1. Determine which of the following limits exist as a finite number, which are $\infty$, which are $-\infty$, and which do not exist at all. Where possible, compute the limit.

(a) 
\[
\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x}
\]

\[
= \lim_{x \to 0} \frac{1}{2\sqrt{1 + x}} = \frac{1}{2},
\]
on applying the $0/0$ form of L’Hôpital’s Rule.

Alternatively, we recognize the limit to be the derivative of $f(x) = \sqrt{1 + x}$ with respect to $x$ at $x = 0$. Since $f'(x) = \frac{1}{2\sqrt{1 + x}}$, we see that $f'(0) = 1/2$.

(b) 
\[
\lim_{x \to 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right)
\]

We find
\[
= \lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} = \lim_{x \to 0} \frac{\sin x}{\cos x + \cos x - x \sin x} = \frac{0}{2 - 0} = 0,
\]
on applying the $0/0$ form of L’Hôpital’s Rule twice.

(c) 
\[
\lim_{x \to \infty} \frac{\sin x}{x}
\]

L’Hôpital’s Rule doesn’t apply in this case. However, the limit is easily seen to be 0 since
\[\left| \frac{\sin x}{x} \right| \leq \frac{1}{|x|} \leq \epsilon\]
if $x > 1/\epsilon$. Alternatively, one can apply the Squeeze Principle here.

(d) 
\[
\lim_{x \to 0^+} \frac{\sqrt{1 - \cos x}}{x}
\]
A direct application of L’Hôpital’s Rule doesn’t help here either. However, one may multiply numerator and denominator by $1 + \cos x$:

$$
\lim_{x \to 0^+} \frac{\sqrt{1 - \cos^2 x}}{x \sqrt{1 + \cos x}} = \lim_{x \to 0^+} \frac{|\sin x|}{x \sqrt{1 + \cos x}} = \lim_{x \to 0^+} \frac{|\sin x|}{x \sqrt{1 + \cos x}} = \lim_{x \to 0^+} \frac{\sin x}{x \sqrt{1 + \cos x}} = \frac{1}{\sqrt{2}}.
$$

Alternatively, a double-angle formula may be used to rewrite the limit as

$$
\lim_{x \to 0^+} \sqrt{2} \frac{|\sin \frac{x}{2}|}{x} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.
$$

Alternatively, one may first compute the square of the limit,

$$
\lim_{x \to 0^+} \frac{1 - \cos x}{x^2} = \lim_{x \to 0^+} \frac{\sin x}{2x} = \frac{1}{2}
$$

and then take the square root to get the original limit, $1/\sqrt{2}$.

2. Let

$$
f(x) = \begin{cases} 
  x^4 \sin^2 \left(\frac{1}{x}\right) & \text{if } x \neq 0, \\
  0 & \text{if } x = 0.
\end{cases}
$$

(a) Show that $f$ has a local minimum at 0.

Since $f(x) \geq 0 = f(0)$ for all real $x$, we see that $f$ in fact has a global minimum at 0.

(b) Is it possible to find a number $\delta > 0$ such that $f$ is decreasing on $(-\delta, 0)$ and increasing on $(0, \delta)$?

No. The function oscillates between 0 and $x^4$ infinitely often on $(-\delta, 0)$ and $(0, \delta)$ for any $\delta > 0$; there are infinitely many regions within these intervals where it is increasing and infinitely many other regions where it is decreasing. For example, the even function $f$ satisfies $f\left(\frac{2}{(2n+1)\pi}\right) > 0$ and $f\left(\frac{1}{n\pi}\right) = 0$ for $n = \lceil(\pi \delta)^{-1}\rceil$.

(c) Compute $f'(x)$.

$$
f'(x) = \begin{cases} 
  4x^3 \sin^2 \left(\frac{1}{x}\right) - 2x^2 \sin \left(\frac{1}{x}\right) \cos \left(\frac{1}{x}\right) & \text{if } x \neq 0, \\
  0 & \text{if } x = 0.
\end{cases}
$$

(d) Compute $f''(0)$.

$$
\lim_{x \to 0} \frac{4x^3 \sin^2 \left(\frac{1}{x}\right) - 2x^2 \sin \left(\frac{1}{x}\right) \cos \left(\frac{1}{x}\right)}{x} = 0.
$$
(e) Determine whether \( f \) has an inflection point at \( x = 0 \).

For \( x \neq 0 \) we know from part (c) that

\[
f'(x) = 4x^3 \sin^2 \left( \frac{1}{x} \right) - x^2 \sin \left( \frac{2}{x} \right),
\]

and hence

\[
f''(x) = 12x^2 \sin^2 \left( \frac{1}{x} \right) + 4x^3 \sin \left( \frac{1}{x} \right) \cos \left( \frac{1}{x} \right) \left( \frac{-1}{x^2} \right) - 2x \sin \left( \frac{2}{x} \right) - x^2 \cos \left( \frac{2}{x} \right) \left( \frac{-2}{x^2} \right).
\]

In particular \( f'' \left( \frac{1}{n\pi} \right) = 2 \) and \( f'' \left( \frac{2}{(2n+1)\pi} \right) = \frac{48}{n^2\pi^2} - 2 < 0 \) for sufficiently large natural numbers \( n \). So, \( f'' \) takes on both positive and negative values at positive points arbitrarily close to zero and (similarly at negative points close to zero).

