Math 100: Calculus I

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Preface

These notes were developed for a first-year engineering mathematics course on differential and integral calculus at the University of Alberta. The author would like to thank Andy Hammerlindl and Tom Prince for coauthoring the high-level graphics language `Asymptote` (freely available at `http://asymptote.sourceforge.net`) that was used to draw the mathematical figures in this text. The code to lift TEX characters to three dimensions and embed them as surfaces in `PDF` files was developed with in collaboration with Orest Shardt.

Chapter 0

Real Numbers

Mathematics deals with different types of numbers:

$\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of natural (counting) numbers;

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the set of integers;

$\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0\}$, the set of rational numbers (fractions);

$\mathbb{R}$, the set of all real numbers.

Notice that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Remark: The decimal expansion of a rational number ends in a repeating pattern of digits:

$$1/2 = 0.5000\ldots = 0.5\bar{0}$$

$$1/3 = 0.3333\ldots = 0.\bar{3}$$

$$2/7 = 0.285714285714\ldots = 0.2\overline{85714}$$

Remark: The real numbers are those numbers like $\sqrt{2} = 1.414213562373\ldots$ and $\pi = 3.1415926535897\ldots$ that do not end in a repeating pattern and thus cannot be represented as a ratio of two integers.

0.1 Open and Closed Intervals

Let $a, b \in \mathbb{R}$ and $a < b$. There are 4 types of (finite) intervals:

$$\begin{aligned}
[a, b] &= \{x : a \leq x \leq b\}, \leftarrow \textit{closed} \text{ (contains both endpoints)}\\
(a, b) &= \{x : a < x < b\}, \leftarrow \quad \textit{open} \text{ (excludes both endpoints)}\\
[a, b) &= \{x : a \leq x < b\},\\
(a, b] &= \{x : a < x \leq b\}.
\end{aligned}$$

It is convenient to also define:

$$\begin{aligned}
&(-\infty, \infty) = \mathbb{R}, \\
&[a, \infty) = \{x : x \geq a\}, \\
&(a, \infty) = \{x : x > a\}, \\
&(-\infty, a] = \{x : x \leq a\}, \\
&(-\infty, a) = \{x : x < a\}.
\end{aligned}$$

0.2 Inequalities

- $a < b \Rightarrow a + c < b + c$
- $a < b$ and $c < d \Rightarrow a + c < b + d$
- $a < b$ and $c > 0 \Rightarrow ac < bc$
- $a < b$ and $c < 0 \Rightarrow ac > bc$
- $0 < a < b \Rightarrow 1/a > 1/b$

- To determine the set of x values for which $x^2 - 5x + 6 < 0$, we factor $x^2 - 5x + 6 = (x - 2)(x - 3)$ and consider the following table:

Interval	$x - 2$	$x - 3$	$(x - 2)(x - 3)$
$x < 2$	$-$	$-$	$+$
$2 < x < 3$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$

We thus see that $x^2 - 5x + 6 = (x - 2)(x - 3) < 0$ if and only if $2 < x < 3$, in other words, when $x \in (2, 3)$.

0.3 Absolute Value

The fact that for any nonzero real number either $x > 0$ or $-x > 0$ makes it convenient to define an *absolute value* function:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Properties: Let x and y be any real numbers.

(A1) $|x| \geq 0$.

(A2) $|x| = 0 \iff x = 0$.

(A3) $|-x| = |x|$.

(A4) $|xy| = |x|\,|y|$.

(A5) If $a \geq 0$, then

$$|x| \leq a \iff -a \leq x \leq a.$$

Proof: First note the equivalence

$$\begin{aligned} |x| \leq a \quad &\iff \quad 0 \leq x \leq a \ \text{ or } \ 0 < -x \leq a \\ &\iff \quad -a \leq x \leq a. \end{aligned}$$

(A6) $-|x| \leq x \leq |x|$.

Proof: Apply (A5) with $a = |x|$.

(A7)

$$|x + y| \leq |x| + |y|. \qquad \textit{(Triangle Inequality)}$$

Proof:

$$\textbf{(A6)} \Rightarrow \begin{cases} -|x| \leq x \leq |x| \\ -|y| \leq y \leq |y| \end{cases}$$

$$\begin{aligned} &\Rightarrow -(|x| + |y|) \leq x + y \leq |x| + |y| = a \\ &\textbf{(A5)} \Rightarrow |x + y| \leq |x| + |y|. \end{aligned}$$

Remark: On letting $y \to -y$, we can use **(A3)** to rewrite the Triangle Inequality as

$$|x - y| \leq |x| + |y|.$$

- If $|u-1| < 0.1$ and $|v-1| < 0.2$ then

$$\begin{aligned} |u-v| = |(u-1)-(v-1)| &\leq |u-1| + |v-1| \\ &< 0.1 + 0.2 = 0.3. \end{aligned}$$

Remark: For all real x, $\sqrt{x^2} = |x| \geq 0$. By definition, $\sqrt{x}$ and $x^{1/2}$ denote the **non-negative** square root of x. For example, $\sqrt{4} = 2$, not ± 2.

Remark: Note the following equivalences:

- $|x| = a \iff x = \pm a$;
- $|x| < a \iff -a < x < a$;
- $|x| > a \iff x < -a$ or $x > a$.

Problem 0.1: Find the set of x such that $|x-1| \leq x$.

To handle the absolute value, we must break the problem into cases:

We know that the argument $x-1$ of the absolute value is non-negative when $x \geq 1$. In this case, the inequality reduces to

$$x - 1 \leq x,$$

which always holds. The solutions set for this case is thus $[1, \infty)$.

In the remaining case, where $x < 1$, the inequality becomes

$$-x + 1 \leq x,$$

which means that $1 \leq 2x$, or equivalently, $x \geq 1/2$. But this is still under the restriction that $x < 1$ so our solution set for this case is the interval $[1/2, 1)$.

The complete solution to the original inequality $|x-1| \leq x$ is the union of our two solutions sets, namely $[1, \infty) \cup [1/2, 1) = [1/2, \infty)$.

Chapter 1

Functions

1.1 Examples of Functions

Definition: A *function* f is a rule that associates a real number y to each real number x in some subset $\mathcal{D}$ of $\mathbb{R}$. The set $\mathcal{D}$ is called the *domain* of f.

Definition: The *range* $f(\mathcal{D})$ of f is the set $\{f(x) : x \in \mathcal{D}\}$.

- $f(x) = x^2$ on domain $\mathcal{D} = [0, 2)$:

$$f(\mathcal{D}) = \{x^2 : x \in [0, 2)\} = [0, 4).$$

An equivalent definition is:

Definition: A *function* is a collection of pairs of numbers (x, y) such that if (x, y_1) and (x, y_2) are in the collection, then $y_1 = y_2$. That is,

$$x_1 = x_2 \Rightarrow f(x_1) = f(x_2).$$

This can be restated as the *vertical line test*: an set of ordered pairs (x, y) is a function if every vertical line intersects their graph at most once.

Definition: If a function f has domain A and range B, we write $f : A \to B$.

Definition: *Constant functions* are functions of the form $f(x) = c$, where c is a constant.

- $f(x) = 1$ is a constant function.

Definition: *Polynomials* are functions of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0.$$

When $a_n \neq 0$, we say that the *degree* of f is n and write $\deg f = n$. While a nonzero constant function has degree 0, it turns out to be convenient to define the degree of the zero function $f(x) = 0$ to be $-\infty$.

- $f(x) = x^2 + 1$ and $f(x) = 3x^2 - 1$ are polynomials of degree 2.

Note that a polynomial $f(x)$ with only even-degree terms (all the odd-degree coefficients are zero) satisfies the property $f(-x) = f(x)$, while a polynomial $f(x)$ with only odd-degree terms satisfies $f(-x) = -f(x)$. We generalize this notion with the following definition.

Definition: A function f is said to be *even* if $f(-x) = f(x)$ for every x in the domain of f.

Definition: A function f is said to be *odd* if $f(-x) = -f(x)$ for every x in the domain of f.

- The functions x, x^3, and $\sin x$ are odd.

- The functions 1, x^2, and $\cos x$ are even.

- The functions $x + 1$, $\log x$, e^x are neither even nor odd.

Problem 1.1: Show that an odd function f with domain $\mathbb{R}$ satisfies $f(0) = 0$.

Definition: *Rational functions* are functions of the form $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials. They are defined on the set of all x for which $Q(x) \neq 0$.

- $\dfrac{1}{x}$ and $\dfrac{x^3 + 3x^2 + 1}{x^2 + 1}$ are both rational functions.

Composition Once we have defined a few elementary functions, we can create new functions by combining them together using $+$, $-$, $\cdot$, $\div$, or by introducing the *composition* operator $\circ$.

Definition: If $f : A \to B$ and $g : B \to C$ then we define $g \circ f : A \to C$ to be the function that takes $x \in A$ to $g(f(x)) \in C$.

- $$\begin{array}{ll} f(x) = x^2 + 1 & f : \mathbb{R} \to [1, \infty), \\ g(x) = 2\sqrt{x} & g : [1, \infty) \to [2, \infty), \\ g(f(x)) = 2\sqrt{x^2+1} & g \circ f : \mathbb{R} \to [2, \infty). \end{array}$$

  Note however that $f(g(x)) = 4x + 1$, so that $f \circ g : [0, \infty) \to [1, \infty)$.

- $$\begin{array}{ll} f(x) = x^2 + 1 & f : \mathbb{R} \to [1, \infty), \\ g(x) = \frac{1}{x} & g : [1, \infty) \to (0, 1], \\ g(f(x)) = \frac{1}{x^2+1} & g \circ f : \mathbb{R} \to (0, 1]. \end{array}$$

  One can also build new functions from old ones using *cases*, or *piecewise* definitions:

- $$f(x) = \begin{cases} 0 & x < 0, \\ \frac{1}{2} & x = 0, \\ 1 & x > 0. \end{cases}$$

- $$f(x) = |x| = \begin{cases} x & x \geq 0, \\ -x & x < 0. \end{cases}$$

  Cases can sometimes introduce jumps in a function.

Problem 1.2: Graph the function

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ 2 - x & \text{if } 1 < x \leq 2. \end{cases}$$

Definition: A function is said to be *increasing* (*decreasing*) on an interval I if

$$x, y \in I, x \leq y \Rightarrow f(x) \leq f(y) \quad (f(x) \geq f(y))$$

and *strictly increasing* (*strictly decreasing*) if

$$x, y \in I, x < y \Rightarrow f(x) < f(y) \quad (f(x) > f(y)).$$

Note that a strictly increasing function is increasing.

1.2 Transformations of Functions

Transformations can be used to obtain new functions from old ones:

- To obtain the graph of $y = f(x) + c$, shift the graph of $f(x)$ a distance c upwards.
- To obtain the graph of $y = f(x - c)$, shift the graph of $f(x)$ a distance c rightwards.
- To obtain the graph of $y = cf(x)$, stretch the graph of $f(x)$ vertically by the factor $c > 0$.
- To obtain the graph of $y = f(x/c)$, stretch the graph of $f(x)$ horizontally by the factor $c > 0$.
- To obtain the graph of $y = -f(x)$, reflect the graph of $f(x)$ about the x axis.
- To obtain the graph of $y = f(-x)$, reflect the graph of $f(x)$ about the y axis.

Problem 1.3: Sketch the graphs of $|x - 1|$ and x on the same axes and use your graph to verify the results of Prob. 0.1.

1.3 Trigonometric Functions

Trigonometric functions are functions relating the shape of a right-angle triangle to one of its other angles.

Definition: If we label one of the non-right angles by θ, the length of the hypotenuse by hyp, and the lengths of the sides opposite and adjacent to x by opp and adj, respectively, then

$$\sin\theta = \frac{\text{opp}}{\text{hyp}},$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}},$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}}.$$

Note here that since θ is one of the nonright angles of a right-angle triangle, these definitions apply only when $0 < \theta < 90°$. Note also that $\tan\theta = \sin\theta/\cos\theta$. Sometimes it is convenient to work with the reciprocals of these functions:

$$\csc\theta = \frac{1}{\sin\theta},$$

$$\sec\theta = \frac{1}{\cos\theta},$$

$$\cot\theta = \frac{1}{\tan\theta}.$$

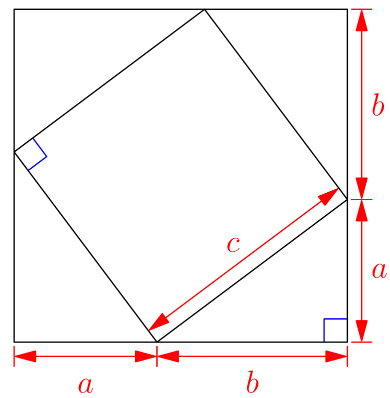


Figure 1.1: Pythagoras' Theorem

Pythagoras' Theorem states that the square of the length c of the hypotenuse of a right-angle triangle equals the sum of the squares of the lengths a and b of the other two sides. A simple geometric proof of this important result is illustrated in Figure 1.1. Four identical copies of the triangle, each with area $ab/2$, are placed around a square of side c, so as to form a larger square with side $a+b$. The area c^2 of the inner square is then just the area $(a+b)^2 = a^2 + 2ab + b^2$ of the large square minus the total area $2ab$ of the four triangles. That is, $c^2 = a^2 + b^2$.

