1. For \( n = 0, 1, 2, \ldots \), find a reduction formula for

\[
I_n = \int_1^e \log^n x \, dx
\]

(where \( \log \) denotes the natural logarithm).

For \( n = 0 \), we see that \( I_0 = e - 1 \).

For \( n \geq 1 \), we find on integrating by parts that

\[
I_n = \int_1^e \log^n x \, dx = [x \log x]_1^e - \int_1^e x \log^{n-1} x \cdot \frac{1}{x} \, dx = e - n I_{n-1}.
\]

An alternative reduction formula for \( n \geq 2 \) is

\[
I_n = \int_1^e \log x \cdot \log^{n-1} x \, dx
\]

\[
= [(x \log x - x) \log^{n-1} x]_1^e - \int_1^e (x \log x - x)(n-1) \log^{n-2} x \cdot \frac{1}{x} \, dx
\]

\[
= -(n-1) \int_1^e (\log x - 1) \log^{n-2} x \, dx
\]

\[
= (n-1)(I_{n-2} - I_{n-1}).
\]

We can use this formula together with \( I_0 \) and \( I_1 = [x \log x - x]_1^e = 1 \) recursively to find \( I_n \) for any integer \( n \geq 2 \).

2. Establish the results (which you can use on Homework 2):

\[
\cosh^2 x = \frac{\cosh 2x + 1}{2},
\]

\[
\sinh^2 x = \frac{\cosh 2x - 1}{2},
\]

\[
2 \sinh x \cosh x = \sinh 2x.
\]

\[
\cosh^2 x = \frac{(e^x + e^{-x})^2}{4} = \frac{e^{2x} + 2 + e^{-2x}}{4} = \frac{\cosh 2x + 1}{2},
\]

\[
\sinh^2 x = \frac{(e^x - e^{-x})^2}{4} = \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{\cosh 2x - 1}{2},
\]

and

\[
2 \sinh x \cosh x = \frac{2(e^x - e^{-x})(e^x + e^{-x})}{4} = \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x.
\]
3. Let \( f \) be a strictly increasing differentiable function on \([a,b]\) with inverse \( g \). Consider the statement:
\[
2\pi \int_a^b x[f(x) - f(a)] \, dx = \pi \int_{f(a)}^{f(b)} [b^2 - g^2(y)] \, dy.
\]

(a) Establish this result with a picture for the case where \( a > 0 \) and \( f(a) > 0 \) by illustrating each integral as a volume.

Consider the volume obtained by revolving the hatched area in the diagram below about the \( y \) axis. The left-hand integral slices this volume along the red vertical lines; this corresponds to the method of shells. The right-hand integral slices the very same volume along the blue horizontal lines; this corresponds to the method of cross sections. For illustration purposes we have revolved only from 0 to \( 4\pi/3 \) radians.

(b) Perform a substitution followed by an integration by parts to provide an independent proof of this result. Hint: try substituting \( x = g(y) \) in the right-hand side.

On substituting \( x = g(y) \) in the integral on the right-hand side, and integrating by parts, we obtain
\[
\pi \int_a^b [b^2 - x^2] f'(x) \, dx = \pi \left[ (b^2 - x^2) f(x) \right]_a^b + \pi \int_a^b 2xf(x) \, dx = -\pi (b^2 - a^2) f(a) + 2\pi \int_a^b xf(x) \, dx
\]
\[
= -2\pi f(a) \int_a^b x \, dx + 2\pi \int_a^b xf(x) \, dx = 2\pi \int_a^b x[f(x) - f(a)] \, dx.
\]