There is no change from being concave on one side to convex on the other, so there is no inflection point at \( x = 0 \).

(f) Can either of the First or Second Derivative Tests be used to show that \( f \) has a minimum at \( x = 0 \)?

No. From part (b), we see that the conditions of the First Derivative test do not hold and in part (d), we found that \( f''(0) = 0 \), so the Second Derivative Test gives us no useful information.

3. Consider the function \( f(x) = \sin(x^3) \) on \([-\sqrt{\pi}, \sqrt{\pi}]\).

(a) Find all critical points of \( f \).

The critical points of \( f \) are at 0 and \( \pm \sqrt[3]{\frac{\pi}{2}} \).

(b) Determine on which intervals \( f \) is increasing and on which intervals \( f \) is decreasing.

Since \( f'(x) = 3x^2 \cos(x^3) \), we see that \( f \) is decreasing on \([-\sqrt{\pi}, -\sqrt[3]{\frac{\pi}{2}}]\), increasing on \([-\sqrt[3]{\frac{\pi}{2}}, \sqrt{\pi}]\), and decreasing on \([\sqrt[3]{\frac{\pi}{2}}, \sqrt{\pi}]\).

(c) Find and classify (local/global, interior/endpoint, maximum/minimum) all extrema of \( f \) and the points at which they occur.

We see from the First Derivative Test that \( f \) has an endpoint local maximum (of 0) at \(-\sqrt{\pi}\) and an endpoint local minimum value of 0 at \(\sqrt{\pi}\). There is also an interior global minimum value of \(-1\) at \(-\sqrt[3]{\frac{\pi}{2}}\) and an interior global maximum value of 1 at
$\sqrt[3]{\pi/2}$. The critical point at $x = 0$ is neither a maximum nor a minimum since $f$ is strictly increasing on $[-\sqrt[3]{\pi/2}, \sqrt[3]{\pi/2}]$.

(d) Compute $\lim_{x \to 0} \frac{f''(x)}{x}$.

$$
\lim_{x \to 0} \frac{f''(x)}{x} = \lim_{x \to 0} \frac{-9x^4 \sin(x^3) + 6x \cos(x^3)}{x} = 6.
$$

(e) In view of part (d), does $f$ have an inflection point at $x = 0$? Why or why not?

Letting $\epsilon = 1$, we see from part (d) that $f''$, like the identity function, changes sign at 0. So $f$ is concave on $[-\delta, 0]$ and convex on $[0, \delta]$ for some $\delta > 0$. Thus $f$ does indeed have an inflection point at $x = 0$.

(f) Sketch a graph of $f$ using the above information.

4. Consider the function $f(x) = x - \cos x$.

(a) Find $f'(x)$.

$$
f'(x) = 1 + \sin x.
$$
(b) Show that $f$ is one-to-one on $\mathbb{R}$.

For $x \notin \left\{ \frac{3\pi}{2} + 2n\pi : n \in \mathbb{Z} \right\}$, we know that $\sin x > -1$ and hence $f'(x) > 0$. Thus (from the Mean Value Theorem), $f$ is strictly increasing on each closed interval $[\frac{3\pi}{2} + 2n\pi, \frac{3\pi}{2} + 2(n+1)\pi]$, for $n \in \mathbb{Z}$, and hence on all of $\mathbb{R}$. Therefore, $f$ is one-to-one on $\mathbb{R}$.

(c) Let $b = \frac{\pi}{6} - \sqrt{3}$. Find a number $a \in \mathbb{R}$ for which $f(a) = b$.

$$a = \frac{\pi}{6}.$$

(d) Is the value for $a$ in part (c) unique?

Yes since $f$ is 1-1.

(e) Let $g$ be the inverse function of $f$. Find $g'(b)$, for the value $b$ given in part (c).

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{f'(\frac{\pi}{6})} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}.$$

5. (a) Write down the Taylor expansion of $f(x) = \cos x$ about $a = 0$ to $n = 6$ terms, including an expression for the remainder term $R_6$.

Since $f^{(0)}(0) = \cos 0 = 1$, $f^{(1)}(0) = -\sin 0 = 0$, $f^{(2)}(0) = -\cos 0 = -1$, $f^{(3)}(0) = \sin 0 = 0$, $f^{(4)}(0) = \cos 0 = 1$, $f^{(5)}(0) = -\sin 0 = 0$, $f^{(6)}(c) = -\cos c$, the Taylor expansion of $f$ about $a = 0$ is

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + R_6,$$

where $R_6 = -\frac{x^6}{720} \cos c$ for some $c \in (0, x)$.