Remark: If we scale a right-angle triangle with angle θ and $90^\circ - \theta$, so that the hypotenuse $c = 1$, the length of the sides opposite and adjacent to the angle θ are $\sin\theta$ and $\cos\theta$, respectively. Pythagoras' Theorem then leads to the following important identity:

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1. \tag{1.1}$$

Other useful identities result from dividing both sides of this equation either by $\sin^2\theta$:

$$1 + \cot^2\theta = \csc^2\theta,$$

or by $\cos^2\theta$:

$$\tan^2\theta + 1 = \sec^2\theta.$$

Note that Eq. (1.1) implies both that $|\sin\theta| \le 1$ and $|\cos\theta| \le 1$.

Definition: We define the number π to be the area of a *unit circle* (a circle with radius 1).

Definition: Instead of using degrees, in our development of calculus it will be more convenient to measure angles in terms of the area of the sector they subtend on the unit circle. Specifically, we define an angle measured in *radians* to be twice[1] the area of the sector that it subtends, as shown in Figure 1.2. For example, our definition of π says that a full unit circle (360°) has area π; the corresponding angle in radians would then be 2π. Thus, we can convert between radians and degrees with the formula

$$\pi \text{ radians} = 180^\circ.$$

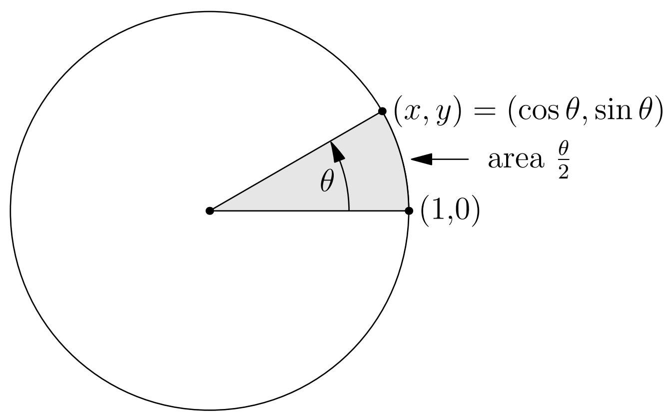


Figure 1.2: The unit circle

The coordinates x and y of a point P on the unit circle are related to θ as follows:

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x,$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y.$$

Complementary Angle Identities:

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right),$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta.$$

[1] The reason for introducing the factor of two in this definition is to make the angle x expressed in radians equal to the length of the arc it subtends on the unit circle, as we will see later using integral calculus, once we have developed the notion of the length of an arc. For example, the circumference of a full circle of unit radius will be found to be precisely 2π.

Supplementary Angle Identities:

$$\sin(\pi - \theta) = \sin\theta,$$

$$\cos(\pi - \theta) = -\cos\theta.$$

Symmetries:

$$\sin(-\theta) = -\sin\theta,$$
$$\cos(-\theta) = \cos\theta,$$
$$\sin(\theta + 2\pi) = \sin\theta,$$
$$\cos(\theta + 2\pi) = \cos\theta.$$

Problem 1.4: We thus see that $\sin\theta$ is an odd periodic function of θ and $\cos\theta$ is an even periodic function of θ, both with period 2π. Use these facts to prove that $\tan\theta$ is an odd periodic function of θ with period π.

Special Values:

$$\begin{aligned}
\sin(0) &= \cos\left(\frac{\pi}{2}\right) = 0,\\
\sin\left(\frac{\pi}{2}\right) &= \cos(0) = 1,\\
\sin\left(\frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}},\\
\sin\left(\frac{\pi}{6}\right) &= \cos\left(\frac{\pi}{3}\right) = \frac{1}{2},\\
\sin\left(\frac{\pi}{3}\right) &= \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.
\end{aligned}$$

Addition Formulae:

Claim:

$$\cos(A - B) = \cos A\cos B + \sin A\sin B.$$

Proof: Consider the points $P = (\cos A, \sin A)$, $Q = (\cos B, \sin B)$, and $R = (1, 0)$ on the unit circle, as illustrated in Fig. 1.3. We can use Pythagoras' Theorem to obtain a formula for the length (squared) of a chord subtended by an angle:

$$\overline{QR}^2 = (1 - \cos B)^2 + \sin^2 B = 1 - 2\cos B + \cos^2 B + \sin^2 B = 2 - 2\cos B.$$

For example, since the angle subtended by $\overline{PQ}$ is $A - B$,

$$\overline{PQ}^2 = 2 - 2\cos(A - B).$$

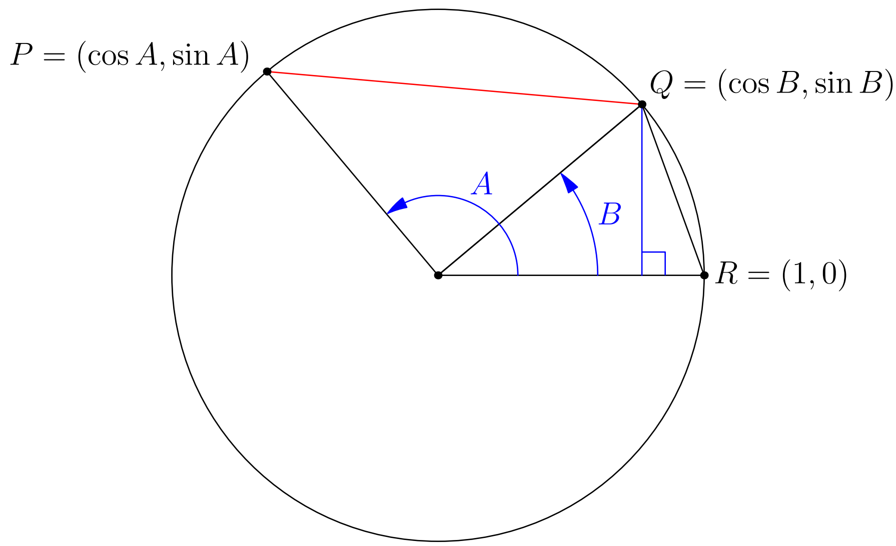


Figure 1.3: The unit circle with points $P = (\cos A, \sin A)$, $Q = (\cos B, \sin B)$, and $R = (1, 0)$

Alternatively, we could compute $\overline{PQ}^2$ directly:

$$\begin{aligned}\overline{PQ}^2 &= (\cos A - \cos B)^2 + (\sin A - \sin B)^2 \\ &= \cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A - 2 \sin A \sin B + \sin^2 B \\ &= 2 - 2(\cos A \cos B + \sin A \sin B).\end{aligned}$$

On comparing these two results, we conclude that

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

The claim thus holds.

Remark: Other trigonometric addition formulae follow easily from the above result:

$$\begin{aligned}\cos(A + B) &= \cos(A - (-B)) \\ &= \cos A \cos(-B) + \sin A \sin(-B) \\ &= \cos A \cos B - \sin A \sin B.\end{aligned}$$

$$\begin{aligned}\sin(A + B) &= \cos\left[\frac{\pi}{2} - (A + B)\right] \\ &= \cos\left[\left(\frac{\pi}{2} - A\right) - B\right] \\ &= \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B \\ &= \sin A \cos B + \cos A \sin B.\end{aligned}$$

$$\begin{aligned}\sin(A-B) &= \sin(A-(-B)) \\ &= \sin A\cos(-B)+\cos A\sin(-B) \\ &= \sin A\cos B-\cos A\sin B.\end{aligned}$$

$$\begin{aligned}\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B} \\ &= \frac{(\sin A\cos B+\cos A\sin B)\cdot\frac{1}{\cos A\cos B}}{(\cos A\cos B-\sin A\sin B)\cdot\frac{1}{\cos A\cos B}} \\ &= \frac{\tan A+\tan B}{1-\tan A\tan B}, \qquad \text{provided } A,\ B,\ A+B \text{ are not odd multiples of } \tfrac{\pi}{2}.\end{aligned}$$

Double-Angle Formulae:

$$\begin{aligned}\sin 2A &= \sin(A+A) \\ &= \sin A\cos A+\sin A\cos A \\ &= 2\sin A\cos A.\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos(A+A) \\ &= \cos A\cos A-\sin A\sin A \\ &= \cos^2 A-\sin^2 A \\ &= \cos^2 A-(1-\cos^2 A) \\ &= 2\cos^2 A-1 \\ &= (1-\sin^2 A)-\sin^2 A \\ &= 1-2\sin^2 A.\end{aligned}$$

Also, if A is not an odd multiple of $\pi/4$ or $\pi/2$,

$$\begin{aligned}\tan 2A &= \tan(A+A) \\ &= \frac{\tan A+\tan A}{1-\tan A\tan A} \\ &= \frac{2\tan A}{1-\tan^2 A}.\end{aligned}$$

Inequalities: We have already seen that $|\sin x|\le 1$ and $|\cos x|\le 1$. Our development of trigonometric calculus will rely on the following additional key result:

$$\sin x\le x\le\tan x \quad \text{for all } x\in\left[0,\frac{\pi}{2}\right).$$

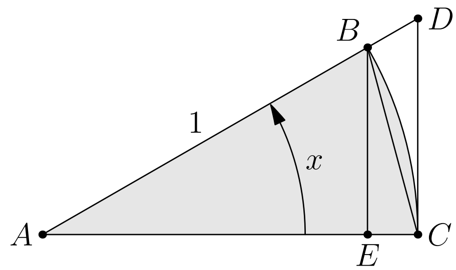
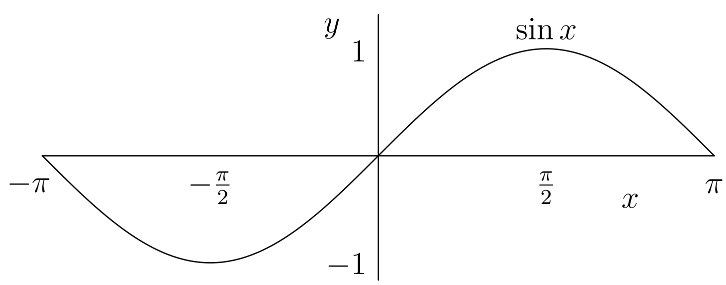


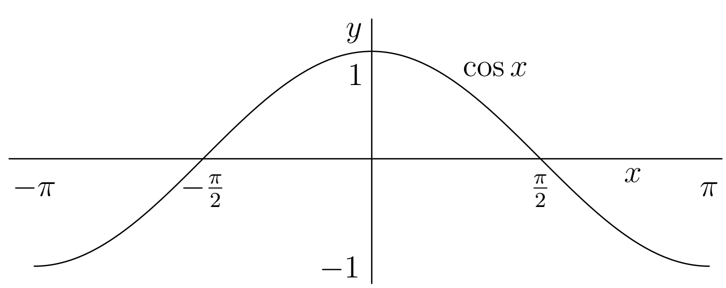
Figure 1.4: Geometric proof of $\sin x \le x \le \tan x$

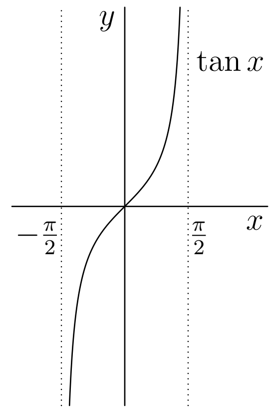
We establish this result geometrically, referring to the arc of unit radius in Fig 1.4. The shaded area of the sector ABC subtended by the angle x (measured in radians) is $x/2$. Since $BE = \sin x$ and $DC = \tan x$, we deduce

$$\begin{aligned} &\text{Area}_{\triangle_{ABC}} \le \text{Area}_{\text{Sector}_{ABC}} \le \text{Area}_{\triangle_{ADC}} \\ \Rightarrow\ &\frac{1}{2}(1)\sin x \le \frac{x}{2} \le \frac{1}{2}(1)\tan x \\ &\Rightarrow \sin x \le x \le \tan x \quad \text{for all } x \in \left[0, \frac{\pi}{2}\right). \end{aligned}$$

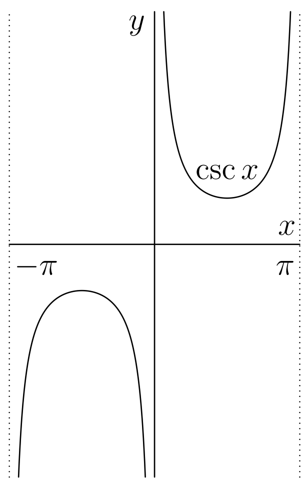
Problem 1.5: Verify that the graphs of the functions $y = \sin x$, $y = \cos x$, and $y = \tan x$ are periodic extensions of the illustrated graphs.

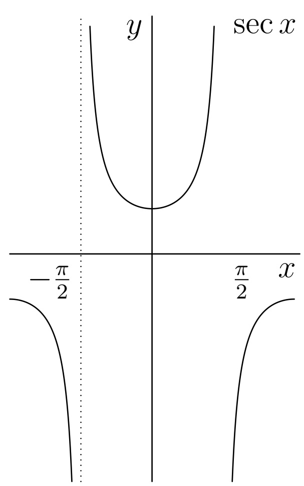


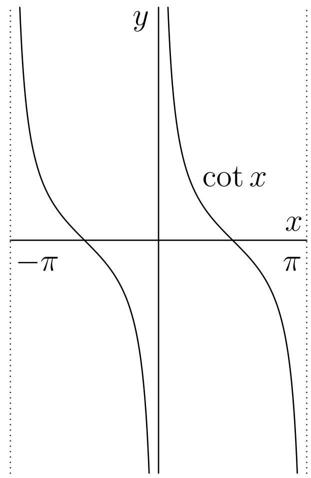




Problem 1.6: Verify that the graphs of the functions $y = \csc x = 1/\sin x$, $y = \sec x = 1/\cos x$, and $y = \cot x = 1/\tan x$ are periodic extensions of the illustrated graphs.

