \([-a, a] \), then

\[
\int_{-a}^{a} f_e(x) \, dx = 2 \int_{0}^{a} f_e \, dx.
\]

\[
\int_{-a}^{a} f_e(x) \, dx = \int_{0}^{a} f_e(x) \, dx + \int_{0}^{a} f_e(x) \, dx = -\int_{a}^{0} f_e(-x) \, dx + \int_{0}^{a} f_e(x) \, dx \\
= \int_{0}^{a} f_e(x) \, dx + \int_{0}^{a} f_e(x) \, dx = 2 \int_{0}^{a} f_e.
\]

(c) Show that if \( f_o \) is an odd integrable function on \([-a, a] \) that

\[
\int_{-a}^{a} f_o = 0.
\]

\[
\int_{-a}^{a} f_o(x) \, dx = \int_{-a}^{0} f_o(x) \, dx + \int_{0}^{a} f_o(x) \, dx = -\int_{a}^{0} f_o(-x) \, dx + \int_{0}^{a} f_o(x) \, dx \\
= -\int_{0}^{a} f_o(x) \, dx + \int_{0}^{a} f_o(x) \, dx = 0.
\]
(d) Deduce that
\[ \int_0^a f + \int_{-a}^0 f = 2 \int_0^a f_e \]
and
\[ \int_0^a f - \int_{-a}^0 f = 2 \int_0^a f_o. \]
We find
\[ \int_0^a f + \int_{-a}^0 f = \int_0^a (f_e + f_o) + \int_{-a}^0 (f_e + f_o) = \int_{-a}^a f_e + \int_{-a}^a f_o = 2 \int_0^a f_e \]
and
\[ \int_0^a f - \int_{-a}^0 f = \int_0^a (f_e + f_o) - \int_{-a}^0 (f_e + f_o) = \int_0^a f_e - \int_0^a f_e + \int_0^a f_o + \int_0^a f_o = 2 \int_0^a f_o. \]

2. Evaluate
(a) \[ \int_1^2 \frac{1}{u^2} du \]
\[ = \left[ -\frac{1}{u} \right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}. \]

(b) \[ \int_0^1 (2x - 4x^4 + 5) \, dx \]
\[ = \int_0^1 (2x - 4x^4 + 5) \, dx = \left[ x^2 - \frac{4}{5}x^5 + 5x \right]_0^1 = 1 - \frac{4}{5} + 5 - 0 = \frac{26}{5}. \]

(c) \[ \int_{-1}^1 (t - 1)(t + 3) \, dt \]
\[ = \int_{-1}^1 (t^2 + 2t - 3) \, dt = 2 \int_0^1 (t^2 - 3) \, dt = 2 \left[ \frac{t^3}{3} - 3t \right]_0^1 = \frac{2}{3} - 6 = -\frac{16}{3}. \]
Note that we have used the fact that the integral of an odd function on \([-1, 1]\) vanishes and that the integral of an odd function on \([-1, 1]\) is twice its integral on \([0, 1]\).
(d) \[ \int_0^1 \sqrt{u} \, du \]
\[ = \left[ \frac{2}{3} u^{3/2} \right]_0^1 = \frac{2}{3}. \]

(e) \[ \frac{d}{dx} \int_2^{x^3} \frac{du}{\log u} \]
\[ = \frac{1}{\log x^3} \cdot 3x^2 = \frac{x^2}{\log x}. \]

(f) \[ \int_0^1 2x \sin(1 - x^2) \, dx \]

Hint: To find an antiderivative use the fact that
\[ \frac{d}{dx} f(g(x)) = f'(g(x))g'(x). \]

What expressions should you identify with \( f \) and \( g \)?
\[ \int_0^1 2x \sin(1 - x^2) \, dx = \left[ \cos(1 - x^2) \right]_0^1 = \cos 0 - \cos 1 = 1 - \cos 1. \]

Here we identified \( f(u) = \cos u \) and \( g(x) = 1 - x^2 \).

3. Use the Fundamental Theorem of Calculus to prove this slightly stronger version of the Mean Value Theorem for Integrals: if \( f \) is continuous on \([a, b] \), then
\[ \int_a^b f = f(c)(b - a) \]

for some number \( c \in (a, b) \).

