1. Determine which of the following limits exist as a finite number, which are $\infty$, which are $-\infty$, and which do not exist at all. Where possible, compute the limit.

(a) \[ \lim_{x \to 1} \frac{x^7 - 1}{x^{11} - 1} \]

\[
\lim_{x \to 1} \frac{x^7 - 1}{x^{11} - 1} = \lim_{x \to 1} \frac{7x^{7-1}}{11x^{11-1}} = \frac{7}{11}.
\]

(b) \[ \lim_{n \to \infty} \cos(n\pi) \quad (\text{here } n \in \mathbb{Z}) \]

We have seen that \( \lim_{n \to \infty} (-1)^n \) does not exist.

(c) \[ \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \]

This is just the definition of the derivative of \( f(x) = \sqrt{x} \) at \( x = 1 \); the limit therefore evaluates to \( f'(1) = 1/2 \).

(d) \[ \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} \]

Hint: first find the square of this limit.

Since

\[ \lim_{x \to \infty} \frac{x^2}{x^2 + 1} = 1. \]
and the square root function is continuous, we know that
\[
\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \to \infty} \sqrt{\frac{x^2}{x^2 + 1}} = \sqrt{\lim_{x \to \infty} \frac{x^2}{x^2 + 1}} = 1.
\]

(e) \[
\lim_{x \to 3} \frac{x + 3}{\tan \pi x}
\]
does not exist, by Corollary 2.2.1 and Theorem 3.1, noting that \(3 + 3 = 6 \neq 0\) but \(\lim_{x \to 3} \tan \pi x = 0\). Note that the limit is not \(\infty\) or \(-\infty\) since \(\tan \pi x\) does not have a definite sign for values of \(x\) near (but not equal to) 3.

(f) \[
\lim_{h \to 0} \frac{\arctan(x + h) - \arctan x + \arctan \frac{1}{x + h} - \arctan \frac{1}{x}}{h} = \frac{d}{dx} \left( \arctan x + \arctan \frac{1}{x} \right) = \frac{d}{dx} \left( \frac{1}{1 + x^2} + \frac{1}{1 + \frac{1}{x^2}} \left( \frac{-1}{x^2} \right) \right) = \frac{d}{dx} \left( \frac{1}{1 + x^2} + \frac{-1}{x^2 + 1} \right) = 0.
\]

2. Find the maximum value of \(f(x) = \cos^4 x \sin x\) on \(\mathbb{R}\).

Hint: first let \(u = \sin x\). What are the possible values of \(u\)?

Letting \(u = \sin x\), we see that we need to maximize \(g(u) = (1 - u^2)^2 u\) on \([-1, 1]\]. Since \(g(u) = u^5 - 2u^3 + u\), we see that \(g'(u) = 5u^4 - 6u^2 + 1 = (5u^2 - 1)(u^2 - 1)\), which has roots when \(u = \pm 1/\sqrt{5}\) and \(u = \pm 1\). The continuous function \(g\) must take on global maximum and minimum values somewhere in \([-1, 1]\). At the endpoints, where \(u = \pm 1\), we see that \(g(u) = 0\). At the remaining critical points, we find that \(g(1/\sqrt{5}) = -g(-1/\sqrt{5}) = \frac{16}{25\sqrt{5}}\). The maximum value of \(f\) on \(\mathbb{R}\) is thus \(\frac{16}{25\sqrt{5}}\).

3. (a) Use the Mean Value Theorem to prove for all real \(a\) and \(b\) that
\[
|\cos a - \cos b| \leq |a - b|.
\]

The Mean Value theorem guarantees for any real numbers \(a\) and \(b\) that
\[
\cos a - \cos b = -\sin c(a - b)
\]
for some \(c \in (a, b)\). Since \(|\sin c| \leq 1\) we deduce
\[
|\cos a - \cos b| = |\sin c||a - b| \leq |a - b|.
\]
(b) Use the Mean Value Theorem to prove for all real $a$ and $b$ that

$$\left| \cos^5 a - \cos^5 b \right| \leq \frac{16}{5\sqrt{5}} |a - b|.$$ 

Hint: use your result from Question 2.

The Mean Value theorem implies for any real numbers $a$ and $b$ that

$$\left| \cos^5 a - \cos^5 b \right| = \left| 5 \cos^4 c \sin c \right| |a - b|$$

for some $c \in (a,b)$. We have already shown in Question 2 that the odd function $f(x) = \cos^4 x \sin x$ (and hence $|f(x)|$) is bounded above by $\frac{16}{25\sqrt{5}}$. Thus

$$\left| \cos^5 a - \cos^5 b \right| \leq \frac{16}{5\sqrt{5}} |a - b|.$$

4. Consider the function $f(x) = \arctan x - x$.

(a) Find $f'(x)$.

$$f'(x) = \frac{1}{1 + x^2} - 1$$

(b) Show that $f$ is one-to-one on $\mathbb{R}$.

Part (a) shows that $f'(x) < 0$ for all $x \neq 0$. This means that the continuous function $f$ is strictly decreasing on $(-\infty, 0]$ and $[0, \infty)$ and hence on all of $\mathbb{R}$.