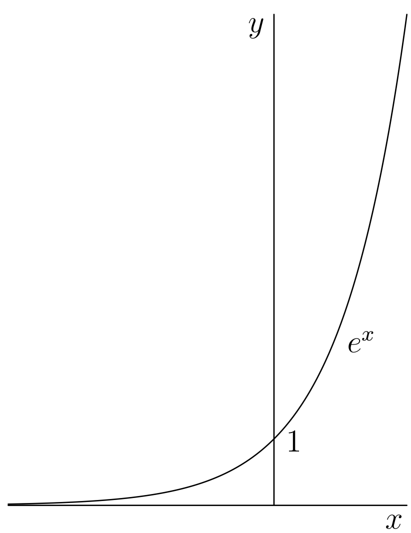






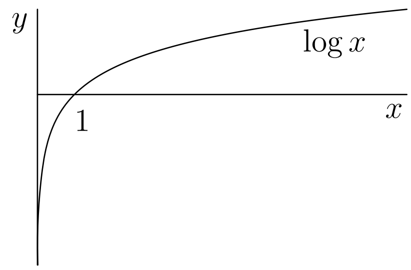
1.4 Exponential and Logarithmic Functions

The graph of the natural exponential function e^x, sometimes written $\exp(x)$, is shown below:



Remark: Notice that $e^x > 0$ for all real x.

The inverse of the exponential function is the natural logarithm $\log x$, sometimes written $\ln x$. It is defined for all positive x:



Remark: There are other exponential function (e.g. 10^x or 2^x) corresponding to other choices of the base (e.g. 10 or 2). The natural logarithm corresponds to the base $e \approx 2.718281828459\ldots$.

Definition: The general exponential function to the base b is defined as

$$b^x \doteq e^{x \log b}.$$

(We use the symbol $\doteq$ to emphasize a definition, although the notation $:=$ is more common.)

Remark: For a positive base b and real x and y:

1. $b^{x+y} = b^x b^y$.

2. $b^{x-y} = \dfrac{b^x}{b^y}$.

3. $(b^x)^y = b^{xy}$.

4. $(ab)^x = a^x b^x$.

Remark: We can also define a logarithm function to the base b:

$$\log_b x \doteq \frac{\log x}{\log b}.$$

Remark: If x, y, and b are positive numbers,

1. $\log_b(xy) = \log_b x + \log_b y$.

2. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$.

3. $\log_b(x^r) = r\log_b x$.

1.5 Induction

Suppose that the weather office makes a long-term forecast consisting of two statements:

(A) If it rains on any given day, then it will also rain on the following day.

(B) It will rain today.

What would we conclude from these two statements? We would conclude that it will rain every single day from now on!

Or, consider a secret passed along an infinite line of people, $P_1P_2\ldots P_nP_{n+1}\ldots$, each of whom enjoys gossiping. If we know for every $n \in \mathbb{N}$ that P_n will always pass on a secret to P_{n+1}, then the mere act of telling a secret to the first person in line will result in everyone in the line eventually knowing the secret!

These amusing examples encapsulate the axiom of *Mathematical Induction*:

If a subset $\mathcal{S} \subset \mathbb{N}$ satisfies

(i) $1 \in \mathcal{S}$,

(ii) $k \in \mathcal{S} \Rightarrow k+1 \in \mathcal{S}$,

then $\mathcal{S} = \mathbb{N}$.

For example, suppose we wish to find the sum of the first n natural numbers. For small values of n, we could just compute the total of these n numbers directly. But for large values of n, this task could become quite time consuming! The great mathematician and physicist Carl Friedrich Gauss (1777–1855) at age 10 noticed that the rate of increase of the terms in the sum

$$1 + 2 + \ldots + n$$

could be exactly compensated by first writing the sum backwards, as

$$n + (n-1) + \ldots + 1,$$

and then averaging the two equal expressions term-by-term to obtain a sum of n identical terms:

$$\underbrace{\frac{n+1}{2}+\frac{n+1}{2}+\ldots+\frac{n+1}{2}}_{n\text{ terms}}=n\left(\frac{n+1}{2}\right).$$

We will use mathematical induction to verify Gauss' claim that

$$1+2+\ldots+n \equiv \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{1.2}$$

Let $\mathcal{S}$ be the set of numbers n for which Eq. (1.2) holds.

Step 1: Check $1 \in \mathcal{S}$:

$$1=\frac{1(1+1)}{2}=1.$$

Step 2: Suppose $k \in \mathcal{S}$, i.e.

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

Then

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \left(\sum_{i=1}^{k} i\right)+(k+1)\\ &= \frac{k(k+1)}{2}+(k+1)\\ &= (k+1)\left(\frac{k}{2}+1\right)\\ &= \frac{(k+1)(k+2)}{2}.\end{aligned}$$

Hence $k+1 \in \mathcal{S}$.
That is, $k \in \mathcal{S} \Rightarrow k+1 \in \mathcal{S}$.

By the Axiom of Mathematical Induction, we know that $\mathcal{S}=\mathbb{N}$.
In other words,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \text{for all } n \in \mathbb{N}.$$

- Prove that for all natural numbers n,

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}. \tag{1.3}$$

Step 1: We see for $n = 1$ that $1 = 1^2(1+1)^2/4$.

Step 2: Suppose

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \doteq S_n.$$

Then

$$\begin{aligned}\sum_{i=1}^{n+1} i^3 &= \left(\sum_{i=1}^{n} i^3\right) + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2}{4}(n^2+4n+4) \\ &= \frac{(n+1)^2(n+2)^2}{4} = S_{n+1}.\end{aligned}$$

Hence by induction, Eq. (1.3) holds.

Problem 1.7: Use induction to prove that $22^n - 15$ is a multiple of 7 for every natural number n.

Step 1: We see for $n = 1$ that $22 - 15 = 7$ is a multiple of 7.
Step 2: Assume that $22^n - 15$ is a multiple of 7, say $7m$. We need only show that $22^{n+1} - 15$ is also a multiple of 7:

$$22^{n+1} - 15 = 22^n \cdot 22 - 15 = (7m + 15) \cdot 22 - 15 = 7m \cdot 22 + 15 \cdot 21 = 7(m \cdot 22 + 15 \cdot 3),$$

which is indeed a multiple of 7. By mathematical induction, we see that $22^n - 15$ is multiple of 7 for every $n \in \mathbb{N}$.

1.6 Summation Notation

Recall

$$\sum_{k=1}^{k=n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

Q. What is $\sum_{k=0}^{k=n} k$?

A.

$$\sum_{k=0}^{k=n} k = 0 + \sum_{k=1}^{k=n} k = 0 + \frac{n(n+1)}{2} = \frac{n(n+1)}{2}.$$

Q. How about $\displaystyle\sum_{k=1}^{k=n+1} k$?

A.

$$\sum_{k=1}^{k=n+1} k = \left(\sum_{k=1}^{k=n} k\right) + (n+1) = \frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2}.$$

Q. How about $\displaystyle\sum_{k=1}^{k=n}(k+1)$?

A.

Method 1:

$$\sum_{k=1}^{k=n}(k+1) = \sum_{k=1}^{k=n} k + \sum_{k=1}^{k=n} 1 = \frac{n(n+1)}{2} + n = \frac{n(n+3)}{2}.$$

Method 2: First, let $k' = k+1$:

$$\sum_{k=1}^{k=n}(k+1) = \sum_{k'=2}^{k'=n+1} k'.$$

Next, it is convenient to replace the symbol k' with k (since it is only a dummy index anyway):

$$\sum_{k'=2}^{k'=n+1} k' = \sum_{k=2}^{k=n+1} k = \left(\sum_{k=1}^{k=n+1} k\right) - 1 = \frac{(n+1)(n+2)}{2} - 1 = \frac{n(n+3)}{2}.$$

In general,

$$\boxed{\sum_{k=L}^{k=U} a_{k+m} = \sum_{k=L+m}^{k=U+m} a_k.}$$

Verify this by writing out both sides explicitly.

Problem 1.8: For any real numbers a_1, a_2, …, a_n, b_1, b_2, …, b_n, and c prove that

$$\sum_{k=1}^{n} c(a_k + b_k) = c\sum_{k=1}^{n} a_k + c\sum_{k=1}^{n} b_k.$$

- *Telescoping sum*:

$$\begin{aligned}
\sum_{k=1}^{n}(a_{k+1}-a_k) &= \sum_{k=1}^{n} a_{k+1} - \sum_{k=1}^{n} a_k \\
&= \sum_{k=2}^{n+1} a_k - \sum_{k=1}^{n} a_k \\
&= \cancel{\sum_{k=2}^{n}} a_k + a_{n+1} - \left(a_1 + \cancel{\sum_{k=2}^{n}} a_k \right) \\
&= a_{n+1} - a_1.
\end{aligned}$$

Chapter 2

Limits

2.1 Sequence Limits

Definition: A *sequence* is a function on the domain $\mathbb{N}$. The value of a function f at $n \in \mathbb{N}$ is often denoted by a_n,

$$a_n = f(n).$$

The consecutive function values are often written in a list:

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \ldots\} \quad \leftarrow \text{Repeated values are allowed.}$$

- $a_n = f(n) = n^2$,
  $\{a_n\}_{n=1}^{\infty} = \{1, 4, 9, 16 \ldots\}$.

- The *Fibonacci* sequence,

  $$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\},$$

  begins with the numbers 1 and 1, with subsequent numbers defined as the sum of the two immediately preceding numbers.

- $$\{\cos(n\pi)\}_{n=0}^{\infty} = \{(-1)^n\}_{n=0}^{\infty} = \{1, -1, 1, -1, \ldots\}.$$

- $$\{\sin(n\pi)\}_{n=0}^{\infty} = \{0, 0, 0, 0, \ldots\}.$$

- $\left\{\dfrac{(-1)^n}{n}\right\}_{n=1}^{\infty} = \left\{-1, \dfrac{1}{2}, -\dfrac{1}{3}, \dfrac{1}{4}, \ldots\right\}.$
  Notice that as n gets large, the terms of this sequence get closer and closer to zero. We say that they *converge* to 0. However, a_n is not equal to 0 for any $n \in \mathbb{N}$.

We can formalize this observation with the following concept:

Definition: The sequence $\{a_n\}_{n=1}^{\infty}$ is *convergent* with limit L if, for each $\epsilon > 0$, there exist a number N such that

$$n > N \Rightarrow |a_n - L| < \epsilon.$$

We abbreviate this as: $\lim\limits_{n\to\infty} a_n = L$.

If no such number L exists, we say $\{a_n\}_{n=1}^{\infty}$ *diverges.*

Remark: The statement $\lim\limits_{n\to\infty} a_n = L$ means that $|a_n - L|$ can be made as small as we please, simply by choosing n large enough.

Remark: Equivalently, as illustrated in Fig. 2.1, $\lim\limits_{n\to\infty} a_n = L$ means that any open interval about L contains all but a finite number of terms of $\{a_n\}_{n=1}^{\infty}$.

Remark: If a sequence $\{a_n\}_{n=1}^{\infty}$ converges to L, the previous remark implies that every open interval $(L - \epsilon, L + \epsilon)$ will contain an infinite number of terms of the sequence (there cannot be only a finite number of terms inside the interval since a sequence has infinitely many terms and only finitely many of them are allowed to lie outside the interval).

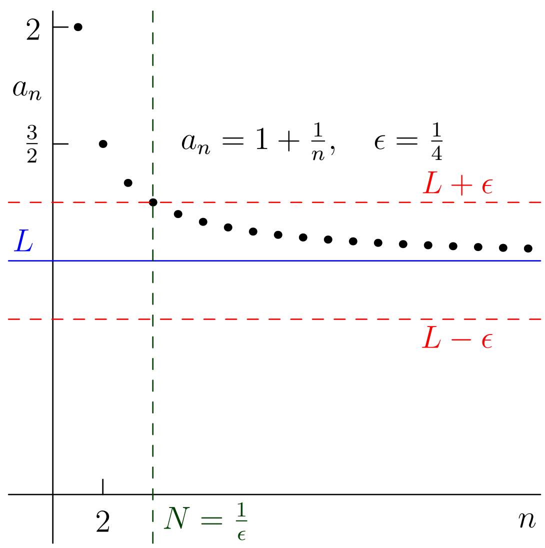


Figure 2.1: Limit of a sequence

- Let $a_n = 1$, for all $n \in \mathbb{N}$
  i.e. $\{1, 1, 1, \ldots\}$.
  Let $\epsilon > 0$. Choose $N = 1$.

$$n > 1 \Rightarrow |a_n - 1| = |1 - 1| = 0 < \epsilon.$$

That is, $L = 1$. Write $\lim\limits_{n\to\infty} a_n = 1$.

Remark: Here N does not depend on ϵ, but normally it will.