The continuous function \( f \) has an antiderivative \( F \) on \([a, b] \), and we know
\[ \frac{1}{b - a} \int_a^b f = \frac{F(b) - F(a)}{b - a} = F'(c) = f(c) \]

for some \( c \in (a, b) \), by the Mean Value Theorem for derivatives, noting that \( F \) is differentiable and hence continuous on \([a, b] \).
4. Let \( f \) be a continuous function on \([a, b]\).

(a) If there exists a point \( x_0 \in [a, b] \) such that \( f(x_0) > 0 \), show that there exists a \( \gamma > 0 \) such that \( f(x) > 0 \) for all \( x \in [x_0 - \gamma, x_0 + \gamma] \cap [a, b] \).

Consider \( \epsilon = f(x_0) > 0 \). From the continuity of \( f \) we know that there exists a \( \delta > 0 \) such that

\[ x \in (x_0 - \delta, x_0 + \delta) \cap [a, b] \Rightarrow |f(x) - f(x_0)| < f(x_0), \]

which in particular implies that \( f(x) - f(x_0) > -f(x_0) \). Hence

\[ x \in (x_0 - \delta, x_0 + \delta) \cap [a, b] \Rightarrow f(x) > 0. \]

If we let \( \gamma = \delta/2 \) we see that \( f \) is positive on the closed interval \([x_0 - \gamma, x_0 + \gamma] \cap [a, b] \).

(b) If it is also known that \( f \) is non-negative on \([a, b]\), prove that either \( \int_a^b f > 0 \) or else \( f(x) = 0 \) for all \( x \in [a, b] \).

**Proof 1:** We are given that \( f(x) \geq 0 \) for all \( x \in [a, b] \). If \( f(x) \) is identically zero on \([a, b]\) we are done. Otherwise \( f(x_0) > 0 \) for some \( x_0 \in [a, b] \). From part (a), we know that there exists a \( \gamma > 0 \) such that \( f \) is nonzero on the closed interval \([x_0 - \gamma, x_0 + \gamma] \cap [a, b] \), which we simply denote as \([c, d]\). We know that \( f \) must achieve its minimum value \( m > 0 \) on \([c, d]\). Hence by Theorem 5.5, \( \int_c^d f \geq m(d - c) > 0 \). Now

\[ \int_a^b f = \int_a^c f + \int_c^d f + \int_d^b f. \]

The first and third integral on the right hand side are certainly non-negative, by Corollary 5.5.1, and the second integral is positive, so we see that \( \int_a^b f > 0 \).

**Proof 2:** Consider \( \epsilon = f(x_0)/2 > 0 \). From the continuity of \( f \) we know that there exists a \( \delta > 0 \) such that

\[ x \in (x_0 - \delta, x_0 + \delta) \cap [a, b] \Rightarrow |f(x) - f(x_0)| < f(x_0)/2, \]

which in particular implies that \( f(x) - f(x_0) > -f(x_0)/2 \). Hence

\[ x \in (x_0 - \delta, x_0 + \delta) \cap [a, b] \Rightarrow f(x) > f(x_0)/2. \]

If we let \( \gamma = \delta/2 \) we see that \( f(x) > f(x_0)/2 \) on the closed interval \([x_0 - \gamma, x_0 + \gamma] \cap [a, b] \), which we denote \([c, d]\). Hence by Theorem 5.5, \( \int_c^d f \geq \frac{f(x_0)}{2}(d - c) > 0 \). Now

\[ \int_a^b f = \int_a^c f + \int_c^d f + \int_d^b f. \]

The first and third integral on the right hand side are certainly non-negative, by Corollary 5.5.1, and the second integral is positive, so we see that \( \int_a^b f > 0 \).

**Proof 3:** From Corollary 5.8.1 we know that \( f \) has an antiderivative \( F(x) \) such that \( F'(x) = f(x) \geq 0 \) for all \( x \in [a, b] \). Thus \( F \) is an increasing function on \([a, b]\). But the FTC implies, for \( b > a \), that \( \int_a^b f = F(b) - F(a) \geq 0 \). If \( \int_a^b f = 0 \) then \( F(b) = F(a) \) and hence the increasing function \( F \) must be constant on \([a, b]\), so that \( f(x) = F'(x) = 0 \) for all \( x \in [a, b] \). Otherwise \( \int_a^b f > 0 \).
5. Let $f$ be a continuous function on $[a, b]$ such that $\int_a^b f(x)g(x) \, dx = 0$ for all continuous functions $g$ satisfying $g(a) = g(b) = 0$. Prove that $f$ is identically zero on $[a, b]$. Hint: consider $g(x) = f(x)(x - a)(b - x)$.