(c) Let $b = \pi/4 - 1$. Find a real number $a$ such that $f(a) = b$.

$$a = 1.$$ 

(d) Is the value for $a$ in part (c) unique? Circle the correct answer: [Yes] / No.

(e) Let $g$ be the inverse function of $f$. Find $g'(b)$, for the value $b$ given in part (c).

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{f'(1)} = \frac{1}{-\frac{2}{3}} = -2.$$
5. Consider the function \( f(x) = \frac{-x^5}{20} + \frac{x^4}{12} \) on \( \mathbb{R} \).

(a) Determine on which intervals \( f \) is increasing and on which intervals \( f \) is decreasing.

Since
\[
f'(x) = -\frac{x^4}{4} + \frac{x^3}{3} = x^3 \left( -\frac{x}{4} + \frac{1}{3} \right),
\]
we see that \( f \) is decreasing on \((-\infty, 0]\), increasing on \([0, 4/3]\), and decreasing on \([4/3, \infty)\).

(b) Does \( f \) have any interior local extrema? If so, where do these occur? Do they correspond to a local maximum or minimum of \( f \)?

Note that \( f \) has two critical points: \( x = 0 \) and \( x = 4/3 \). By the First Derivative Test, \( f \) has a local minimum at \( x = 0 \) and a local maximum at \( x = 4/3 \).

(c) Does the continuous function \( f \) achieve global minimum or maximum values on \( \mathbb{R} \)? If so, at what points do these occur and what are the corresponding function values? Justify your answer.

No, \( f \) does not have a global maximum or minimum on \( \mathbb{R} \) since \( \lim_{x \to -\infty} f = \infty \) and \( \lim_{x \to \infty} f = -\infty \).

(d) Determine on which intervals \( f \) is convex and on which intervals \( f \) is concave.

Since \( f''(x) = -3x^2 + 2x = -x^2(x-1) \), we see that \( f \) is convex on \((-\infty, 1]\) and concave on \([1, \infty)\).

(e) Does \( f \) have any inflection points? If so, where?

Yes: \( f \) has an inflection point at \( x = 1 \) since its behaviour changes from convex to concave at this point.

(f) Sketch a graph of \( f \) using the above information.
6. Suppose $f$ is differentiable on $[a, b]$ but $f'$ is not necessarily continuous, with $f'(a) = m < M = f'(b)$. Let $v \in (m, M)$ and define $g(x) = f(x) - vx$.

(a) Show that $g$ achieves a global minimum value on $[a, b]$ at some point $c \in [a, b]$.

Since $g$ is differentiable we know that it is continuous and hence must achieve a minimum value on the closed interval $[a, b]$.

(b) Express $g'(x)$ in terms of $f'(x)$.

$$g'(x) = f'(x) - v.$$ 

(c) Prove that $g$ is strictly decreasing near $a$ and strictly increasing near $b$.

This follows from the facts that

$$g'(a) = f'(a) - v = m - v < 0$$

and

$$g'(b) = f'(b) - v = M - v > 0.$$ 

(d) Show that the number $c$ in part (a) cannot be equal to either $a$ or $b$.

This follows from part (c), which precludes there being a minimum at $a$ or at $b$. 

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(e) Prove that \( f'(c) = v \).

Since \( g \) has a local interior minimum at \( c \in (a, b) \), we know that \( g'(c) = 0 \). Hence \( f'(c) = v \). This result, known as the **Darboux theorem**, shows that derivatives, even discontinuous ones, necessarily obey the intermediate value property.

7. (a) Write down the Taylor expansion of \( f(x) = \sin^2 x \) about \( a = 0 \) to \( n = 2 \) terms, including an expression for the remainder term \( R_2 \).

We see that \( f^{(0)}(x) = \sin^2 x, f^{(1)}(x) = 2 \sin x \cos x = \sin 2x \), and \( f^{(2)}(x) = 2 \cos 2x \). Since \( f^{(0)}(0) = f^{(1)}(0) = 0 \), the Taylor expansion of \( f \) is just the remainder term:

\[
\sin^2 x = \frac{2 \cos(2c)}{2} x^2 = \cos(2c)x^2,
\]

where \( c \) is between 0 and \( x \).

(b) For \( x \in [0, \pi/2] \), use part (a) to establish the inequality

\[
\sin x \geq \frac{x}{\sqrt{1 + 2x^2}}
\]

Hint: express the remainder term in terms of \( \sin c \) and estimate it.

On noting that \( \cos(2c) = 1 - 2 \sin^2 c \) is decreasing on \( [0, x] \subset [0, \pi/2] \), we see that

\[
\cos(2c) \geq 1 - 2 \sin^2 x.
\]

Part (a) thus implies that

\[
\sin^2 x \geq (1 - 2 \sin^2 x)x^2,
\]

from which we obtain

\[
(1 + 2x^2) \sin^2 x \geq x^2.
\]

This proves that

\[
\sin x \geq \frac{x}{\sqrt{1 + 2x^2}} \quad \forall x \in [0, \pi/2].
\]