1. Prove that $\sqrt{3}$ is irrational.

Suppose that there existed integers $p$ and $q$ such that $p^2 = 3q^2$. Without loss of generality we may assume that $p$ and $q$ are not both divisible by 3 (otherwise we could cancel out the common factor of 3). We note that $p^2$ is divisible by 3.

Express $p = 3n + r$ where $r = 0, 1, 2$. Then $p^2 = 9n^2 + 6nr + r^2$. If $r = 1$ or $r = 2$, then $p^2$ is not a multiple of 3. The only way that $p^2$ can be divisible by 3 is if $p$ is itself a multiple of 3. (Alternatively, consider the prime factorization of $p$. Since 3 is prime, the only way it can be a factor of $p^2$ is if it is also a factor of $p$.)

Hence $9n^2 = 3q^2$, or $3n^2 = q^2$. Replacing $p$ by $q$ in the above argument, we see that $q$ is also divisible by 3. This contradicts the fact that $p$ and $q$ are not both divisible by 3.

2. Let $r$ and $s$ be rational numbers.

(a) Is $r + s$ necessarily a rational number? (Prove or provide a counterexample.)

Yes, let $r = p/q$ and $s = m/n$, where $p, m \in \mathbb{Z}$ and $q, n \in \mathbb{N}$; we may always write

$$\frac{p}{q} + \frac{m}{n} = \frac{pn + mq}{qn}.$$

(b) Is $r - s$ necessarily a rational number?

Yes, we may always write

$$\frac{p}{q} - \frac{m}{n} = \frac{pm - mq}{qn}.$$

(c) Is $rs$ necessarily a rational number?

Yes, we may always write

$$\frac{pm}{qn} = \frac{pm}{qn}.$$

(d) Is $r/s$ necessarily a rational number?

No, consider $r = 1$, $s = 0$. There is no rational number $r/s$.

3. Let $x$ and $y$ be irrational numbers. Prove or provide a counterexample:

(a) Is $x + y$ necessarily an irrational number?

No, consider $x = \sqrt{2}$, $y = -\sqrt{2}$.
(b) Is $x - y$ necessarily an irrational number?
No, consider $x = \sqrt{2}$, $y = \sqrt{2}$.

(c) Is $xy$ necessarily an irrational number?
No, consider $x = \sqrt{2}$, $y = \sqrt{2}$.

(d) Is $x/y$ necessarily an irrational number?
No, consider $x = \sqrt{2}$, $y = \sqrt{2}$.

4. Let $r$ be a rational number and $x$ be an irrational number. Prove or provide a counterexample:

(a) Is $r + x$ necessarily an irrational number?
Yes, let $y = r + x$; if $y$ were rational then $y - r$ would be rational.

(b) Is $r - x$ necessarily an irrational number?
Yes, let $y = r - x$; if $y$ were rational then $r - y$ would be rational.

(c) Is $rx$ necessarily an irrational number?
No, consider $r = 0$, $x = \sqrt{2}$.

(d) Is $r/x$ necessarily an irrational number?
No, consider $r = 0$, $x = \sqrt{2}$.

5. Prove that if the decimal expansion of a real number ends in a repeating pattern of digits, the number must be rational.

(a) First show that numbers of the form $0.d_1d_2d_3\ldots d_n$, are rational, where $d_i$, $i = 1 \ldots n$, are decimal digits. Hint: what happens when you multiply such a number by $10^n$?

Let $x$ be a number of this form. Then $10^n x = d_1d_2d_3\ldots d_n + x$. Solving for $x$, we find that $x$ is the ratio of the integer with digits $d_1d_2d_3\ldots d_n$ to the integer $10^n - 1$. Hence $x$ is a rational number.

(b) Now generalize your result in part (a) to show that the statement holds for any real number that has a decimal expansion ending in a repeating pattern.

Denote the number by $x$ and express its decimal expansion as

$$x = a_1a_2\ldots a_kb_1b_2\ldots b_l d_1d_2d_3\ldots d_n.$$  

From part (a), we know that $0.d_1d_2d_3\ldots d_n = p/q$ for some integers $p$ and $q$. Upon multiplying $x$ by $10^k$ we see that

$$10^k x = a_1a_2\ldots a_kb_1b_2\ldots b_l + p/q.$$  

Hence

$$x = \frac{(a_1a_2\ldots a_kb_1b_2\ldots b_l) \times q + p}{10^k q},$$

so $x$ is indeed a rational number.
6. (a) Consider the quadratic equation $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are real numbers. For which values of $a$, $b$, and $c$ does this equation have (i) one root $x$; (ii) two roots; (iii) no roots; (iv) infinitely many roots? For cases (i) and (ii), derive the formula that determines the roots. Hint: complete the square.