(b) Use part (a) to prove that

$$\frac{14}{90} < \cos \sqrt{2} < \frac{15}{91}.$$

On setting $x = \sqrt{2}$ we find

$$\cos \sqrt{2} = 1 - \frac{2}{2} + \frac{4}{24} - \frac{8}{720} \cos c = \frac{1}{6} - \frac{1}{90} \cos c$$

for some $c \in (0, \sqrt{2})$. Since $\cos c$ is strictly decreasing on this interval (noting that $\sqrt{2} < \pi$), we know that $-1 < -\cos c < -\cos \sqrt{2}$. Thus

$$\frac{1}{6} - \frac{1}{90} < \cos \sqrt{2} < \frac{1}{6} - \frac{1}{90} \cos \sqrt{2},$$

from which we find

$$\frac{14}{90} < \cos \sqrt{2} < \frac{90}{91} \cdot \frac{1}{6} = \frac{15}{91}.$$
6. Suppose that a function $f$ is differentiable on $\mathbb{R}$ and $f'(x) = f(x)$ for all $x \in \mathbb{R}$.

(a) Use induction to prove that the $n$-th derivative $f^{(n)}(x) = f(x)$ for all $n \in \mathbb{N}$.

We are told that the desired result holds for $n = 1$. Assume that it holds for $n$. Then

$$f^{(n+1)}(x) = (f^n(x))' = f'(x) = f(x).$$

Hence the result holds for all $n \in \mathbb{N}$.

(b) Let $x \in \mathbb{R}$. Show for any $n \in \mathbb{N}$ that the value of $f$ at a point $x \neq 0$ can be expressed in terms of $f(0)$ and a remainder term $R_n$,

$$f(x) = f(0)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!}\right) + R_n,$$

where

$$R_n = \frac{f^{(n)}(c_n)}{n!} x^n$$

for some point $c_n \in (0, x)$. (If $x < 0$ then by $(0, x)$ we mean the interval $(x, 0)$. Note that $c_n$ depends on $n$.)

This is just the Taylor’s expansion of $f$ at $x$ about the point $a = 0$.

(c) Show that $f$ is bounded on $[0, x]$.

Since $f$ is differentiable on $\mathbb{R}$, it must be continuous on $\mathbb{R}$. Hence $f$ is bounded on any closed interval (and in particular on $[0, x]$).

(d) Using part (c) and the fact that for any fixed $x$,

$$\lim_{n \to \infty} \frac{x^n}{n!} = 0,$$

show that $\lim_{n \to \infty} R_n = 0$.

From part (c) we know that there exists a number $M$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. From part (a) we have that

$$|R_n| \leq \frac{|f^{(n)}(c_n)|}{n!} x^n \leq M \frac{|x|^n}{n!}.$$

Given $\epsilon > 0$ we can find a number $N$ such that $\frac{|x|^n}{n!} \leq \frac{\epsilon}{M}$ for $n > N$, so that $|R_n| < \epsilon$. Thus $\lim_{n \to \infty} R_n = 0$.

(e) If $f(0) = 1$ show that

$$f(x) = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{x^k}{k!}.$$

In Math 118, we will establish the existence of $f$, which we will come to know as the exponential function $\exp(x)$.

$$\lim_{n \to \infty} \sum_{k=0}^{n} \frac{x^k}{k!} = \lim_{n \to \infty} \sum_{k=0}^{n-1} \frac{x^k}{k!} = \lim_{n \to \infty} f(x) - \lim_{n \to \infty} R_n = f(x).$$
7. Suppose that a positive function $f$, with $f(0) = 1$, is differentiable on $\mathbb{R}$, with derivative $f'(x) = f(x)$ for all $x \in \mathbb{R}$.

(a) For each $n \in \mathbb{N}$ show for some number $A_n \in (1, f(1))$ that

$$f(1) = \sum_{k=0}^{n-1} \frac{1}{k!} + \frac{A_n}{n!}.$$  

From Taylor’s Theorem about the point $a = 0$ we know that

$$f(1) = \sum_{k=0}^{n-1} \frac{1}{k!} f(0) + \frac{1}{n!} f(c_n)$$

for some $c_n \in (0, 1)$. Since $f'(x) = f(x) > 0$ on $\mathbb{R}$, we see that $f$ is strictly increasing on $\mathbb{R}$. Thus

$$0 < c_n < 1 \implies 1 < f(c_n) < f(1).$$

The desired result then follows on letting $A_n = f(c_n)$ and noting that $f(0) = 1$.

(b) Use part (a) to prove for every integer $n > f(1)$ that

$$0 < \frac{1}{n} < (n - 1)! f(1) - \sum_{k=0}^{n-1} \frac{(n-1)!}{k!} < \frac{f(1)}{n} < 1.$$  

This follows on multiplying the result

$$\frac{1}{n!} < f(1) - \sum_{k=0}^{n-1} \frac{1}{k!} = \frac{A_n}{n!} < \frac{f(1)}{n!}$$

by $(n - 1)!$.

(c) Deduce from part (b) that $f(1)$ is irrational. Hint: what would part (b) imply for large $n$ if $f(1)$ were rational?

Each term of the summation in part (b) is an integer. If $f(1)$ were rational, this would imply for $n$ sufficiently large that there exists an integer in the interval $(0, 1)$, which is absurd! Hence $f(1)$ must be irrational. In Math 118, we will show that

$$f(1) = e \triangleq \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$