- The sequence $a_n = \dfrac{1}{n}$ converges to 0 since given $\epsilon > 0$, we may force $|a_n - 0| < \epsilon$ for $n > N$ simply by picking $N \geq \dfrac{1}{\epsilon}$:

$$n > N \Rightarrow |a_n - 0| = \frac{1}{n} < \frac{1}{N} \leq \epsilon.$$

- $a_n = \dfrac{(-1)^n}{n}$.

  $\lim_{n\to\infty} a_n = 0$ since $|a_n - 0| = \left|\dfrac{(-1)^n}{n}\right| = \dfrac{1}{n} < \dfrac{1}{N}$ if $n > N$.

  So, given $\epsilon > 0$, we may force $|a_n - 0| < \epsilon$ for $n > N$ simply by picking $N \geq \dfrac{1}{\epsilon}$:

$$n > N \Rightarrow |a_n - 0| = \frac{1}{n} < \frac{1}{N} \leq \epsilon.$$

- The sequence $a_n = \dfrac{1}{n+1}$ converges to 0 since given $\epsilon > 0$, we may force $|a_n - 0| < \epsilon$ for $n > N$ simply by picking $N \geq \dfrac{1}{\epsilon}$:

$$n > N \Rightarrow |a_n - 0| = \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N} \leq \epsilon.$$

Problem 2.1: Show that the sequence

$$a_n = \frac{n}{n+1}$$

converges to 1.

Given $\epsilon > 0$, we may force $|a_n - 1| < \epsilon$ for $n > N$ simply by picking $N \geq \dfrac{1}{\epsilon}$:

$$n > N \Rightarrow |a_n - 1| = \left|\frac{n}{n+1} - 1\right| = \left|\frac{n}{n+1} - \frac{n+1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N} \leq \epsilon.$$

Thus $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Remark: Such limit calculations can quickly become quite technical. Fortunately, many limit questions can be greatly simplified using the following properties.

Properties of Limits: Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences. Denote $L = \lim\limits_{n\to\infty} a_n$ and $M = \lim\limits_{n\to\infty} b_n$.

$$\lim_{n\to\infty} (a_n + b_n) = L + M;$$

$$\lim_{n\to\infty} a_n b_n = LM;$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M} \text{ if } M \neq 0.$$

- An easier way to show that $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1$ is to use the fact that the limit of a difference is the difference of limits, **provided that each individual limit exists:**

$$\begin{aligned} \lim_{n\to\infty} \frac{n}{n+1} &= \lim_{n\to\infty} \frac{n+1-1}{n+1} = \lim_{n\to\infty} \left[\frac{n+1}{n+1} - \frac{1}{n+1}\right] \\ &= \lim_{n\to\infty} \frac{n+1}{n+1} - \lim_{n\to\infty} \frac{1}{n+1} = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1 - 0 = 1. \end{aligned}$$

- Another way to show that $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1$ is divide numerator and denomator by the highest power of n appearing (in this case, n^1) and use the fact that the ratio of limits is the limit of the ratio, **provided that each individual limit exists:**

$$\begin{aligned} \lim_{n\to\infty} \frac{n}{n+1} &= \lim_{n\to\infty} \frac{1}{1+1/n} = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} 1+1/n} \\ &= \frac{1}{1+\lim\limits_{n\to\infty} 1/n} = \frac{1}{1+\lim\limits_{n\to\infty} 1/n} = \frac{1}{1+0} = 1. \end{aligned}$$

Problem 2.2: Show by first dividing numerator and denominator by n^2 that

$$\lim_{n\to\infty} \frac{2n^2-5}{3n^2+n+1} = \frac{2}{3}.$$

Problem 2.3: Find $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$.

We find

$$\begin{aligned}\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) &= \lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)\\ &= \lim_{n\to\infty}(n+1-n)\cdot\frac{1}{\sqrt{n+1}+\sqrt{n}}\\ &= \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}\\ &= 0\end{aligned}$$

since the final fraction is less than ϵ whenever $n > 1/\epsilon^2$.

Remark: The *Squeeze Theorem* states that if $x_n \le z_n \le y_n$ for all $n \in \mathbb{N}$ and the sequences $\{x_n\}$ and $\{y_n\}$ both converge **to the same number** c, then $\{z_n\}$ is also convergent to c.

- The Squeeze Theorem provides an alternative means to show $\lim_{n\to\infty} a_n = \frac{(-1)^n}{n} = 0$: Since
$$-\frac{1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}$$
and $\lim_{n\to\infty} -\frac{1}{n} = -\lim_{n\to\infty}\frac{1}{n} = -0 = 0 = \lim_{n\to\infty}\frac{1}{n}$, the Squeeze Theorem guarantees that $\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$ too.

Definition: A sequence is *bounded* if there exists a number B such that
$$|a_n| \le B \quad \text{for all } n \in \mathbb{N}.$$
Recall that this means that a_n lies within some interval $(-B, B)$.

Definition: A sequence $\{a_n\}_{n=1}^{\infty}$ is *increasing* if
$$a_1 \le a_2 \le a_3 \le \dots, \quad \text{i.e.} \quad a_n \le a_{n+1} \quad \text{for all } n \in \mathbb{N}$$
and *decreasing* if
$$a_1 \ge a_2 \ge a_3 \ge \dots, \quad \text{i.e.} \quad a_n \ge a_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

Definition: A sequence is *monotone* if it is either (i) an increasing sequence or (ii) a decreasing sequence.

The following theorem can be helpful in establishing the convergence of monotone sequences:

Remark: Every bounded, monotone sequence is convergent.

- The sequence $a_n = \sum_{k=0}^{n} 2^{-k} = 1 + 1/2 + 1/4 + \ldots + 1/2^n$ is bounded below by 1 and above by 2. Since each new term in the sum is positive, a_n is also a monotone (increasing) sequence and is thus convergent.

Problem 2.4: Let $x \in [0, 1]$. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined inductively by $a_1 = x$, and $a_{n+1} = a_n(1 - a_n)$ for $n \geq 1$.

(a) Show that $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence.

$$a_{n+1} - a_n = -a_n^2 \leq 0.$$

(b) Prove that $\{a_n\}_{n=1}^{\infty}$ is bounded.

We show that $0 \leq a_n \leq 1$ for all n. For $n = 1$, we are given that $a_1 = x \in [0, 1]$. Suppose that $0 \leq a_n \leq 1$. Then $0 \leq 1 - a_n \leq 1$ and hence $a_{n+1} = a_n(1 - a_n)$ is also in $[0, 1]$. By induction, we see that $a_n \in [0, 1]$ for all n.

(c) Does $\{a_n\}_{n=1}^{\infty}$ converge? Why or why not? If it does converge, find its limit.

The sequence converges because it is a decreasing bounded sequence. To find the limit, let

$$L = \lim_{n\to\infty} a_n.$$

The sequences $\{a_{n+1}\}$ and $\{a_n\}$ converge to the same limit since $n + 1 \to \infty$ as $n \to \infty$. Hence

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} [a_n(1 - a_n)] = \lim_{n\to\infty} a_n \lim_{n\to\infty} (1 - a_n) = L(1 - L) = L - L^2.$$

This implies that $L^2 = 0$, from which we deduce $\lim_{n\to\infty} a_n = 0$.

2.2 Function Limits

Consider the function $f(x) = \frac{1}{x}$ $(x \neq 0)$.

Notice that as x gets large, $f(x)$ gets closer to, but never quite reaches, 0, very much like the terms of the sequence $\{\frac{1}{n}\}$ as $n \to \infty$. In fact, at integer values of x, f evaluates to a member of the sequence $\{\frac{1}{n}\}$:

$$f(n) = \frac{1}{n} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Unlike a sequence, f is defined also for nonintegral values of x. We therefore need to generalize our definition of a limit:

Definition: We say $\lim_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$ we can find a real number N such that

$$x > N \Rightarrow |f(x) - L| < \epsilon.$$

- Let $f(x) = 1/x$. Given any $\epsilon > 0$, we can make

$$|f(x) - 0| = \left|\frac{1}{x}\right| < \frac{1}{N} = \epsilon$$

for $x > N$ simply by picking $N = \dfrac{1}{\epsilon}$.
Hence $\lim_{x\to\infty} f(x) = 0$.

- $$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}.$$

Remark: As with sequence limits, we have the following properties:

Properties: Suppose $L = \lim_{x\to\infty} f(x)$ and $M = \lim_{x\to\infty} g(x)$. Then

$$\lim_{x\to\infty} (f(x) + g(x)) = L + M;$$

$$\lim_{x\to\infty} f(x)g(x) = LM;$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{L}{M} \text{ if } M \neq 0;$$

Remark: We can also introduce the notion of a limit of a function $f(x)$ as x approaches some real number a.

- Cnsider the function $f(x) = \sin x$. Notice for all real numbers near $x = 0$ that $\sin x$ is very close to 0. That is, if δ is a small positive number, the value of $\sin x$ is very close to zero for all $x \in (-\delta, \delta)$. Given $\epsilon > 0$, we can in fact always find a small region $(-\delta, \delta)$ about the origin such that

$$x \in (-\delta, \delta) \Rightarrow |\sin x| < \epsilon.$$

For example, we could choose $\delta = \epsilon$ since we have already shown that $|\sin x| \leq |x|$ for all real x:

$$|x| < \delta \Rightarrow |\sin x| \leq |x| < \delta = \epsilon.$$

We express this fact with the statement $\lim_{x\to 0} \sin x = 0$.

Definition: We say $\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0$ we can find a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

Remark: In the previous example we see that $a = 0$ and the limit L is 0. Notice in this case that $\lim_{x \to 0} f(x) = 0 = f(0)$. However, this is not true for all functions f. The value of a limit as $x \to a$ might be quite different from the value of the function at $x = a$. Sometimes the point a might not even be in the domain of the function, but the limit may still be defined. This is why we restrict $0 < |x - a|$ (that is, $x \neq a$) in the above definition.

Remark: The value of a function at a itself is irrelevant to its limit at a. We don't need to evaluate the function at $x = a$ any more than we need to evaluate the function $f(x) = 1/x$ at $x = \infty$ to find $\lim_{x \to \infty} f(x) = 0$.

- Let
$$f(x) = \begin{cases} x & \text{if } x > 0, \\ -x & \text{if } x < 0. \end{cases}$$
When we say $\lim_{x \to 0} f(x) = 0$ we mean the following. Given $\epsilon > 0$, we can make
$$|f(x)| < \epsilon$$
for all x satisfying $0 < |x| < \delta$ just by choosing $\delta = \epsilon$. That is,
$$0 < |x| < \delta \Rightarrow |f(x)| = |x| < \delta = \epsilon.$$

- How about
$$f(x) = |x| = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -x & \text{if } x < 0, \end{cases}$$
Is $\lim_{x \to 0} f(x) = 0$? Yes, the value of f at $x = 0$ does not matter.

- Consider now
$$f(x) = \begin{cases} x & \text{if } x > 0, \\ 1 & \text{if } x = 0, \\ -x & \text{if } x < 0. \end{cases}$$
Is $\lim_{x \to 0} f(x) = 0$? Yes, the value of f at $x = 0$ does not matter.

- Let
$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{2} & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$
This function is defined everywhere. Does $\lim_{x\to 0} f(x)$ exist?

No, given $\epsilon = \frac{1}{2}$, there are values of $x \neq 0$ in every interval $(-\delta, \delta)$ with very different values of f:
$$f\left(\frac{\delta}{2}\right) = 1,$$
$$f\left(-\frac{\delta}{2}\right) = 0.$$
Thus $\lim\limits_{x\to 0} f(x)$ does not exist.

- Let $f(x) = 7x - 3$. Show that $\lim\limits_{x\to 1} f(x) = 4$.
Let $\epsilon > 0$. Our task is to produce a $\delta > 0$ such that
$$0 < |x - 1| < \delta \Rightarrow |f(x) - 4| < \epsilon.$$
Well, $|f(x) - 4| = |7x - 7| = 7|x - 1| < 7\delta$.
How can we make $|f(x) - 4| < \epsilon$?
No matter what ϵ we are given, we can easily choose $\delta = \epsilon/7$, so that $7\delta = \epsilon$.

Q. Suppose
$$f(x) = \begin{cases} 7x - 3 & x \neq 1, \\ 5 & x = 1. \end{cases}$$
What is $\lim\limits_{x\to 1} f(x)$?

A. The limit is still 4; the value of $f(x)$ at $x = 1$ is completely irrelevant. The function need not even be defined at $x = 1$.

Remark: $\lim\limits_{x\to a}$ describes the behaviour of a function near a, not at a.

- Let $f(x) = x^2$, $x \in \mathbb{R}$.
Show $\lim\limits_{x\to 3} f(x) = 9$.
$$\begin{aligned} |x - 3| < \delta \Rightarrow |f(x) - 9| = \left|x^2 - 9\right| &= |x - 3|\,|x + 3| = |x - 3|\,|x - 3 + 6| \\ &< \delta(\delta + 6) \quad \text{from the Triangle Inequality.} \end{aligned}$$

We could solve the quadratic equation $\delta(\delta+6) = \epsilon$, but it is easier to restrict $\delta \le 1$ so that

$$\delta(\delta+6) \le \delta(1+6) = 7\delta \le \epsilon \ \text{ if } \ \delta \le \frac{\epsilon}{7}.$$

Note here that we must allow for the possibility that $\delta < \epsilon/7$ instead of just setting $\delta = \epsilon/7$, in order to satisfy our simplifying restriction that $\delta \le 1$.

