1. (a) A spherical balloon is being inflated at the rate of 10 cm$^3$/s. Given that the volume $V$ of the balloon is related to the radius by $V = \frac{4}{3}\pi r^3$, use the Chain Rule to compute how fast the radius of the balloon is growing when the volume has reached 100 cm$^3$.

The rate of inflation $r(t)$ must equal the rate of volume increase: $dV/dt = 10$ cm$^3$/s. We will need to know the formula for $r$ in terms of $V$,

$$r = \left(\frac{3}{4\pi}\right)^{1/3} V^{1/3}.\]$$

and its derivative,

$$\frac{dr}{dV} = \left(\frac{3}{4\pi}\right)^{1/3} \frac{1}{3} V^{-2/3} = \left(\frac{1}{36\pi}\right)^{1/3} V^{-2/3}.\]$$

When the volume of the balloon is 100 cm$^3$, we can use the Chain Rule to determine that the radius is growing at the rate

$$\frac{dr}{dt} = \frac{dr}{dV} \frac{dV}{dt} = \left(\frac{1}{36\pi}\right)^{1/3} (100 \text{ cm}^3)^{-2/3} \times 10 \frac{\text{cm}^3}{\text{s}} = 0.096 \frac{\text{cm}}{\text{s}}.$$

Incidentally, the derivative of $r$ with respect to $V$ can also be calculated by first calculating $dV/dr = 4\pi r^2$, taking the reciprocal to get $dr/dV$, and finally expressing the result in terms of $V$. (What justifies that one can calculate $dr/dV$ in this way?)

(b) Suppose now that the (constant) inflation rate of the balloon is unknown, but it is known that when the volume is 100 cm$^3$, the radius is growing at a rate of 1 cm/s. How fast is the radius of the balloon growing when the volume has reached 1000 cm$^3$?

We are given that at a time $t_1$, the volume $V(t_1) = 100$ cm$^3$ and $\frac{dr}{dt} \big|_{t_1} = 1$ cm/s. By the Chain Rule,

$$\frac{dr}{dt} \big|_{t_1} = \frac{dr}{dV} \big|_{t_1} \frac{dV}{dt}$$

and

$$\frac{dr}{dt} \big|_{t_2} = \frac{dr}{dV} \big|_{t_2} \frac{dV}{dt},$$

since the inflation rate $\frac{dV}{dt}$ is constant. Hence

$$\frac{dr}{dt} \big|_{t_2} = \left(\frac{dr}{dV} \big|_{t_2}\right) \frac{dr}{dt} \big|_{t_1} = \left(\frac{V(t_2)}{V(t_1)}\right)^{-2/3} \frac{dr}{dt} \big|_{t_1} = \left(\frac{1}{10}\right)^{2/3} \times 1 \frac{\text{cm}}{\text{s}} = 0.215 \frac{\text{cm}}{\text{s}}.$$
2. If a function is continuous on a closed interval, we know from Theorem 3.4 that it must achieve global maximum and minimum values somewhere in the interval. We know from Theorem 4.4 that if these extrema occur in the interior of the interval, then the derivative of the function must vanish there. However, it is also possible that the global maximum or minimum occurs at one of the endpoints of the intervals, in which case it is not necessary that the derivative vanish there.

Find the maximum and minimum values (if they exist) of the following functions by finding the points in the interval where the derivative is 0 and comparing the function values at these points with the function values at the end points. Use this information to sketch each of the functions.

(a) \( f(x) = 2x^3 - 3x^2 + 1 \) on \([\frac{1}{2}, 2]\).

We find \( f'(x) = 6x^2 - 6x = 6x(x - 1) \). The only critical points are at \( x = 0 \) and \( x = 1 \). The point \( x = 1 \) is in the given interval, with \( f(1) = 0 \). We also see that \( f(1/2) = 1/2 \) and \( f(2) = 5 \). So the global maximum value of 5 occurs at \( x = 2 \) and the global minimum value of 0 occurs at the \( x = 1 \).

(b) \( f(x) = x^5 + x + 1 \) on \([-1, 1]\).

There are no critical points. The global minimum value is \(-1\), which occurs at \( x = -1 \) and the global maximum value is 3, which occurs at \( x = 1 \).

(c) \( f(x) = \frac{1}{x^5 + x + 1} \) on \([-\frac{1}{2}, 1]\).

There are no critical points on \([-\frac{1}{2}, 1]\). The global maximum value is \(32/15\), which occurs at \( x = -1/2 \) and the global minimum value is \(1/3\), which occurs at \( x = 1 \).

(d) \( f(x) = \frac{x}{x^2 - 1} \) on \([0, 5]\).

The only critical point on \([0, 5]\) is at \( x = 1 \), where the function is not even defined. There is no global maximum or minimum value (the function is not continuous on \([0, 5]\)).
3. A closed box of dimensions $x \times 2x \times h$ is to be constructed out of sheets of plywood. If one square metre of material is used to construct the box, determine the dimensions $x$ and $h$ that maximize the volume of the box. Determine the maximal volume. Prove that your answer is actually the global maximum.

Expressing length in m, area in m$^2$, and volume in m$^3$, the total surface area is

$$1 = A(x) = 2(xh + 2xh + 2x^2) = 6xh + 4x^2,$$

from which we see we can eliminate $h$:

$$h = \frac{1 - 4x^2}{6x}.$$

Note that for $h$ to be non-negative we must have $x \leq 1/2$. Thus the volume is

$$V(x) = 2x^2h = \frac{1}{3}x(1 - 4x^2) = \frac{1}{3}(x - 4x^3).$$

At an interior local extremum, the differentiable function $V$ must have derivative 0, which implies that $1 - 12x^2 = 0$, or $x = \frac{1}{2\sqrt{3}}$.

