1. Consider the function $f(x) = 1 - |x|$ on $[-1, 1]$.

(a) Construct a uniform partition $P$ on $[-1, 1]$ by dividing $[-1, 1]$ into 10 subintervals of equal widths.

$$P = \left\{ -1, -\frac{8}{10}, -\frac{6}{10}, -\frac{4}{10}, -\frac{2}{10}, 0, \frac{2}{10}, \frac{4}{10}, \frac{6}{10}, \frac{8}{10}, 1 \right\}.$$ 

(b) Compute $L(P, f)$.

Since the partition is uniform,

$$L(P, f) = \frac{2}{10} \left( \frac{0}{10} + \frac{2}{10} + \frac{4}{10} + \frac{6}{10} + \frac{8}{10} + \frac{8}{10} + \frac{6}{10} + \frac{4}{10} + \frac{2}{10} + 0 \right) = \frac{8}{100}(1+2+3+4) = \frac{4}{5}.$$ 

(c) Compute $U(P, f)$.

$$U(P, f) = \frac{2}{10} \left( \frac{2}{10} + \frac{4}{10} + \frac{6}{10} + \frac{8}{10} + 1 + 1 + \frac{8}{10} + \frac{6}{10} + \frac{4}{10} + \frac{2}{10} \right) = \frac{8}{100}(1+2+3+4+5) = \frac{6}{5}.$$ 

(d) Using the formula for the area of a triangle, show that

$$L(P, f) < A < U(P, f),$$

where $A$ denotes the area bounded by $y = f(x)$, $y = 0$, $x = -1$, and $x = 1$.

The total bounded area is the sum of the areas of two isosceles right-angle triangles with unit side: $A = \frac{1}{2} + \frac{1}{2} = 1$; we observe that

$$\frac{4}{5} = L(P, f) < 1 < U(P, f) = \frac{6}{5}.$$ 

2. (a) Consider the function $f(x) = x^2$ on $[0, 1]$. Find a sequence of lower sums and a sequence of upper sums for $f$ on $[0, 1]$ that converge to the same value.

Consider uniform partitions $P_n = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1 \right\}$ for $n = 1, 2, \ldots$. Then noting that $x^2$ is an increasing function we find

$$L(P_n, f) = \sum_{i=1}^{n} \left( \frac{i-1}{n} \right)^2 \left( \frac{1}{n} \right) = \frac{1}{n^3} \sum_{i=1}^{n} (i-1)^2 = \frac{1}{n^3} \sum_{i=0}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6n^3} = \frac{(n-1)(2n-1)}{6n^2}$$
and
\[
U(P, f) = \sum_{i=1}^{n} \left( \frac{i}{n} \right)^2 \left( \frac{1}{n} \right) = \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6n^3} = \frac{(n + 1)(2n + 1)}{6n^2}.
\]

We see that \( \lim_{n \to \infty} L(P, f) = \frac{1}{3} = \lim_{n \to \infty} U(P, f) \).

(b) Use your answer in part (a) to compute \( \int_{0}^{1} x^2 \).

By Theorem 5.1, we see that \( \int_{0}^{1} x^2 \) exists and equals \( \frac{1}{3} \).

3. Let
\[
f(x) = \begin{cases} 
\sin x & x \neq 0, \\
1 & x = 0 
\end{cases}
\]
and \( P \) be the nonuniform partition \( \{0, \frac{\pi}{6}, \frac{\pi}{2}\} \).

(a) Show that \( f \) is monotonic on \([0, \frac{\pi}{2}]\).

For \( x \in (0, \frac{\pi}{2}) \), we find
\[
f'(x) = \frac{x \cos x - \sin x}{x^2} < 0
\]
since \( x < \tan x \). Hence, from the Mean Value Theorem, the continuous function \( f \) is in fact strictly decreasing on \([0, \pi/2]\).

(b) Compute the lower sum \( L(P, f) \).

Since the lower sums are evaluated at the right-hand endpoints, we find
\[
L(P, f) = \frac{1}{6} \cdot \frac{\pi}{6} + \frac{2}{\pi} \cdot \frac{\pi}{3} = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}.
\]

(c) Compute the upper sum \( U(P, f) \).