Hence

$$|x-3| < \min\left(1, \frac{\epsilon}{7}\right) \Rightarrow |f(x) - 9| < \epsilon.$$

- Let $f(x) = \frac{1}{x}$, $x \neq 0$.

  Show $\lim_{x\to 2} f(x) = \dfrac{1}{2}$.

  Given $\epsilon > 0$, try to find a δ such that

$$0 < |x-2| < \delta \Rightarrow \left| f(x) - \frac{1}{2} \right| < \epsilon.$$

Note $\left| f(x) - \dfrac{1}{2} \right| = \left| \dfrac{1}{x} - \dfrac{1}{2} \right| = \left| \dfrac{2-x}{2x} \right|$ becomes very large near $x = 0$.

Is this a problem? No, we are only interested in the behaviour of the function near $x = 2$.

Let us restrict $\delta \le 1$, to keep the factor $2x$ in the denominator from getting really small (and hence the whole expression from getting really large). Then

$$|x-2| < 1 \Rightarrow -1 < x-2 < 1 \Rightarrow 1 \le x \le 3 \Rightarrow \frac{1}{x} \le 1.$$

So

$$\left| f(x) - \frac{1}{2} \right| = \left| \frac{2-x}{2x} \right| \le \frac{1}{2} |x-2| < \frac{1}{2}\delta \le \epsilon,$$

if we take $\delta = \min(1, 2\epsilon)$.

Properties: Suppose $L = \lim_{x\to a} f(x)$ and $M = \lim_{x\to a} g(x)$. Then

$$\lim_{x\to a} (f(x) + g(x)) = L + M;$$

$$\lim_{x\to a} f(x)g(x) = LM;$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M} \text{ if } M \neq 0;$$

Remark: If $M = 0$, we need to simplify a result before we can use the final property:

- $$\lim_{x\to 1}\frac{x-1}{x^2-1}=\lim_{x\to 1}\frac{x-1}{(x-1)(x+1)}=\lim_{x\to 1}\frac{1}{(x+1)}=\frac{\lim\limits_{x\to 1}1}{\lim\limits_{x\to 1}(x+1)}=\frac{1}{2}.$$

Remark: The *Squeeze Theorem for Functions* states that if $f(x)\le h(x)\le g(x)$ when $x\in(a-\delta,a+\delta)$, for some $\delta>0$, then

$$\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=L\Rightarrow\lim_{x\to a}h(x)=L.$$

Remark: If $a>0$, then $\lim\limits_{x\to a}\sqrt{x}=\sqrt{a}$. To see this, consider

$$0\le\left|\sqrt{x}-\sqrt{a}\right|=\left|\sqrt{x}-\sqrt{a}\right|\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}=\frac{|x-a|}{\sqrt{x}+\sqrt{a}}\le\frac{|x-a|}{\sqrt{a}}.$$

The Squeeze Theorem then implies that

$$\lim_{x\to a}\left|\sqrt{x}-\sqrt{a}\right|=0,$$

or equivalently,

$$\lim_{x\to a}\left(\sqrt{x}-\sqrt{a}\right)=0.$$

Thus

$$\lim_{x\to a}\sqrt{x}=\lim_{x\to a}\sqrt{a}.$$

Remark: Similarly, it can be shown that

$$\lim_{x\to a}\sqrt{g(x)}=\sqrt{\lim_{x\to a}g(x)}$$

for any non-negative function $g(x)$.

Problem 2.5: Let $f(x)>0$ be a positive function, defined everywhere except perhaps at $x=0$. Suppose that

$$\lim_{x\to 0}\left(f(x)+\frac{1}{f(x)}\right)=2.$$

Prove that $\lim\limits_{x\to 0}f(x)$ exists and equals 1. Hint: First note that

$$\lim_{x\to 0}\left(\sqrt{f(x)}\pm\frac{1}{\sqrt{f(x)}}\right)=\sqrt{\lim_{x\to 0}\left(\sqrt{f(x)}\pm\frac{1}{\sqrt{f(x)}}\right)^2}.$$

Then sum these two expressions to find $\lim\limits_{x\to 0}\sqrt{f(x)}$.

Definition: If for every $M > 0$ there exists a $\delta > 0$ such that $x \in (a-\delta, a+\delta) \Rightarrow f(x) > M$, we say $\lim_{x\to a} f(x) = \infty$.

-
$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

Definition: We say $\lim_{x\to\infty} f(x) = \infty$ if for every $M > 0$ we can find a real number N such that
$$x > N \Rightarrow f(x) > M.$$

-
$$\lim_{x\to\infty} e^x = \infty.$$

2.3 Continuity

Definition: Let $D \subset \mathbb{R}$. A point c is an *interior point* of D if it belongs to some open interval (a, b) entirely contained in D: $c \in (a, b) \subset D$.

- $\frac{1}{10}, \frac{1}{2}, \frac{2}{3}, \frac{9}{10}$ are interior points of $[0, 1]$ but 0 and 1 are not.

- All points of $(0, 1)$ are interior points of $(0, 1)$.

Recall that the value of f at $x = a$ is completely irrelevant to the value of its limit as $x \to a$. Sometimes, however, these two values will happen to agree. In that case, we say that $f(x)$ is *continuous* at $x = a$.

Definition: A function f is *continuous* at an interior point a of its domain if
$$\lim_{x\to a} f(x) = f(a).$$

Remark: Otherwise, if

(a) the limit fails to exist, or

(b) the limit exists and equals some number $L \neq f(a)$,

the function is said to be *discontinuous*.

Remark: f is continuous at $a \iff$ for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

Note that when $x = a$ we have $|f(a) - f(a)| = 0 < \epsilon$.

- $f(x) = x$ is continuous at every point a of its domain ($\mathbb{R}$) since $\lim_{x \to a} x = a = f(a)$ for all $a \in \mathbb{R}$.

- $f(x) = x^2$ is continuous at all points a since

$$\lim_{x \to a} f(x) = \lim_{x \to a} x^2 = \lim_{x \to a} x \cdot \lim_{x \to a} x = a \cdot a = a^2 = f(a).$$

- Likewise, we see that every polynomial is continuous at all real numbers a.

Remark: Suppose f and g are continuous at a. Then $f+g$ and fg are continuous at a and f/g is continuous at a if $g(a) \neq 0$.

Remark: A rational function is continuous at all points of its domain.

- $f(x) = \dfrac{1}{x}$ is continuous at all $x \neq 0$.

- $f(x) = \dfrac{1}{x^2+1}$ is continuous everywhere.

- $f(x) = \dfrac{1}{x^2-1}$ is continuous on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

- We have seen that $\lim_{x \to a} \sqrt{x} = \sqrt{a}$. This means that $f(x) = \sqrt{x}$ is continuous at all $a > 0$.

Remark: If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

2.4 One-Sided Limits

Definition: We write $\lim\limits_{x\to a^+} f(x) = L$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{0 < x - a < \delta}_{\text{i.e. } a<x<a+\delta} \Rightarrow |f(x) - L| < \epsilon.$$

- For the function
$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$
we see that $\lim\limits_{x\to 0^+} H(x) = 1$.

Definition: We write $\lim\limits_{x\to a^-} f(x) = L$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < a - x < \delta \Rightarrow |f(x) - L| < \epsilon.$$

- In the above example, we see that $\lim\limits_{x\to 0^-} H(x) = 0$.

Remark: $\lim\limits_{x\to a} f(x) = L \iff \lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = L$.

Definition: A function f is *continuous from the right* at a if

$$\lim_{x\to a^+} f(x) = f(a).$$

Definition: A function f is *continuous from the left* at a if

$$\lim_{x\to a^-} f(x) = f(a).$$

- $f(x) = \sqrt{x}$ is continuous from the right at $x = 0$.

Remark: A function is continuous at an interior point a of its domain if and only if it is continuous both from the left and from the right at a.

Definition: A function f is said to be *continuous on* $[a, b]$ if f is continuous at each point in (a, b) and continuous from the right at a and from the left at b.

- $f(x) = \sqrt{x}$ is continuous on $[\,0, \infty)$.

Remark: Continuous functions are free of sudden jumps. This property may be exploited to help locate the roots of a continuous function. Suppose we want to know whether the continuous function $f(x) = x^3 + x^2 - 1$ has a root in $(0, 1)$. We might notice that $f(0)$ is negative and $f(1)$ is positive. Since f has no jumps, it would then seem plausible that there exists a number $c \in (0, 1)$ where $f(c) = 0$. The following theorem establishes that this is indeed the case.

Theorem 2.1 (Intermediate Value Theorem [IVT]): *Suppose*

(i) *f is continuous on [a,b],*

(ii) *$f(a) < y < f(b)$.*

Then there exists a number $c \in (a, b)$ such that $f(c) = y$.

Problem 2.6: Show that $f(x) = x^7 + x^5 + 2x - 1$ has at least one real root in $(0, 1)$.

Since f is a polynomial, it is continuous. Noting that $-1 = f(0) < f(1) = 3$, we then know by the Intermediate Value Theorem that there exists an $c \in (0, 1)$ for which $f(c) = 0$.

Problem 2.7: Let $f(x) = 2x^3 + x^2 - 1$. Show that there exists $x \in (0, 1)$ such that $f(x) = x$.

Consider the continuous function $g(x) = f(x) - x$. We see that $g(0) = -1$ and $g(1) = 1$. By the Intermediate Value Theorem, there exists a point $c \in (0, 1)$ for which $g(c) = 0$, so that $f(c) = c$.

Chapter 3

Differentiation

3.1 Toangent Lines

Definition: Given a function f and a fixed point a of its domain, we can construct the *secant line* joining the points $(a, f(a))$ and $(b, f(b))$ for every point $b \neq a$ in the domain of f. The precise equation for this line depends on the value of b:

$$y = f(a) + m(b) \cdot (x - a),$$

where the slope $m(b)$ is

$$m(b) = \frac{f(b) - f(a)}{b - a}.$$

- If $f(x) = x^2$, the equation of the secant line through $(3, 9)$ and (b, b^2) for $b \neq 3$ is

$$y = 9 + \frac{b^2 - 9}{b - 3}(x - 3).$$

Definition: The *tangent line* of f at an interior point a of its domain, is obtained as the limit of the *secant line* as b approaches a:

$$y = f(a) + m \cdot (x - a),$$

where the limiting slope m is a number (independent of b):

$$m = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}.$$

- If $f(x) = x^2$, the slope of the tangent line through $(3, 9)$ is

$$m = \lim_{b \to 3} \frac{b^2 - 9}{b - 3} = \lim_{b \to 3} \frac{(b - 3)(b + 3)}{b - 3} = \lim_{b \to 3}(b + 3) = 6.$$

The equation of the tangent line to f through $(2, 4)$ is thus

$$y = 9 + 6(x - 3).$$

3.2 The Derivative

Definition: Let a be an interior point of the domain of a function f. If

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

exists, then f is said to be *differentiable* at a. The limit is denoted $f'(a)$ and is called the *derivative* of f at a. If f is differentiable at every point a of its domain, we say that f is differentiable.

Written in this way, we see that the derivative is the limit of the *slope*

$$m(x)=\frac{f(x)-f(a)}{x-a}$$

of a secant line joining the points $(a, f(a))$ and $(x, f(x))$, where $x \neq a$. The limit is taken as x gets closer to a; that is,

$$f'(a)=\lim_{x\to a} m(x).$$

Remark: Using the substition $h = x - a$, we see that

$$\lim_{x\to a} m(x) = L \iff \lim_{h\to 0} m(a+h) = L.$$

This subsitution allows us to rewrite the definition of a derivative as

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

- Let $f(t)$ be the position of a particle on a curve at time t. The *average velocity* of the particle between time t and $t+h$ is the ratio of the distance travelled over the time interval, h:

$$\frac{\text{change in position}}{\text{change in time}}=\frac{f(t+h)-f(t)}{h} \quad (h\neq 0).$$

The *instantaneous velocity* at t is calculated by taking the limit $h \to 0$:

$$\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}=f'(t).$$

- If $f(x) = c$, where c is a constant, then

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{c-c}{h}=0 \quad \text{for all } a\in\mathbb{R}.$$

- The derivative of the *affine* function $f(x) = mx + b$, where m and b are constants, (the graph of which is a straight line) has the constant value m:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{m(a+h) - ma}{h} = m.$$

In the case where $b = 0$, the function $f(x) = mx$ is said to be *linear*. A function that is neither linear nor affine is said to be *nonlinear*.

Remark: The derivative is the natural generalization of the slope of linear and affine functions to nonlinear functions. In general, the value of the local (or instantaneous) slope of a nonlinear function will depend on the point at which it is evaluated.

- Consider the function $f(x) = x^2$. Then

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{a^2} + 2ha + h^2 - \cancel{a^2}}{h} \\ &= \lim_{h \to 0} (2a + h) = 2a. \end{aligned}$$

We see here that the value of the derivative of f at the point a depends on a. Note that

$$\begin{aligned} f'(a) &< 0 \text{ for } \ a < 0, \\ f'(a) &= 0 \text{ for } \ a = 0, \\ f'(a) &> 0 \text{ for } \ a > 0. \end{aligned}$$

It is convenient to think of the derivative as a function on its own, which in general will depend on exactly where we evaluate it. We emphasize this fact by writing the derivative in terms of a dummy argument such as a or x. In this case, we can express this functional relationship as $f'(a) = 2a$ for all a, or with equal validity, $f'(x) = 2x$ for all x.