First of all we see that $g$ is continuous and $g(a) = g(b) = 0$. Thus

$$0 = \int_a^b f(x)g(x) \, dx = \int_a^b f^2(x)(x - a)(b - x) \, dx.$$ 

Since $f^2(x)(x - a)(b - x)$ is continuous and non-negative on $[a, b]$, we know from Question 4(b) that $f^2(x)(x - a)(b - x)$ must be identically zero on $[a, b]$. Hence $f^2(x) = 0$ on $(a, b)$. By continuity, we see that $f^2$ and hence $f$ must vanish on all of $[a, b]$. This is known as the Fundamental Lemma of the Calculus of Variations.

6. Let $f$ be a function on $[a, b]$ and $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a, b]$.

(a) For each $i = 1, \ldots, n$ define the numbers

$$m_i = \inf \{f(x) : x \in [x_{i-1}, x_i]\},$$
$$M_i = \sup \{f(x) : x \in [x_{i-1}, x_i]\},$$
$$\underline{m}_i = \inf \{|f(x)| : x \in [x_{i-1}, x_i]\},$$
$$\underline{M}_i = \sup \{|f(x)| : x \in [x_{i-1}, x_i]\}.$$

Show for each $i = 1, \ldots, n$ that $\underline{M}_i - \underline{m}_i \leq M_i - m_i$.

Denote $I_i = [x_{i-1}, x_i]$. Since $|f(x) - f(y)| \leq M_i - m_i$ for all $x, y \in I_i$ we know from the triangle inequality that

$$|f(x)| - |f(y)| \leq |f(x) - f(y)| \leq M_i - m_i \quad \forall x, y \in I_i.$$

Hence

$$\underline{M}_i - \underline{m}_i = \sup_{x \in I_i} |f(x)| - \inf_{y \in I_i} |f(y)| = \sup_{x \in I_i, y \in I_i} \inf |f(x)| - |f(y)| | \leq M_i - m_i.$$ 

(Alternatively, one can consider the four cases $m_i \geq 0$, $M_i \leq 0$, $-M_i \leq m_i < 0$, and $m_i < -M_i < 0$ separately.)

(b) Suppose $\int_a^b f$ exists. Use part (a) to show that $\int_a^b |f|$ exists.

Given $\epsilon > 0$ we know from Theorem 5.2 that there exists a partition $P$ of $[a, b]$ such that

$$U(P, f) - L(P, f) = \sum_{i=1}^n (M_i - m_i)(x_{i+1} - x_i) < \epsilon.$$

But then

$$U(P, |f|) - L(P, |f|) = \sum_{i=1}^n (\underline{M}_i - \underline{m}_i)(x_{i+1} - x_i) \leq \sum_{i=1}^n (M_i - m_i)(x_{i+1} - x_i) < \epsilon.$$ 

Hence, by Theorem 5.2, we see that $\int_a^b |f|$ exists.
7. Let $f$ be an integrable function on $[a, b]$ with $|f| \leq M$ on $[a, b]$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a, b]$. 

(a) Suppose also that $f$ is non-negative on $[a, b]$. For each $i = 1, \ldots, n$ define

$$m_i \doteq \inf \{f(x) : x \in [x_{i-1}, x_i]\},$$

$$M_i \doteq \sup \{f(x) : x \in [x_{i-1}, x_i]\}.$$

Show that

$$M_i^2 - m_i^2 \leq 2M(M_i - m_i), \quad i = 1, \ldots, n$$

and use this result to show that $f^2$ is integrable on $[a, b]$.

For each $i = 1, \ldots, n$ we note that

$$M_i^2 - m_i^2 = (M_i + m_i)(M_i - m_i) \leq 2M(M_i - m_i)$$

since $m_i \leq M$ and $M_i \leq M$.