The upper sums are evaluated at the left-hand endpoints:
\[
U(P, f) = \frac{\pi}{6} + \frac{1}{6} \left( \frac{\pi}{2} - \frac{\pi}{6} \right) = \frac{\pi}{6} + 1.
\]

(d) Use parts (b) and (c) to find lower and upper bounds for \( \int_{0}^{\pi/2} f \).

\[
\frac{7}{6} = L(P, f) \leq \int_{0}^{\pi/2} f \leq U(P, f) = 1 + \frac{\pi}{6}.
\]
4. Let \( S_m(n) = \sum_{k=1}^{n} k^m \). Recall that
\[
S_m(n) = \sum_{k=1}^{n} k^m = \frac{n^{m+1}}{m+1} + P_m(n)
\]
for all integers \( m \geq 0 \), where the notation \( P_m(n) \) represents a polynomial of degree \( m \) in the variable \( n \). Let \( f(x) = x^m \). Use the above result to show that
\[
\int_0^1 f = \frac{1}{m+1}
\]
for all integers \( m \geq 0 \).

Construct uniform partitions \( Q_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\} \) for \( n = 1, 2, \ldots \). We know for \( m \geq 0 \) that \( x^m \) is an increasing function, so
\[
L(Q_n, f) = \sum_{k=1}^{n} \left( \frac{k-1}{n} \right)^m \left( \frac{1}{n} \right) = \frac{1}{n^{m+1}} \sum_{k=0}^{n-1} k^m = \frac{1 + S_m(n-1)}{n^{m+1}} = \frac{1}{m+1} + \frac{P_m(n)}{n^{m+1}}
\]
and
\[
U(Q_n, f) = \sum_{k=1}^{n} \left( \frac{k}{n} \right)^m \left( \frac{1}{n} \right) = \frac{1}{n^{m+1}} \sum_{k=1}^{n} k^m = \frac{S_m(n)}{n^{m+1}} = \frac{1}{m+1} + \frac{P'_m(n)}{n^{m+1}},
\]
for two polynomials \( P_m(n) \) and \( P'_m(n) \) of degree \( m \). Since \( \lim_{n \to \infty} P_m(n)/n^{m+1} = 0 \) for any polynomial \( P_m(n) \) we see that \( \lim_{n \to \infty} L(Q_n, f) = \lim_{n \to \infty} U(Q_n, f) = 1/(m+1) \); thus, Theorem 5.1 implies that \( \int_0^1 f \) exists and equals \( 1/(m+1) \).

5. In this problem we demonstrate an alternate way of computing the integral of \( f(x) = x^m \) on the interval \([a, b] \), where \( 0 < a < b \) and \( m \in \mathbb{N} \), using the nonuniform partition \( P = \{ae^{i/n} : i = 0, 1, \ldots, n\} \), with \( c = b/a \).

(a) Show that
\[
U(P, f) = a^{m+1}(1 - c^{-1/n}) \sum_{i=1}^{n} (c^{(m+1)/n})^i = (a^{m+1} - b^{m+1})c^{(m+1)/n} \left( \frac{1 - c^{-1/n}}{1 - c^{(m+1)/n}} \right)
\]
\[
= (b^{m+1} - a^{m+1})c^{m/n} \left( \frac{1}{1 + c^{1/n} + \ldots + c^{m/n}} \right).
\]
\[ U(P, f) = \sum_{i=1}^{n} a^m c^{i m/n} [ac^{i/n} - ac^{(i-1)/n}] \]
\[ = a^{m+1} (1 - c^{-1/n}) \sum_{i=1}^{n} c^{(m+1)/n} \]
\[ = a^{m+1} (1 - c^{-1/n}) c^{(m+1)/n} \sum_{i=0}^{n-1} c^{i+1} \]
\[ = a^{m+1} (1 - c^{-1/n}) c^{(m+1)/n} \frac{1 - c^{m+1}}{1 - c^{(m+1)/n}} \]
\[ = (a^{m+1} - b^{m+1}) c^{m/n} \frac{c^{1/n} - 1}{1 - c^{(m+1)/n}}. \]

Letting \( x = c^{1/n} \) we recall that may factorize the expression in the denominator as \( (1 - x^{m+1}) = (1 - x)(1 + x + x^2 + \ldots + x^m) \). Thus
\[ U(P, f) = (b^{m+1} - a^{m+1}) c^{m/n} \left( \frac{1}{1 + c^{1/n} + \ldots + c^{m/n}} \right). \]

(b) Show that \( L(P, f) = c^{-m/n} U(P, f) \).

\[ L(P, f) = \sum_{i=1}^{n} a^m c^{(i-1)m/n} [ac^{i/n} - ac^{(i-1)/n}] \]
\[ = c^{-m/n} \sum_{i=1}^{n} a^m c^{i m/n} [ac^{i/n} - ac^{(i-1)/n}] \]
\[ = c^{-m/n} U(P, f). \]