- If $f(x) = \sqrt{x}$, we find that

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h \to 0} \left(\frac{x+h-x}{h} \cdot \frac{1}{\sqrt{x+h} + \sqrt{x}} \right) = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

- Let $f(x) = x^n$, where $n \in \mathbb{N}$. Then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^n} + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + h^n - \cancel{x^n}}{h} \\
&= \lim_{h\to 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + h^{n-1} \right] \\
&= nx^{n-1}.
\end{aligned}$$

Remark: An alternative proof of the above result relies on the factorization

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + xa^{n-2} + a^{n-1}),$$

which may be established either by long division, summing a geometric series, or by multiplying out the right-hand-side, exploiting the collapse of this Telescoping sum to just two end terms:

$$\begin{aligned}
(x-a)\sum_{k=0}^{n-1} x^{n-1-k}a^k &= \sum_{k=0}^{n-1} x^{n-k}a^k - \sum_{k=0}^{n-1} x^{n-1-k}a^{k+1} = \sum_{k=0}^{n-1} x^{n-k}a^k - \sum_{k=1}^{n} x^{n-k}a^k \\
&= x^n - a^n.
\end{aligned}$$

For example, when $n = 2$ we recover the result $x^2 - a^2 = (x-a)(x+a)$ and when $n = 3$ we obtain $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$.

- If $f(x) = x^n$, we then find that

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x\to a} \frac{x^n - a^n}{x - a} \\
&= \lim_{x\to a} \left(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + xa^{n-2} + a^{n-1} \right) \\
&= na^{n-1}.
\end{aligned}$$

- We can compute the derivative of the function $f(x) = x^{1/n}$ where $x > 0$ and $n \in \mathbb{N}$

by applying the above factorization to $x-a=(x^{1/n})^n-(a^{1/n})^n$:

$$\begin{aligned}
f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\\
&= \lim_{x\to a}\frac{x^{1/n}-a^{1/n}}{x-a}\\
&= \lim_{x\to a}\frac{x^{1/n}-a^{1/n}}{(x^{1/n}-a^{1/n})(x^{(n-1)/n}+x^{(n-2)/n}a^{1/n}+\ldots+x^{1/n}a^{(n-2)/n}+a^{(n-1)/n})}\\
&= \frac{1}{\underbrace{\lim_{x\to a}x^{(n-1)/n}+\lim_{x\to a}x^{(n-2)/n}a^{1/n}+\ldots+\lim_{x\to a}a^{(n-1)/n}}_{n\text{ terms}}}\\
&= \frac{1}{na^{(n-1)/n}} = \frac{1}{n}a^{\frac{1-n}{n}}\\
&= \frac{1}{n}a^{\frac{1}{n}-1}.
\end{aligned}$$

Remark: The derivative of an exponential function $f(x)=b^x$ can be found by using the properties of exponentials:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{b^{x+h}-b^x}{h}=b^x\lim_{h\to 0}\frac{b^h-1}{h}=b^x f'(0),$$

on noting that $f'(0)=\frac{b^h-1}{h}$. This emphasizes the important property that the slope of an exponential function is proportional to the value of the function itself.

Remark: For the special choice of base $b=e$, this proportionality constant equals one; that is, $f'(0)=1$. That is, if $f(x)=e^x$, then $f'(x)=e^x$. The natural exponential function is thus its own derivative.

Problem 3.1: At what point on the graph of $f(x)=e^x$ is the tangent line parallel to the line $y=2x$?

On setting $f'(x)=e^x=2$ we find that $x=2$. The required point is thus $(\log 2, 2)$.

Q. Are all functions differentiable?

A. No, consider

$$f(x)=\begin{cases}0 & \text{if } x<0,\\ 1 & \text{if } x\ge 0.\end{cases}$$

We see that

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{1-1}{x}=\lim_{x\to 0^+}\frac{0}{x}=0,$$

but

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{0-1}{x} \text{ does not exist.}$$

So $\lim_{x\to 0} \frac{f(x)-1}{x-0}$ does not exist. It appears, at the very least, that we must avoid jumps, as the following theorem points out.

Theorem 3.1 (Differentiable $\Rightarrow$ Continuous): *If f is differentiable at a then f is continuous at a.*

Proof: For $x \neq a$, we may write

$$f(x) = f(a) + \frac{f(x)-f(a)}{x-a}(x-a).$$

If $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, then

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a} f(a) + \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a}(x-a)\\ &= f(a) + f'(a)\cdot 0\\ &= f(a),\end{aligned}$$

so f is continuous at a.

Q. Are all **continuous** functions differentiable?

A. No, consider $f(x) = |x|$:

$$\frac{f(x)-f(0)}{x-0} = \frac{|x|-0}{x-0} = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x>0,\\ -1 & \text{if } x<0.\end{cases}$$

Hence $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ does not exist; f is not differentiable at 0, even though f is continuous at 0.

Derivative Notation

Three equivalent notations for the derivative have evolved historically. Letting $y = f(x)$, $\Delta y = f(x+h) - f(x)$, and $\Delta x = (x+h) - x = h$, we may write

$$f'(x) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}.$$

To help us remember this, we sometimes denote the derivative by dy/dx (Leibnitz notation).

The operator notation Df (or $D_x f$, which reminds us that the derivative is with respect to x) is also occasionally used to emphasize that the derivative Df is a function derived from the original function, f.

Remark: When the derivative f' of a function f is itself differentiable we will use either the notation f'' or $f^{(2)}$ to denote the second derivative of f. In general, we will let $f^{(n)}$ denote the n-th derivative of f, obtained by differentiating f with respect to its argument n times (the parentheses help us avoid confusion with powers). It is also convenient to define $f^{(0)} = f$ itself.

Remark: Observe that $f^{(n+1)} = (f^{(n)})'$ and that if $f^{(n+1)}$ exists at a point x, then $f^{(n)}$ and all lower-order derivatives must also exist at x.

3.3 Properties

Theorem 3.2 (Properties of Differentiation): *If f and g are both differentiable at a, then*

(a) $(f+g)'(a) = f'(a) + g'(a)$,

(b) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$,

(c) $\left(\dfrac{f}{g}\right)'(a) = \dfrac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}$ *if* $g(a) \neq 0$.

Proof: We are given that $f'(a) = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists and $g'(a) = \lim\limits_{x\to a} \dfrac{g(x)-g(a)}{x-a}$ exists.

(a)

$$\begin{aligned}
\lim_{x\to a} \frac{(f+g)(x)-(f+g)(a)}{x-a} &= \lim_{x\to a} \frac{f(x)+g(x)-f(a)-g(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} + \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \\
&= f'(a) + g'(a).
\end{aligned}$$

(b)

$$\begin{aligned}
&\lim_{x\to a} \frac{(fg)(x)-(fg)(a)}{x-a} = \lim_{x\to a} \frac{f(x)g(x)-f(a)g(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x)g(x)-f(a)g(x)+f(a)g(x)-f(a)g(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \underbrace{\lim_{x\to a} g(x)}_{\text{exists } =g(a) \text{ by Theorem 3.1}} + f(a) \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \\
&= f'(a)g(a) + f(a)g'(a).
\end{aligned}$$

(c) Let $h(x) = \dfrac{1}{g(x)}$. Then

$$\begin{aligned}
h'(a) &= \lim_{x\to a} \frac{h(x)-h(a)}{x-a} = \lim_{x\to a} \frac{\frac{1}{g(x)} - \frac{1}{g(a)}}{x-a} \\
&= \lim_{x\to a} \frac{\frac{g(a)-g(x)}{g(x)g(a)}}{x-a} \\
&= -\frac{1}{g^2(a)} \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = -\frac{g'(a)}{g^2(a)}.
\end{aligned}$$

Then from **(b)**,

$$\begin{aligned}
\left(\frac{f}{g}\right)'(a) = (fh)'(a) &= f'(a)h(a) + f(a)h'(a) \\
&= \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{g^2(a)} \\
&= \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.
\end{aligned}$$

Remark: Any polynomial is differentiable on $\mathbb{R}$.

Remark: A rational function is differentiable at every point of its domain.

Problem 3.2: If $f(x) = e^x - x$, find f' and $f'' \doteq (f')'$.

Since $f'(x) = e^x - 1$, we see that $f''(x) = e^x$.

Problem 3.3: Use the quotient rule to show that the rule $dx^n/dx = nx^{n-1}$ is valid for all $n \in \mathbb{Z}$, including $n = 0$ and $n < 0$.

For $n = 0$, the derivative evaluates to $\displaystyle\lim_{h\to 0} \frac{1-1}{h} = 0$. For $n < 0$, we have

$$\frac{d}{dx}x^n = \frac{d}{dx}\frac{1}{x^{-n}} = \frac{0\cdot x^{-n} - 1\cdot \frac{d}{dx}x^{-n}}{(x^{-n})^2} = \frac{-(-n)x^{-n-1}}{(x^{-n})^2} = nx^{n-1}.$$

Problem 3.4: Use the following procedure to show that the derivative of $\sin x$ is $\cos x$.

(a) Use the inequality $\sin x \le x \le \tan x$ for $0 \le x < \pi/2$ to prove that

$$\cos x \le \frac{\sin x}{x} \le 1 \text{ for } 0 < |x| < \frac{\pi}{2}.$$

For $0 < x < \pi/2$, we know that both x and $\cos x$ are positive, which allows us rewrite the inequalities $x \le \tan x$ and $\sin x \le x$ as

$$\cos x \le \frac{\sin x}{x} \le 1.$$

Since each of these expressions are even functions of x, the inequality also holds for $-\pi/2 < x < 0$.

(b) Prove that

$$\lim_{x\to 0} \frac{\sin x}{x}$$

exists and evaluate the limit.

Since $\cos x$ is a continuous function, we know that $\lim_{x\to 0} \cos x = \cos 0 = 1$. We then deduce from the Squeeze Theorem that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

(c) Prove that

$$1 - \cos x \le \frac{x^2}{2} \quad \text{for all } x \in \mathbb{R}.$$

Hint: Try replacing x by $2x$.

$$1 - \cos x = 2\sin^2 \frac{x}{2} = 2\left|\sin \frac{x}{2}\right|^2 \le 2\left|\frac{x}{2}\right|^2 = \frac{x^2}{2}.$$

(d) Prove that

$$\lim_{x\to 0} \frac{1 - \cos x}{x}$$

exists and evaluate the limit.

Since $\cos x \le 1$ for all x, we know for $x \ne 0$ that

$$\left|\frac{1 - \cos x}{x}\right| = \frac{1 - \cos x}{|x|} \le \frac{|x|}{2}$$

and hence

$$-\frac{|x|}{2} \le \frac{1 - \cos x}{x} \le \frac{|x|}{2}.$$

As $x \to 0$, we deduce from the Squeeze Principle that

$$\lim_{x\to 0} \frac{1 - \cos x}{x} = 0.$$

Alternatively,

$$\begin{aligned}
\lim_{x\to 0} \frac{1 - \cos x}{x} &= \lim_{x\to 0} \frac{(1 - \cos x)(1 + \cos x)}{x(1 + \cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1 + \cos x)} \\
&= \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{\sin x}{1 + \cos x} = 1 \cdot \frac{0}{2} = 0.
\end{aligned}$$

(e) Use the above results to prove that $\sin x$ is differentiable at any real number a and find its derivative. That is, show that

$$\lim_{h\to 0} \frac{\sin(a+h)-\sin a}{h}$$

exists and evaluate the limit.

$$\lim_{h\to 0} \frac{\sin a\cos h + \cos a \sin h - \sin a}{h} = \sin a \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos a \lim_{h\to 0} \frac{\sin h}{h} = 0 + \cos a = \cos a.$$

Problem 3.5: Compute

$$\lim_{x\to\infty} x \tan\left(\frac{1}{x}\right)$$

Hint: let $y = 1/x$. As $x \to \infty$, what happens to y?

$$= \lim_{y\to 0^+} \frac{\tan y}{y} = \lim_{y\to 0^+} \left(\frac{\sin y}{y}\right) \lim_{y\to 0^+} \left(\frac{1}{\cos y}\right) = 1 \cdot 1 = 1.$$

Theorem 3.3 (Chain Rule): *Suppose $h = f \circ g$, i.e. $h(x) = f(g(x))$. Let a be an interior point of the domain of h and define $b = g(a)$. If $f'(b)$ and $g'(a)$ both exist, then h is differentiable at a and*

$$h'(a) = f'(b)g'(a).$$

That is, if $y = f(u)$ and $u = g(x)$, then

$$\left.\frac{dy}{dx}\right|_a = \left.\frac{dy}{du}\right|_b \left.\frac{du}{dx}\right|_a .$$

- Consider that $\frac{d}{dx}(x^2+1)^2 = \frac{d}{dx} f(g(x))$, where $u = g(x) = x^2 + 1$ and $f(u) = u^2$. We let $h(x) = f(g(x))$:

$$\begin{aligned} h'(x) &= f'(u)g'(x) \\ &= 2u \cdot 2x \\ &= 2(x^2+1)\cdot 2x = 4x^3 + 4x. \end{aligned}$$

As a check, we could also work out this derivative directly:

$$\frac{d}{dx}(x^2+1)^2 = \frac{d}{dx}(x^4 + 2x^2 + 1) = 4x^3 + 4x.$$

- The Chain Rule makes it easy to find

$$\begin{aligned}\frac{d}{dx}(x^3+1)^{100} &= 100(x^3+1)^{99}3x^2\\ &= 300x^2(x^3+1)^{99}.\end{aligned}$$

- Let$f(u) = u^{\frac{1}{n}} \Rightarrow f'(u) = \frac{1}{n}u^{\frac{1}{n}-1}$ and $g(x) = x^m \Rightarrow g'(x) = mx^{m-1}$.