If $M = 0$ then $f = 0$ and $f^2 = 0$ is integrable on $[0, 1]$. Otherwise, given $\epsilon > 0$ we know from Theorem 5.2 that there exists a partition $P$ of $[a, b]$ such that

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i)(x_{i+1} - x_i) < \frac{\epsilon}{2M}.$$

But then

$$U(P, f^2) - L(P, f^2) = \sum_{i=1}^{n} (M_i^2 - m_i^2)(x_{i+1} - x_i) \leq \sum_{i=1}^{n} 2M(M_i - m_i)(x_{i+1} - x_i) < 2M \frac{\epsilon}{2M} = \epsilon.$$

Hence, by Theorem 5.2, we see that $\int_{a}^{b} f^2$ exists.

(b) Now extend your result in part (a) to show that $f^2$ is integrable even if we drop the condition that $f$ is non-negative. Hint: what do you know about $\int_{a}^{b} |f|$?

From Question 6(b) and part (a) we know that

$$\int_{a}^{b} f \exists \Rightarrow \int_{a}^{b} |f| \exists \Rightarrow \int_{a}^{b} f^2 = \int_{a}^{b} |f|^2 \exists.$$

(c) Suppose that $\int_{a}^{b} f$ and $\int_{a}^{b} g$ both exist. Use part (a) to show that $\int_{a}^{b} fg$ exists. Hint: Consider the functions $(f + g)^2$ and $(f - g)^2$.

From part (b) and the linearity of the integral operator we know that

$$\int_{a}^{b} (f + g)^2 - (f - g)^2 = 4 \int_{a}^{b} fg$$

exists. Thus $\int_{a}^{b} fg$ exists.
8. Let $f$ and $g$ be integrable functions on $[a, b]$.

(a) Prove the **Triangle Inequality for Integrals**:

$$
\left| \int_a^b f \right| \leq \int_a^b |f|.
$$

Hint: consider the functions $f(x)$, $-|f(x)|$ and $|f(x)|$.

We have already seen that if $f$ is integrable then so is $|f|$. Now for every $x \in [a, b]$,

$$
-|f(x)| \leq f(x) \leq |f(x)|.
$$

Thus,

$$
-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|.
$$

This means that $\left| \int_a^b f \right| \leq \int_a^b |f|$.

(b) Use part (a) to prove that

$$
\left| \int_a^b (f + g) \right| \leq \int_a^b |f| + \int_a^b |g|.
$$

This follows from integrating the Triangle Inequality:

$$
\left| \int_a^b (f + g) \right| \leq \int_a^b |f + g| \leq \int_a^b (|f| + |g|) = \int_a^b |f| + \int_a^b |g|.
$$

9. Suppose $f$ and $g$ are both integrable on $[a, b]$. Let

$$
Q(x) = \int_a^b (xf(t) + g(t))^2 \, dt
$$

for $x \in \mathbb{R}$ and define

$$
A = \int_a^b f^2(t) \, dt,
$$

$$
B = 2 \int_a^b f(t)g(t) \, dt,
$$

$$
C = \int_a^b g^2(t) \, dt.
$$

(a) Use the properties of integrals to express $Q(x)$ in terms of $x$ and the numbers $A$, $B$, $C$.

$$
Q(x) = Ax^2 + Bx + C.
$$
(b) Prove that

\[ B^2 - 4AC \leq 0. \]

We note that \( Q(x) \geq 0 \) for all \( x \) by Corollary 5.5.1 and hence \( A \geq 0 \). If \( A = 0 \), the fact that \( Bx + C \geq 0 \) for all \( x \) implies that \( B = 0 \) and thus \( B^2 - 4AC = 0 \). Otherwise, if \( A > 0 \), the quadratic equation \( Q(x) = 0 \) can have at most one real root; consequently, the discriminant \( B^2 - 4AC \) must be less than or equal to zero.

(c) Deduce the Cauchy–Bunyakovsky–Schwarz inequality for integrals:

\[
\left| \int_a^b fg \right| \leq \left( \int_a^b f^2 \right)^{1/2} \left( \int_a^b g^2 \right)^{1/2}.
\]

This follow directly from part (b):

\[
\left( 2 \int_a^b f(t)g(t) \, dt \right)^2 \leq 4 \int_a^b f^2(t) \, dt \int_a^b g^2(t) \, dt,
\]

on taking the square root of each side.