1. Extend the result
\[
\lim_{n \to \infty} c^n = \begin{cases} 
0 & \text{if } 0 \leq c < 1, \\
1 & \text{if } c = 1, \\
\not\exists & \text{if } c > 1
\end{cases}
\]
to show that (here \(n\) takes on only integer values)
\[
\lim_{n \to \infty} r^n = \begin{cases} 
0 & \text{if } |r| < 1, \\
1 & \text{if } r = 1, \\
\not\exists & \text{if } r \leq -1 \text{ or } r > 1.
\end{cases}
\]

If \(-1 < r \leq 0\), let \(c = -r\). Then \(0 \leq c < 1\) and \(|r^n| = |c^n|\). Since \(\lim c_n = 0\), we know that given \(\epsilon > 0\) we can always find an \(N\) such that \(n > N \Rightarrow |r^n| = |c^n| < \epsilon\). We deduce that \(\lim_{n \to \infty} r^n = 0\) as well. (Alternatively, note that for \(c > 0\) we have \(-c^n \leq r^n \leq c^n\) and apply the Squeeze Principle.)

For \(r = -1\), we have already seen that \(\lim_{n \to \infty} r^n\) does not exist.

If \(r < -1\), then \(\{r^n\}\) cannot converge, for otherwise the subsequence \(\{r^{2n}\}\) = \(\{c^n\}\), where \(c = r^2 > 1\) must also converge, contradicting the above result.

2. (a) Let \(r\) be a real number. Consider the sequence \(\{S_n\}_{n=0}^\infty\), where \(S_n\) is the partial sum of the geometric series
\[
S_n = \sum_{k=0}^{n} r^k.
\]

For what values of \(r\) does \(S = \lim_{n \to \infty} S_n\) exist? Compute \(S\) (when the limit exists) in terms of \(r\). When the limit exists, we say that the infinite series
\[
\sum_{k=0}^{\infty} r^k
\]
converges and has limit \(S\). Hint: Consider the telescoping sum \(rS_n - S_n\).

\[
rS_n - S_n = \sum_{k=0}^{n} r^{k+1} - \sum_{k=0}^{n} r^k = \sum_{k=1}^{n+1} r^k - \sum_{k=0}^{n} r^k = r^{n+1} - r^0.
\]

Hence if \(r \neq 1\),
\[
S_n = \frac{r^{n+1} - 1}{r - 1}.
\]
For \( r = 1 \), we compute directly that \( S_n = \sum_{k=0}^{n} 1 = n + 1 \). In this case, the sequence of partial sums \( \{S_n\} = \{n + 1\} \) diverges.

Otherwise, we know from the properties of limits that \( S_n = \frac{(r^{n+1} - 1)}{(r - 1)} \) converges whenever \( r^n \) converges. Likewise, since (for \( r \neq 0 \))

\[
r^n = \frac{(r - 1)S_n + 1}{r},
\]

we see also that \( S^n \) cannot converge when \( r^n \) diverges, or we would obtain a contradiction. Hence

\[
S = \lim_{n \to \infty} S_n = \begin{cases} 
\frac{1}{1-r} & \text{if } |r| < 1, \\
\notin & \text{if } |r| \geq 1.
\end{cases}
\]

(b) Use your result from part (a) to compute the sum

\[
\sum_{k=1}^{\infty} 9 \times 10^{-k}.
\]

\[
\sum_{k=1}^{\infty} 9 \times 10^{-k} = 9 \sum_{k=0}^{\infty} 10^{-(k+1)} = 9 \frac{1}{10} \sum_{k=0}^{\infty} 10^{-k} = 9 \frac{1}{10} \left( \frac{1}{1 - \frac{1}{10}} \right) = 1.
\]

3. Consider the sequence \( \{a_n\}_{n=1}^{\infty} \) defined inductively by \( a_1 = 1 \), and \( a_{n+1} = \sqrt{2a_n + 8} \) for \( n \geq 1 \).

(a) Use induction to prove that \( \{a_n\}_{n=1}^{\infty} \) is bounded above by 4.

Step 1: For \( n = 1 \), we see that \( a_1 = 1 \leq 4 \).

Step 2: Suppose that \( a_n \leq 4 \) for some \( n \in \mathbb{N} \). Then \( a_{n+1}^2 = 2a_n + 8 \leq 2 \cdot 4 + 8 = 16 \), from which, on observing that \( a_{n+1} \geq 0 \), it follows that \( a_{n+1} \leq 4 \).

(b) Prove that \( \{a_n\}_{n=1}^{\infty} \) is a monotone sequence. Is it an increasing or a decreasing sequence?

Since \( a_n \in [0, 4] \) we find that

\[
a_{n+1}^2 - a_n^2 = 2a_n + 8 - a_n^2 = -(a_n^2 - 2a_n - 8) = -(a_n + 2)(a_n - 4) \geq 0.
\]

Thus \( a_{n+1}^2 \geq a_n^2 \) for all \( n \in \mathbb{N} \). That is, \( \{a_n\}_{n=1}^{\infty} \) is an increasing sequence.