  Then $h(x) = f(g(x)) = x^{\frac{m}{n}} \Rightarrow h'(x) = f'(u)g'(x)$ where $u = g(x)$. Thus

$$\begin{aligned}h'(x) &= f'(g(x))g'(x)\\ &= \frac{1}{n}(x^m)^{\frac{1}{n}-1}mx^{m-1}\\ &= \frac{m}{n}x^{\frac{m}{n}-\cancel{m}+\cancel{m}-1}.\end{aligned}$$

Hence $\frac{d}{dx}x^q = qx^{q-1}$ for all $q \in \mathbb{Q}$.

- Find $\frac{d}{dx}\frac{1}{g(x)}$ (cf. Theorem 3.2(c)).

  Let $f(x) = x^{-1}$, $f'(x) = -x^{-2}$, and $h(x) = \frac{1}{g(x)} = f(g(x))$. Then

$$\begin{aligned}h'(x) &= f'(g(x))g'(x)\\ &= -\frac{1}{[g(x)]^2}g'(x),\end{aligned}$$

We may express this using an alternative notation. Letting $y = \frac{1}{u}$ and $u = g(x)$, we find

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{1}{u^2}g'(x) = -\frac{g'(x)}{g^2(x)}.$$

-

$$\begin{aligned}\frac{d}{dx}\sqrt{\frac{1}{1+x^3}} &= \frac{1}{2\sqrt{\frac{1}{1+x^3}}}\left[-\frac{1}{(1+x^3)^2}\cdot 3x^2\right]\\ &= -\frac{3x^2(1+x^3)^{\frac{1}{2}}}{2(1+x^3)^2} = -\frac{3}{2}x^2(1+x^3)^{-\frac{3}{2}}.\end{aligned}$$

  Here is an even easier way to find this derivative:

$$\begin{aligned}\frac{d}{dx}\sqrt{\frac{1}{1+x^3}} &= \frac{d}{dx}(1+x^3)^{-\frac{1}{2}}\\ &= -\frac{1}{2}(1+x^3)^{-\frac{3}{2}}\cdot 3x^2.\end{aligned}$$

- $$\frac{d}{dx}\sin(\sin(x)) = \cos(\sin(x))\cos(x).$$

- $$\frac{d}{dx}\sin(\sin(\sin(x))) = \cos(\sin(\sin(x)))\cos(\sin(x))\cos(x).$$

Remark: To prove the Chain Rule it is not enough to argue

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}$$

because $\Delta u = g(x) - g(a)$ might be zero for values of x close to (but not equal to) a. However, we can easily fix up this argument as follows.

Proof (of Theorem 3.3):
Let $b = g(a)$ and define

$$m(u) = \begin{cases} \dfrac{f(u)-f(b)}{u-b} & \text{if } u \neq b, \\ f'(b) & \text{if } u = b. \end{cases}$$

Then

$$\begin{aligned} f'(b) \text{ exists} &\Rightarrow \lim_{u\to b} m(u) = f'(b) = m(b) \Rightarrow m \text{ is continuous at } b \\ &\text{and} \\ g'(a) \text{ exists} &\Rightarrow g \text{ is continuous at } a \Rightarrow m\circ g \text{ is continuous at } a, \\ &\Rightarrow \lim_{x\to a} m(g(x)) = m(g(a)) = m(b) = f'(b). \end{aligned}$$

Note that

$$f(u) - f(b) = m(u)(u-b) \text{ for all } u.$$

Letting $u = g(x)$, we then find that

$$\begin{aligned} \lim_{x\to a}\frac{f(g(x)) - f(g(a))}{x-a} &= \lim_{x\to a} m(g(x)) \lim_{x\to a}\frac{g(x)-g(a)}{x-a} \\ &= f'(b)g'(a). \end{aligned}$$

- With the help of the Chain Rule, the derivative of $\cos x$ can be calculated as:

$$\frac{d}{dx}\cos x = \frac{d}{dx}\sin\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2} - x\right)(-1) = -\sin x,$$

- We can also find the derivative of $\tan x$:

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Problem 3.6: Compute

$$\frac{d}{dx}(x\cos x)$$

$$= \cos x - x\sin x.$$

Problem 3.7: Find

$$\frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$= \left(\frac{1}{\cos^2 x}\right)\sin x.$$

Problem 3.8: Let f be a differentiable function. Find the following derivatives

(a)

$$\frac{d}{dx}f(f(f(x)))$$

$$= f'(f(f(x)))f'(f(x))f'(x).$$

(b)

$$\frac{d}{dx}\left[\frac{f^3(x)+1}{f^2(x)}\right]$$

$$= \frac{d}{dx}\left[f(x) + \frac{1}{f^2(x)}\right] = \left[1 - \frac{2}{f^3(x)}\right]f'(x).$$

Problem 3.9: If $f(x) = b^x = e^{x\log b}$, show that $f'(x) = b^x \log b$. Note for $b = e$ that this reduces to $f'(x) = e^x$ since $\log e = 1$.

Problem 3.10: Calculate the following derivatives:

(a)

$$\frac{d}{dx}\frac{x^2}{x^3+1}$$

$$= \frac{2x(x^3+1)-x^2(3x^2)}{(x^3+1)^2} = \frac{-x^4+2x}{(x^3+1)^2}.$$

(b)

$$\frac{d}{dx}\left(\frac{\sqrt{2x+1}}{\sin x}\right)$$

$$\frac{\sin x\frac{1}{\sqrt{2x+1}} - \sqrt{2x+1}\cos x}{\sin^2 x} = \frac{\sin x - (2x+1)\cos x}{\sqrt{2x+1}\sin^2 x}.$$

(c)

$$\frac{d}{dx}\frac{1}{\sin^3(\sin x)}$$

$$\frac{d}{dx}\sin^{-3}(\sin x) = -3\sin^{-4}(\sin x)\cos(\sin x)\cos x = -3\frac{\cos(\sin x)\cos x}{\sin^4(\sin x)}.$$

Problem 3.11: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Consider $g(x) = f(-x)$.

(a) Compute g' in terms of f'.
Using the Chain Rule, we find that $g'(x) = -f'(-x)$.
(b) If f is an even function, show that f' is odd.
If f is even then $g(x) = f(-x) = f(x)$; that is, g and f are the same function. From part (a) we then see that f is odd: $f'(x) = -f'(-x)$.
(c) If f is an odd function, show that f' is even.
If f is odd then $g(x) = f(-x) = -f(x)$. From part (a) we then see that f is even: $f'(x) = f'(-x)$.

Problem 3.12:

(a) A spherical balloon is being inflated at the rate of 10 cm^3/s. Given that the volume V of the balloon is related to the radius by $V = \frac{4}{3}\pi r^3$, use the Chain Rule to compute how fast the radius of the balloon is growing when the volume has reached 100 cm^3.
The rate of inflation $r(t)$ must equal the rate of volume increase: $dV/dt = 10\,\mathrm{cm}^3/\mathrm{s}$. We will need to know the formula for r in terms of V,

$$r = \left(\frac{3}{4\pi}\right)^{1/3} V^{1/3}.$$

and its derivative,

$$\frac{dr}{dV} = \left(\frac{3}{4\pi}\right)^{1/3} \frac{1}{3} V^{-2/3} = \left(\frac{1}{36\pi}\right)^{1/3} V^{-2/3}.$$

When the volume of the balloon is 100 cm^3, we can use the Chain Rule to determine that the radius is growing at the rate

$$\frac{dr}{dt} = \frac{dr}{dV}\frac{dV}{dt} = \left(\frac{1}{36\pi}\right)^{1/3} (100\,\mathrm{cm}^3)^{-2/3} \times 10\,\frac{\mathrm{cm}^3}{\mathrm{s}} = 0.096\frac{\mathrm{cm}}{\mathrm{s}}.$$

Incidentally, the derivative of r with respect to V can also be calculated by first calculating $dV/dr = 4\pi r^2$, taking the reciprocal to get dr/dV, and finally expressing the result in terms of V. (What justifies that one can calculate dr/dV in this way?)

(b) Suppose now that the (constant) inflation rate of the balloon is unknown, but it is known that when the volume is 100 cm^3, the radius is growing at a rate of 1 cm/s. How fast is the radius of the balloon growing when the volume has reached 1000 cm^3?

We are given that at a time t_1, the volume $V(t_1) = 100$ cm^3 and $\frac{dr}{dt}t_1| = 1$ cm/s. By the Chain Rule,

$$\frac{dr}{dt}t_1| = \frac{dr}{dV}t_1|\frac{dV}{dt}$$

and

$$\frac{dr}{dt}t_2| = \frac{dr}{dV}t_2|\frac{dV}{dt},$$

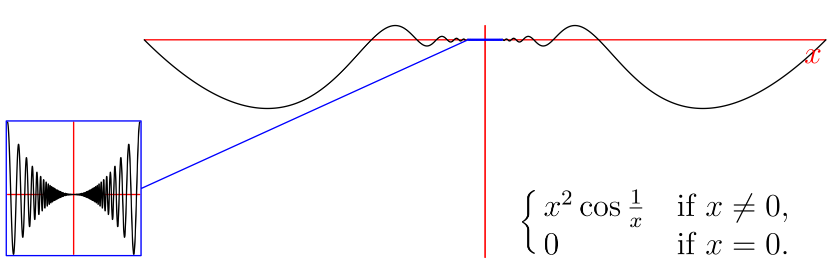
since the inflation rate $\frac{dV}{dt}$ is constant. Hence

$$\frac{dr}{dt}t_2| = \left(\frac{\frac{dr}{dV}t_2|}{\frac{dr}{dV}t_1|}\right)\frac{dr}{dt}t_1| = \left(\frac{V(t_2)}{V(t_1)}\right)^{-2/3}\frac{dr}{dt}t_1| = \left(\frac{1}{10}\right)^{2/3} \times 1\,\frac{\mathrm{cm}}{\mathrm{s}} = 0.215\,\frac{\mathrm{cm}}{\mathrm{s}}.$$

Problem 3.13: Let

$$f(x) = \begin{cases} x^2\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that f is differentiable for all $x \in \mathbb{R}$ and find $f'(x)$. A graph of f is shown below, including an inset where the y axis is stretched to show more detail around the origin.



Since $\cos(x)$ is differentiable at all x and $1/x$ is differentiable on $(-\infty,0)\cup(0,\infty)$, the composite function $\cos(1/x)$, and hence f, is differentiable on $(-\infty,0)\cup(0,\infty)$. Moreover, f is also differentiable at $x=0$, with derivative 0:

$$\lim_{x\to 0}\frac{x^2\cos\left(\frac{1}{x}\right)-0}{x-0}=\lim_{x\to 0}x\cos\left(\frac{1}{x}\right)=0$$

by the Squeeze Theorem since

$$0\le\left|x\cos\left(\frac{1}{x}\right)\right|\le|x|$$

and $\lim\limits_{x\to 0}0=0=\lim\limits_{x\to 0}|x|$.

Hence f is differentiable on $\mathbb{R}$ and

$$f'(x)=\begin{cases}2x\cos\left(\frac{1}{x}\right)+\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0,\\ 0 & \text{if } x=0.\end{cases}$$

Problem 3.14: Suppose that two functions f and g are differentiable n times at the point a. Use induction to prove *Leibnitz's formula*:

$$(fg)^{(n)}(x)=\sum_{k=0}^{n}\binom{n}{k}f^{(n-k)}(x)g^{(k)}(x).$$

3.4 Implicit Differentiation

Suppose that a variable y is defined implicitly in terms of x and we wish to know dy/dx. For example, given the *implicit equation*

$$y^3+3y^2+3y+1=x^5+x, \tag{3.1}$$

we could solve for y to find

$$\begin{aligned}(y+1)^3&=x^5+x\\ \Rightarrow y+1&=(x^5+x)^{\frac{1}{3}}\\ \Rightarrow \frac{dy}{dx}&=\frac{1}{3}(x^5+x)^{-\frac{2}{3}}(5x^4+1).\end{aligned} \tag{3.2}$$

But what happens if you can't (or don't want to) solve for y? You might try first to solve for x in terms of y and then find the derivative dx/dy of the inverse function. But what if this is also difficult?

It is often easier in these cases to differentiate both sides of Eq. (3.1) with respect to x, noting that $y=y(x)$:

$$\frac{d}{dx}\left[y^3(x)+3y^2(x)+3y(x)+1\right]=\frac{d}{dx}(x^5+x).$$

By the Chain Rule, we find

$$\big(3y^2 + 6y + 3\big)y'(x) = 5x^4 + 1,$$

which we can easily solve to obtain dy/dx as a function of x and y,

$$\frac{dy}{dx} = \frac{5x^4 + 1}{3y^2 + 6y + 3} = \frac{5x^4 + 1}{3(y+1)^2}. \tag{3.3}$$

Once we know an (x, y) pair that satisfies Eq. (3.1), we can immediately compute the derivative from Eq. (3.3).