(c) Deduce from parts (a) and (b) that
\[ \int_a^b f = \frac{b^{m+1} - a^{m+1}}{m+1}. \]

Since \( c > 0 \), we know that \( \lim_{n \to \infty} c^{1/n} = 1 \). As \( m \) is fixed, we find
\[ \lim_{n \to \infty} L(P, f) = \lim_{n \to \infty} U(P, f) = (b^{m+1} - a^{m+1}) \left( \frac{1}{m+1} \right). \]

From Theorem 5.1, we deduce
\[ \int_a^b f = \frac{b^{m+1} - a^{m+1}}{m+1}. \]
(d) For \( b = 1 \), determine the limit
\[
\lim_{a \to 0^+} \int_a^1 f.
\]
\[
\lim_{a \to 0^+} \int_a^1 f = \frac{1}{m + 1}.
\]

6. Suppose \( f \) is integrable on \([a, b]\). Let \( g(x) = f(x - c) \) for some \( c \in \mathbb{R} \). By considering lower and upper sums, prove that \( g \) is integrable on \([a + c, b + c]\) and
\[
\int_{a + c}^{b + c} g = \int_a^b f.
\]

From Theorem 5.1, we know that there exists a sequence of partitions \( \{P_n\}_{n=1}^{\infty} \) of \([a, b]\) such that
\[
\lim_{n \to \infty} \mathcal{L}(P_n, f) = \int_a^b f = \lim_{n \to \infty} \mathcal{U}(P_n, f).
\]
Denote the points of \( P_n \) by \( \{x_0, x_1, \ldots, x_n\} \). Let \( Q_n = \{x_0 + c, x_1 + c, \ldots, x_n + c\} \). Note for each \( n \) that \( Q_n \) is a partition of \([a + c, b + c]\), \( \mathcal{L}(Q_n, g) = \mathcal{L}(P_n, f) \), and \( \mathcal{U}(Q_n, g) = \mathcal{U}(P_n, f) \). Hence
\[
\lim_{n \to \infty} \mathcal{L}(Q_n, g) = \lim_{n \to \infty} \mathcal{L}(P_n, f) = \int_a^b f = \lim_{n \to \infty} \mathcal{U}(P_n, f) = \lim_{n \to \infty} \mathcal{U}(Q_n, g).
\]
The desired result then follows upon applying Theorem 5.1.

7. Suppose \( f \) is integrable on \([a, b]\). Let \( g(x) = f(x / c) \) for some \( c > 0 \). By considering lower and upper sums, prove that \( g \) is integrable on \([ac, bc]\) and
\[
\int_{ac}^{bc} g = c \int_a^b f.
\]

From Theorem 5.1, we know that there exists a sequence of partitions \( \{P_n\}_{n=1}^{\infty} \) of \([a, b]\) such that
\[
\lim_{n \to \infty} \mathcal{L}(P_n, f) = \int_a^b f = \lim_{n \to \infty} \mathcal{U}(P_n, f).
\]
Denote the points of \( P_n \) by \( \{x_0, x_1, \ldots, x_n\} \). Let \( Q_n = \{x_0 c, x_1 c, \ldots, x_n c\} \). Note for each \( n \) that \( Q_n \) is a partition of \([ac, bc]\), \( \mathcal{L}(Q_n, g) = c \mathcal{L}(P_n, f) \), and \( \mathcal{U}(Q_n, g) = c \mathcal{U}(P_n, f) \) (note that the interval widths in \( Q_n \) are \( c \) times those of \( P_n \)). Hence
\[
\lim_{n \to \infty} \mathcal{L}(Q_n, g) = c \lim_{n \to \infty} \mathcal{L}(P_n, f) = c \int_a^b f = c \lim_{n \to \infty} \mathcal{U}(P_n, f) = \lim_{n \to \infty} \mathcal{U}(Q_n, g).
\]
The desired result then follows upon applying Theorem 5.1.
8. This exercise provides a third independent way of computing the integral of \( x^n \) on \([0, 1]\). Let \( s_n = \int_0^1 x^n \, dx \) for \( n = 0, 1, 2, \ldots \).