Since for any $a > 0$, $V(x)$ is a continuous function on the domain $[a, 1/2]$, we see that it must achieve its global maximal and minimal value somewhere in this interval. At $x = 1/2$ we see that the volume takes on its minimal value of zero (the box has zero height). As $a \to 0$, we note that the volume $V(a)$ also goes to zero. The maximal volume must therefore occur at the critical point $x = \frac{1}{2\sqrt{3}}$. At this point $h = \frac{2}{3\sqrt{3}}$ and $V(x) = \frac{1}{9\sqrt{3}}$.

4. A canoeist is at the southwest corner of a square lake of side 1 km. She would like to travel to the northeast corner of the lake by rowing to a point on the north shore at a speed of 3 km/h in a straight line at an angle $\theta$ measured relative to north. She then plans to walk east along the north shore at a speed of 6 km/h until she arrives at her destination.

At what angle $\theta$ should the canoeist row in order to arrive at her destination in the shortest possible time? What is this minimum time? Prove that your answer corresponds to a minimum.

The time to reach her destination is

$$T(\theta) = \frac{1}{3\cos \theta} + \frac{1 - \tan \theta}{6}.$$

The continuous function $T$ must achieve global minimum and maximum values on $[0, \frac{\pi}{4}]$. First, we look for critical points of this function on $(0, \frac{\pi}{4})$:

$$0 = T'(\theta) = \frac{\sin \theta}{3\cos^2 \theta} - \frac{1}{6\cos^2 \theta} \Rightarrow \sin \theta = \frac{1}{2}. $$
The only critical point in \((0, \frac{\pi}{4})\) is at \(\theta = \frac{\pi}{6}\). By simply comparing values, we see that the endpoint value \(T(0) = 1/2\) is an exterior global maximum, the endpoint value \(T(\frac{\pi}{4}) = \sqrt{2}/3\) is an exterior local maxima, and \(T(\frac{\pi}{6}) = \frac{1+\sqrt{3}}{6}\) is the global minimum value. Thus the canoeist should row at an angle \(\frac{\pi}{6}\) relative to north.

5. Prove that if \(f\) is continuous on \([a,b]\) and \(f'(x) > 0\) for all \(x \in (a,b)\), then \(f\) is strictly increasing on \([a,b]\):

\[x < y \Rightarrow f(x) < f(y) \quad \forall x, y \in [a,b].\]

Since the conditions of the Mean Value Theorem hold, we know whenever \(x < y\) that

\[\frac{f(y) - f(x)}{y - x} = f'(c) > 0,\]

for some point \(c \in (x,y)\). Hence \(f(y) > f(x)\).

6. If \(f'\) is increasing show that

\[\frac{f(x) - f(x-h)}{h} \leq f'(x) \leq \frac{f(x+h) - f(x)}{h}\]

for every real number \(h > 0\).

By the Mean Value Theorem, we know for some numbers \(c \in (x-h,x)\) and \(d \in (x,x+h)\) that

\[\frac{f(x) - f(x-h)}{h} = f'(c) \leq f'(x) \leq f'(d) = \frac{f(x+h) - f(x)}{h}\]

since \(f'\) is increasing.

7. Prove that the sum of a positive number and its reciprocal is at least 2, using (a) the First Derivative Test.

Let \(f(x) = x + \frac{1}{x}\) for \(x \in (0,\infty)\). We see that \(f'(x) = 1 - 1/x^2\) exists for all \(x \in (0,\infty)\) and there are no endpoints to consider. The only possible extremum is therefore at \(x = 1\), where \(f'(x) = 0\).

To see that \(f\) has a global minimum at \(x = 1\), we can use the First Derivative Test: \(f'(x) < 0\) if \(0 < x < 1\) and \(f'(x) > 0\) if \(x > 1\). The global minimum value of \(f\) is hence \(f(1) = 2\).

(b) the Second Derivative Test. Use Corollary 4.5.3 to show that that the local minimum you find with the Second Derivative Test is actually a global minimum.

Since \(f''(x) = 2/x^3\) is positive at \(x = 1\), Corollary 4.4.8 tells us that \(f\) has a local minimum at \(x = 1\). However, Corollary 4.5.3 tells us even more: since \(f''(x) = 2/x^3 > 0\) on \((0,\infty)\), we can be sure that \(f\) achieves a global minimum value at \(x = 1\).
8. Use L’Hôpital’s Rule to compute the following limits. In each case, first check that L’Hôpital’s Rule applies!

(a) 
\[
\lim_{x \to 1} \frac{x - 1}{\tan(x - 1)} = \lim_{x \to 1} \cos^2(x - 1) = 1.
\]

(b) 
\[
\lim_{x \to 1} \frac{x^m - 1}{x^n - 1} = \lim_{x \to 1} \frac{mx^{m-1}}{nx^{n-1}} = \frac{m}{n} \quad (n \neq 0).
\]

(c) 
\[
\lim_{x \to 0} \frac{\sin x - x \cos x}{x^2} = \lim_{x \to 0} \frac{\cos x - \cos x + x \sin x}{2x} = \lim_{x \to 0} \frac{\sin x}{2} = 0.
\]

(d) 
\[
\lim_{x \to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x \to 0} \frac{\frac{1}{\cos^2 x} - \cos x}{3x^2} = \lim_{x \to 0} \frac{2 \sin x}{\cos^3 x} + \sin x = \lim_{x \to 0} \frac{2 \cos^4 x + 6 \sin^2 x \cos^2 x + \cos x}{6} = \frac{1}{2}.
\]