If $a \neq 0$, we can divide both sides of the equation $ax^2 + bx + c = 0$ by $a$ to obtain

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

We now “complete the square” by writing the first two terms as a perfect square minus a constant:

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0.$$

On solving for $x$ we obtain

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$  

This is the quadratic formula.

Also, if $a = 0$ and $b \neq 0$, then $bx + c = 0$ has the single root $x = -c/b$. Finally, in the case $a = b = 0$ there are infinitely many solutions when $c = 0$ and no solutions when $c \neq 0$.

To summarize:

(i) $a \neq 0$, $b^2 - 4ac = 0$ or $a = 0$, $b \neq 0$;
(ii) $a \neq 0$, $b^2 - 4ac > 0$;
(iii) $a \neq 0$, $b^2 - 4ac < 0$ or $a = b = 0$, $c \neq 0$;
(iv) $a = b = c = 0$.

(b) A rectangular sheet of paper is cut along a line parallel to one side. One of the resulting pieces is square. The other piece has the same aspect ratio (length/width) as the original piece. If the width (shortest dimension) of the original sheet of paper was 1, what was its length?

Denote the length of the original sheet of paper by $L$. We are told that

$$\frac{L - 1}{1} = \frac{1}{L}.$$

Hence $L^2 - L - 1 = 0$. Of the two root to this quadratic equation,

$$L = \frac{1 \pm \sqrt{5}}{2},$$

only the solution $L = \frac{1 + \sqrt{5}}{2}$ is greater than the width 1. This number, which differs from its reciprocal by 1, is the famous golden ratio.
7. Let $a$ and $b$ be real numbers satisfying $0 < a < b$.

(a) Show that $a^2 < b^2$.

On multiplying the inequality $a < b$ respectively by the positive numbers $a$ and $b$, we find that $a^2 < ab$ and $ab < b^2$, so that $a^2 < b^2$.

Alternatively, consider that $0 < (b - a)(b + a) = b^2 - a^2$.

(b) Show that $\sqrt{a} < \sqrt{b}$.

If $\sqrt{a} > \sqrt{b}$, then part (a) would imply that $a > b$. If $\sqrt{a} = \sqrt{b}$, then $a = b$. Given that $a < b$, we can then be sure that $\sqrt{a} < \sqrt{b}$.

Alternatively consider that $0 < b - a = (\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a})$. Hence $0 < \sqrt{b} - \sqrt{a}$ (since the reciprocal of a positive number is positive).

(c) Show that

\[
ab < \frac{a^2 + b^2}{2}.
\]

\[
0 < (a - b)^2 = a^2 - 2ab + b^2
\]

\[
\Rightarrow 2ab < a^2 + b^2,
\]

from which we deduce the desired result.

(d) Show that

\[
a < \sqrt{ab} < \frac{a + b}{2} < b.
\]

We have shown in class that

\[
a < \frac{a + b}{2} < b,
\]

so all we have left to establish is that

\[
a < \sqrt{ab} < \frac{a + b}{2}.
\]

We know from part (b) that $\sqrt{a} < \sqrt{b}$. On multiplying this inequality by $\sqrt{a}$, we deduce $a < \sqrt{ab}$. Finally, from part (c),

\[
2ab < a^2 + b^2 \Rightarrow 4ab < a^2 + 2ab + b^2 = (a + b)^2 \Rightarrow ab < \frac{(a + b)^2}{4} \Rightarrow \sqrt{ab} < \frac{(a + b)}{2},
\]

again using part (b).
8. Use induction to prove the formula, for \( n \in \mathbb{N} \),

\[
\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}.
\]

Step 1: We see for \( n = 1 \) that \( 1 = 1(1 + 1)(2 + 1)/6 \).

Step 2: Suppose

\[
\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6} = S_n.
\]

(where \( \doteq \) denotes a definition). Then

\[
\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n + 1)^2
\]

\[
= \frac{n(n + 1)(2n + 1)}{6} + (n + 1)^2 = \frac{(n + 1)}{6}[n(2n + 1) + 6(n + 1)]
\]

\[
= \frac{(n + 1)}{6}(2n^2 + 7n + 6) = \frac{(n + 1)(n + 2)(2n + 3)}{6} = S_{n+1}.
\]

By induction, we see that the given formula holds for all \( n \in \mathbb{N} \).