(c) Prove that \( \{a_n\}_{n=1}^{\infty} \) converges.

Since \( \{a_n\}_{n=1}^{\infty} \) is an increasing bounded sequence (bounded by 4 from above and \( a_1 \) from below), it must be convergent.
(d) Find \( \lim_{n \to \infty} a_n \). Justify each step in your argument.

From part (c) we know that \( L = \lim_{n \to \infty} a_n \) exists. Moreover \( a_n \geq 0 \) implies that \( L \geq 0 \).
The subsequence \( \{a_{n+1}\}_{n=1}^{\infty} \) of \( \{a_n\}_{n=1}^{\infty} \) must be convergent to the same limit \( L \). By
the properties of limits we know that
\[
L^2 = \lim_{n \to \infty} a_{n+1}^2 = \lim_{n \to \infty} (2a_n + 8) = 2L + 8.
\]
Of the possible solutions to \( 0 = L^2 - 2L - 8 = (L + 2)(L - 4) \) only the solution \( L = 4 \) satisfies
\( L \geq 0 \). Hence \( \lim_{n \to \infty} a_n = 4 \).

4. Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence such that \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = r \), where \( r \in [0, 1) \). Let
\( s = (1 + r)/2 \), so that \( 0 \leq r < s < 1 \).

(a) Show that there exists a number \( N \) such that \( n \geq N \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| \leq s \). Hint:
Consider \( \epsilon = s - r > 0 \).

Let \( \epsilon = s - r > 0 \). Now \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = r \) means in particular that there exists a
number \( N > 0 \) such that
\[
n \geq N \Rightarrow \left| \frac{a_{n+1}}{a_n} - r \right| < s - r.
\]
By the Triangle Inequality, we see that
\[
\left| \frac{a_{n+1}}{a_n} \right| < \left| \frac{a_{n+1}}{a_n} - r \right| + |r| < s - r + r = s.
\]
(b) Use part (a) and induction to show for \( n \geq N \) that
\[
0 \leq |a_n| s^n \leq |a_N| s^n.
\]
The left-hand inequality holds for all \( n \) since \( s > 0 \). The right-hand inequality holds
for \( n = N \). Moreover if \( |a_n| s^N \leq |a_N| s^n \), then
\[
n \geq N \Rightarrow |a_{n+1}| s^N \leq |a_n| s^{n+1} \leq |a_N| s^{n+1},
\]
so the result holds when \( n \) is replaced by \( n + 1 \). By induction, we see that the
right-hand inequality holds for all \( n \geq N \).

(c) Prove that \( \lim_{n \to \infty} a_n = 0 \).

We note that \( \lim_{n \to \infty} |a_N| s^n = |a_N| \lim_{n \to \infty} s^n = 0 \) since \( s < 1 \). The constant sequence
\( \{0\}_{n=1}^{\infty} \) is also convergent to 0. By the Squeeze Principle, noting that \( s^n > 0 \), we
deduce from part (b) that \( \lim_{n \to \infty} |a_n| = 0 \). Directly from the definition of a limit, this
implies that \( \lim_{n \to \infty} a_n = 0 \).
(d) Apply this result to prove for any $x \in \mathbb{R}$ that
\[ \lim_{n \to \infty} \frac{x^n}{n!} = 0. \]
Letting $a_n = \frac{x^n}{n!}$, we see that the result follows from the fact that
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \to \infty} \left| \frac{x}{n+1} \right| = 0. \]

(e) If $r > 1$ show that $\{a_n\}$ is divergent. Hint: consider the sequence $\{1/a_n\}$.
Let $b_n = 1/a_n$. We see that
\[ \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{r}. \]
Since $0 < 1/r < 1$, we know that $\lim_{n \to \infty} b_n = 0$. We then see from Corollary 2.2.1 that $\lim_{n \to \infty} \frac{1}{b_n}$ does not exist. That is, $\{a_n\}$ is divergent.

(f) If $r = 1$ give examples to illustrate that $\{a_n\}$ may be either convergent or divergent.
We see for both the convergent sequence
\[ a_n = \left\{ \frac{1}{n} \right\} \]
and the divergent sequence
\[ a_n = \{n\} \]
that $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If, given any natural number $M$, we can find a number $N$ such that
\[ n > N \Rightarrow a_n > M, \]
we say that $\lim_{n \to \infty} a_n = \infty$.

(a) Suppose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences of real numbers such that $\lim_{n \to \infty} a_n = \infty$ and $\lim_{n \to \infty} b_n = \infty$. Show that
\[ \lim_{n \to \infty} (a_n + b_n) = \infty \]
and
\[ \lim_{n \to \infty} a_n b_n = \infty. \]
We are told that given any natural number $M$, we can find numbers $N_1$ and $N_2$ such that
\[ n > N_1 \Rightarrow a_n > M \]
and
\[ n > N_2 \Rightarrow b_n > M. \]
Let $N = \max(N_1, N_2)$. Then
\[ n > N \Rightarrow a_n + b_n > 2M > M \]
and
\[ n > N \Rightarrow a_nb_n > M^2 \geq M \]
since $M \geq 1$. Hence
\[ \lim_{n \to \infty} (a_n + b_n) = \infty \]
and
\[ \lim_{n \to \infty} a_nb_n = \infty. \]

(b) Under the conditions of part (a), find examples that demonstrate that
\[ \lim_{n \to \infty} (a_n - b_n) \]
and
\[ \lim_{n \to \infty} \frac{a_n}{b_n} \]
may exist as a real number, may have an infinite limit, or may fail to exist at all.