(a) Use Question 6 to show that

\[
\int_0^{2c} x^n \, dx = \int_{-c}^{c} (x + c)^n \, dx.
\]

Let \( f(x) = (x + c)^n \) and \( g(x) = f(x - c) = x^n \). Then

\[
\int_0^{2c} x^n \, dx = \int_0^{2c} g = \int_{-c}^{c} f = \int_{-c}^{c} (x + c)^n \, dx.
\]

(b) Use Question 7 to show that

\[
\int_0^c x^n \, dx = c^{n+1} s_n.
\]

Let \( f(x) = x^n \) and \( g(x) = f(x/c) = x^n/c^n \). Then

\[
\int_0^c x^n \, dx = \int_0^c c^n g = c^{n+1} \int_0^1 f = c^{n+1} s_n.
\]

(c) Question 7 can be extended to the case \( c < 0 \) by interpreting a partition on \([ac, bc]\) as a partition on the interval \([bc, ac]\). Use this extension to show that

\[
\int_{-c}^0 x^k \, dx = (-1)^k \int_0^c x^k \, dx.
\]

Letting \( f(x) = (-x)^k \) and \( g(x) = f(-x) = x^k \), we find

\[
\int_{-c}^0 x^k \, dx = \int_{-c}^0 g = - \int_{-c}^c f = \int_0^c f = (-1)^k \int_0^c x^k \, dx.
\]

(d) Show that \( \int_{-c}^c x^k \, dx \) is equal to 0 for odd \( k \) and \( 2c^{k+1}s_k \) for even \( k \).

This follows from parts (c) and (b) since

\[
\int_{-c}^c x^k \, dx = \int_{-c}^0 x^k \, dx + \int_0^c x^k \, dx = [(-1)^k + 1] \int_0^c x^k \, dx = [(-1)^k + 1]c^{n+1} \int_0^1 x^k \, dx.
\]
(e) Use parts (b), (a), and (d) to prove that

\[ 2^n s_n = \frac{1}{2c^{n+1}} \int_{-c}^{c} (x + c)^n \, dx = \sum_{k=0}^{n} \binom{n}{k} s_k. \]

We find

\[ 2^n s_n = \frac{2^n}{(2c)^{n+1}} \int_{0}^{2c} x^n \, dx = \frac{1}{2c^{n+1}} \int_{-c}^{c} (x + c)^n \, dx \]

\[ = \frac{1}{2c^{n+1}} \int_{-c}^{c} \sum_{k=0}^{n} \binom{n}{k} x^k c^{n-k} \, dx = \frac{1}{2c} \sum_{k=0}^{n} c^{-k} \binom{n}{k} \int_{-c}^{c} x^k \, dx = \sum_{k=0}^{n} \binom{n}{k} s_k. \]

(f) Use the results

\[ \sum_{k=0}^{n} \binom{n}{k} = (1 + 1)^n = 2^n, \quad \sum_{k=0}^{n} \binom{n}{k} (-1)^k = (1 - 1)^n = 0 \]

to prove that

\[ \sum_{k=1}^{n} \binom{n}{k} = 2^{n-1}. \]

Since

\[ \sum_{k=0}^{n} \binom{n}{k} [1 - (-1)^k] = 2^n - 0 = 2^n, \]

we see that

\[ \sum_{k=1}^{n} \binom{n}{k} = 2^{n-1}. \]

(g) Finally, use induction to prove that \( s_n = 1/(n + 1) \) for all \( n \in \mathbb{N} \). It will be necessary to treat the cases where \( n \) is odd and \( n \) is even separately.

We prove this by induction. First we note that \( s_0 = \int_{0}^{1} 1 = 1 = 1/(1 + 0) \). Assume that \( s_k = 1/(k + 1) \) for \( k = 0, 1, \ldots, n - 1 \), where \( n \) is a natural number. We then show that \( s_n = 1/(n + 1) \).

There are two cases to consider. If \( n \) is odd, then no \( s_n \) terms appear on the right-hand side:

\[ 2^n s_n = \sum_{k=0}^{n-1} \binom{n}{k} \frac{1}{k+1} = \sum_{k=0}^{n-1} \binom{n+1}{k+1} \frac{1}{n+1} = \frac{1}{n+1} \sum_{k=1}^{n} \binom{n+1}{k} = \frac{2^n}{n + 1}. \]

Hence \( s_n = 1/(n + 1) \).
If $n$ is an even natural number we first isolate $s_n$ on one side of the equation (being careful not to assume what we are trying to prove):

\[
(2^n - 1)s_n = \sum_{k=0}^{n-2} \binom{n}{k} \frac{1}{k+1} = \sum_{k=0}^{n-2} \binom{n+1}{k+1} \frac{1}{n+1} = \frac{1}{n+1} \sum_{k=1}^{n-1} \binom{n+1}{k}
\]

\[
= \frac{1}{n+1} \left[ \sum_{k=1}^{n+1} \binom{n+1}{k} - \binom{n+1}{n+1} \right] = \frac{1}{n+1}(2^n - 1).
\]

Once again we see that $s_n = 1/(n+1)$. 