<table>
<thead>
<tr>
<th>( a_n - b_n )</th>
<th>( a_n = n, b_n = n )</th>
<th>( a_n = 2n, b_n = n )</th>
<th>( a_n = n + (-1)^n, b_n = n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{a_n}{b_n} )</td>
<td>( a_n = n, b_n = n )</td>
<td>( a_n = n^2, b_n = n )</td>
<td>( a_n = n[2 + (-1)^n], b_n = n )</td>
</tr>
</tbody>
</table>

(c) Show that
\[ \lim_{n \to \infty} a_n = \infty \Rightarrow \lim_{n \to \infty} \frac{1}{a_n} = 0. \]

Given $\epsilon > 0$, let $M = 1/\epsilon$. We know that we can find a number $N$ such that
\[ n > N \Rightarrow a_n > M \Rightarrow |a_n| > M \Rightarrow \left| \frac{1}{a_n} \right| < \frac{1}{M} = \epsilon. \]
Hence \( \lim_{n \to \infty} \frac{1}{a_n} = 0. \)
(d) Does the converse to (c) hold? That is, does
\[
\lim_{n \to \infty} \frac{1}{a_n} = 0 \Rightarrow \lim_{n \to \infty} a_n = \infty?
\]

No, the converse does not hold. Consider \(a_n = -n\). The sequence \(\{\frac{1}{a_n}\}\) converges to 0 as \(n \to \infty\) but \(\lim_{n \to \infty} a_n \neq \infty\) since for \(M = 1\) we have \(a_{N+1} < 0 < M\) no matter which \(N\) we consider. In fact, we say
\[
\lim_{n \to \infty} a_n = -\lim_{n \to \infty} (-a_n) = -\infty.
\]

6. Let \(\{a_n\}_{n=1}^{\infty}\) be a sequence with \(|a_n| \leq B\) for all \(n \in \mathbb{N}\). Consider the sequence \(\{s_n\}_{n=1}^{\infty}\) defined by \(s_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}\) for \(n \in \mathbb{N}\).

(a) Prove that \(\{s_n\}_{n=1}^{\infty}\) is a bounded sequence.

For every \(n \in \mathbb{N}\), we know that \(B\) is an upper bound for the set \(\{a_n, a_{n+1}, a_{n+2}, \ldots\}\), while \(s_n\) is the least upper bound of this set. Hence \(s_n \leq B\) for all \(n \in \mathbb{N}\). Moreover, \(s_n \geq a_n \geq -B\). Thus, \(|s_n| \leq B\) for all \(n \in \mathbb{N}\). That is, \(\{s_n\}_{n=1}^{\infty}\) is a bounded sequence.

(b) Prove that \(\{s_n\}_{n=1}^{\infty}\) is a monotone sequence. Is \(\{s_n\}_{n=1}^{\infty}\) an increasing or a decreasing sequence?

Notice that \(s_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}\) is an upper bound for \(\{a_{n+1}, a_{n+2}, \ldots\}\). But \(s_{n+1} = \sup\{a_{n+1}, a_{n+2}, \ldots\}\) is the least upper bound of \(\{a_{n+1}, a_{n+2}, \ldots\}\). Hence \(s_{n+1} \leq s_n\) for all \(n\). That is, \(\{s_n\}_{n=1}^{\infty}\) is a decreasing sequence.

Alternatively, given a bounded set \(S\), we proved for any nonempty set \(T \subset S\) that \(\sup T \leq \sup S\). Then
\[
s_{n+1} = \sup\{a_{n+1}, a_{n+2}, \ldots\} \leq \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} = s_n.
\]

(c) Is \(\{s_n\}_{n=1}^{\infty}\) necessarily convergent?

Yes, since \(\{s_n\}_{n=1}^{\infty}\) is a decreasing bounded sequence, it must be convergent.

Note: The limit of the sequence \(\{s_n\}_{n=1}^{\infty}\) is known as the limit superior of the sequence \(\{a_n\}_{n=1}^{\infty}\) and is written \(\limsup a_n\). This is just the supremum of the values in the tail of the sequence. In a similar manner, one can define \(\liminf a_n = \lim_{n \to \infty} i_n\), where \(i_n = \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}\). A bounded sequence \(\{a_n\}_{n=1}^{\infty}\) converges \(\iff\) \(\liminf a_n = \limsup a_n\). For example, the bounded sequence \(\{\sin n\}_{n=1}^{\infty}\) does not converge because \(\liminf \sin n = -1\) and \(\limsup \sin n = 1\).