It is instructive to verify that Eqs. (3.2) and (3.3) agree:

$$\frac{dy}{dx} = \frac{5x^4 + 1}{3(y+1)^2} = \frac{5x^4 + 1}{3(x^5 + x)^{\frac{2}{3}}}.$$

Problem 3.15: If $x^2 + y^2 = 25$, find dy/dx. Then find an equation for the tangent line to this circle through the point $(3, 4)$.

On implicitly differentiating both sides with respect to x, we find

$$2x + 2y\frac{dy}{dx} = 0.$$

When $y \neq 0$ we can then solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Since the slope of the tangent line at $(3, 4)$ is $-3/4$, an equation of the tangent line is

$$y - 4 = -\frac{3}{4}(x - 3).$$

Problem 3.16: Find dy/dx if $x^3 + y^3 = 6xy$.

3.5 Inverse Functions and Their Derivatives

This section addresses the question: given a function f, when is it possible to find a function g that undoes the effect of f, so that

$$y = f(x) \iff x = g(y)?$$

Recall that a function is a collection of pairs of numbers (x, y) such that if (x, y_1) and (x, y_2) are in the collection, then $y_1 = y_2$.

Definition: A function $f : A \to B$ is *one-to-one* on its domain A if, whenever (x_1, y) and (x_2, y) are in the collection, then $x_1 = x_2$. That is,

$$x_1 = x_2 \iff f(x_1) = f(x_2).$$

We say that such a function is 1–1 or *invertible*.

This can be restated using the *horizontal line test*: a set of ordered pairs (x, y) is a one-to-one function if every horizontal and every vertical line intersects their graph at most once.

Remark: Equivalently, a 1–1 function f satisfies

$$x_1 \neq x_2 \iff f(x_1) \neq f(x_2).$$

- $f(x) = x$ and $f(x) = x^3$ are 1–1 functions.

- $f(x) = x^2$ and $f(x) = \sin x$ are not 1–1 functions.

Remark: Sometimes a *noninvertible* function can be made invertible by restricting its domain.

- $f = \sin x$ restricted to the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is 1–1.

Remark: If $f : A \to B$ is 1–1 then the collection of pairs of numbers (y, x) such that (x, y) belong to f is also a function.

Definition: The function defined by the pairs $\{(y, x) : (x, y) \in f\}$ is the *inverse* function $f^{-1} : B \to A$ of f.

Problem 3.17: Show that the inverse of a 1–1 function is itself an invertible function; that is, it satisfies both the horizontal and vertical line tests.

- The inverse of the function $\sin x$ restricted to the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is denoted $\arcsin x$ or $\sin^{-1} x$; it is itself a 1–1 function on $[-1, 1]$, yielding values in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Remark: Do not confuse the notation $\sin^{-1} x$ with $\frac{1}{\sin x}$; they are not the same function! Because of this rather unfortunate notational ambiguity, we will use the short-hand notation $f^n(x)$ to denote $(f(x))^n$ only when $n \geq 0$; in particular, we reserve the notation $f^{-1}(x)$ for the inverse of f.

Problem 3.18: Suppose that f and g are inverse functions of each other. Show that $g(f(x)) = x$ for all x in the domain of f and $f(g(y)) = y$ for all y in the range of f.

Theorem 3.4 (Continuous Invertible Functions): *Suppose f is continuous on I. Then f is one-to-one on $I \iff f$ is strictly monotonic on I.*

Theorem 3.5 (Continuity of Inverse Functions): *Suppose f is continuous and one-to-one on an interval I. Then its inverse function f^{-1} is continuous on $f(I) = \{f(x) : x \in I\}$.*

Theorem 3.6 (Differentiability of Inverse Functions): *Suppose f is continuous and one-to-one on an interval I and differentiable at $a \in I$. Let $b = f(a)$ and denote the inverse function of f on I by g. If*

(i) *$f'(a) = 0$, then g is* **not** *differentiable at b;*

(ii) *$f'(a) \neq 0$, then g is differentiable at b and $g'(b) = \dfrac{1}{f'(a)}$.*

- The inverse of the function $f(x) = x^3$ is $f^{-1}(y) = y^{1/3}$ since $y = x^3 \Rightarrow x = y^{\frac{1}{3}}$. Notice that $f'(x) = 3x^2 \neq 0$ for $x \neq 0$ (i.e. $y \neq 0$). We can then verify that

$$\frac{d}{dy}f^{-1}(y) = \frac{1}{3}y^{-\frac{2}{3}} = \frac{1}{3y^{\frac{2}{3}}} = \frac{1}{3[f^{-1}(y)]^2} = \frac{1}{f'(f^{-1}(y))}.$$

- What is the derivative of $y = \arctan x$ (or $y = \tan^{-1} x$, the inverse function of $x = \tan y$?

Theorem 3.6 $\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$ where $x = \tan y$ and $\dfrac{dx}{dy} = \dfrac{1}{\cos^2 y}$. That is,

$$\frac{dy}{dx} = \frac{1}{\frac{1}{\cos^2 y}} = \cos^2 y.$$

Normally, we will want to re-express the derivative in terms of x. Recalling that $\tan^2 y + 1 = \frac{1}{\cos^2 y}$ and $x = \tan y$, we see that

$$\frac{dy}{dx} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

$$\therefore \boxed{\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \text{ on } (-\infty, \infty).}$$

Remark: Although $f(x) = \tan x$ does not satisfy the horizontal line test on $\mathbb{R}$, it does if we restrict $\tan x$ to the domain $(-\frac{\pi}{2}, \frac{\pi}{2})$. We call $\tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ the *principal branch* of $\tan x$, which is sometimes denoted $\operatorname{Tan} x$. Its inverse, which is sometimes written $\operatorname{Arctan} x$ or $\operatorname{Tan}^{-1} x$, maps $\mathbb{R}$ to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

- Consider $f(x) = \sqrt{1-x^2}$, which is 1–1 on $[0, 1]$.

  Note that $f'(x) = -\frac{x}{\sqrt{1-x^2}}$ exists on $[0, 1)$.
  Now $y = \sqrt{1-x^2} \Rightarrow x = \sqrt{1-y^2} \Rightarrow x = f^{-1}(y) = f(y)$.
  In this case f and f^{-1} are identical functions of their respective arguments!

$$\begin{aligned}
\frac{d}{dy} f^{-1}(y) &= \frac{1}{f'(x)} \\
&= -\frac{\sqrt{1-x^2}}{x} \\
&= -\frac{\sqrt{1-[f^{-1}(y)]^2}}{f^{-1}(y)} \\
&= -\frac{\sqrt{1-[f(y)]^2}}{f(y)} \\
&= -\frac{\sqrt{1-(1-y^2)}}{\sqrt{1-y^2}} \\
&= -\frac{y}{\sqrt{1-y^2}} \qquad \text{on } [0, 1).
\end{aligned}$$

- $y = \sin x$ is 1–1 on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

  $\dfrac{dy}{dx} = \cos x \neq 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$.
  The inverse function (as a function of y) is

$$x = \arcsin y \text{ (or } x = \sin^{-1} y),$$

with derivative

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\cos x}.$$

We can express $\cos x$ as a function of y:

$$\begin{aligned}
\cos x &= \sqrt{1-\sin^2 x} \\
&= \sqrt{1-y^2},
\end{aligned}$$

noting that $\cos x > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, to find

$$\frac{d}{dy}\arcsin y = \frac{1}{\sqrt{1-y^2}} \text{ on } (-1,1).$$

That is,

$$\boxed{\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \text{ on } (-1,1).}$$

- $y = \cos x$ is 1–1 on $[0, \pi]$.

  $\dfrac{dy}{dx} = -\sin x \neq 0$ on $(0, \pi)$.
  The inverse function $x = \arccos y$ (or $x = \cos^{-1} y$) has derivative

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{-\sin x},$$

which we can express as a function of y, noting that $\sin x > 0$ on $(0, \pi)$,

$$\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-y^2}.$$

$$\therefore \ \frac{d}{dy}\arccos y = -\frac{1}{\sqrt{1-y^2}},$$

i.e.

$$\boxed{\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}} \text{ on } (-1,1).}$$

It is not surprising that $\frac{d}{dx}\arccos x = -\frac{d}{dx}\arcsin x$ since $\arccos x = \frac{\pi}{2} - \arcsin x$, as can readily be seen by taking the cosine of both sides and using $\cos y = \sin(\frac{\pi}{2} - y)$.

- Prove that $\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$ for all $x \in [-1, 1]$. Let

$$\begin{aligned} f(x) &= \cos^{-1} x + \sin^{-1} x \\ f'(x) &= \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} = 0 \\ \Rightarrow f(x) &= c, \quad \text{a constant.} \end{aligned}$$

  Set $x = 0$ to find c:

$$c = f(0) = \cos^{-1} 0 = \frac{\pi}{2}.$$

$$\therefore \ f(x) = \frac{\pi}{2} \text{ for all } x \in [-1, 1].$$

Problem 3.19: Let $f(x) = \sin^{-1}(x^2 - 1)$. Find

(a) the domain of f;

The inverse function $y = \sin^{-1} x$ has domain $[-1, 1]$, and $x^2 - 1 \in [-1, 1]$ implies $x^2 \in [0, 2]$. Hence, the domain of f is $[-\sqrt{2}, \sqrt{2}]$.

(b) $f'(x)$;

Letting $y = \sin^{-1}(x^2 - 1)$, we first find the derivative for $x > 0$:

$$\begin{aligned}
x^2 - 1 &= \sin y \\
\Rightarrow x &= \sqrt{\sin y + 1} \\
\Rightarrow \frac{dx}{dy} &= \frac{\cos y}{2\sqrt{\sin y + 1}} \\
&= \frac{\sqrt{1-(x^2-1)^2}}{2\sqrt{x^2}} = \frac{\sqrt{2x^2 - x^4}}{2x} \\
\Rightarrow \frac{dy}{dx} &= \frac{2x}{\sqrt{2x^2 - x^4}}.
\end{aligned}$$

Since the derivative of an even function is odd (and *vice-versa*) we see that the same result holds for $x < 0$ as well.

Alternatively, one could use the formula for the derivative of $\sin^{-1} x$ together with the Chain Rule.

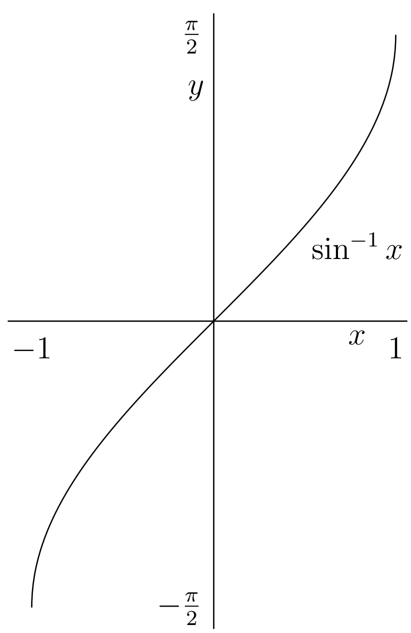
(c) the domain of f'.

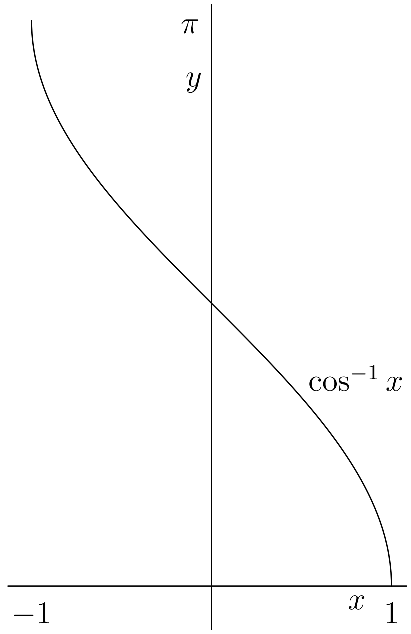
The domain of $f' = dy/dx$ is the set of x such that $2x^2 - x^4 > 0$:

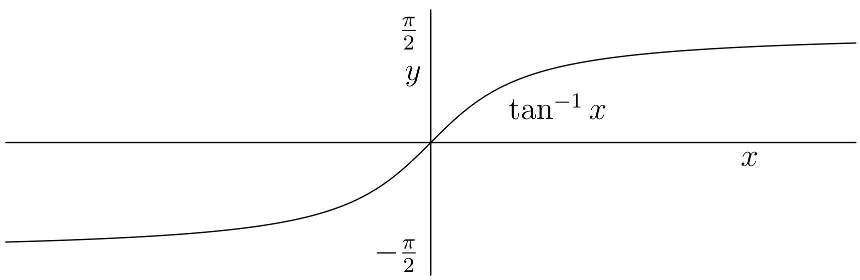
$$2x^2 > x^4 \Rightarrow 2 > x^2 \quad \text{if } x \neq 0.$$

$\therefore$ domain of f' is $\{x : 0 < |x| < \sqrt{2}\} = (-\sqrt{2}, 0) \cup (0, \sqrt{2})$.

Problem 3.20: Verify that the graphs of the functions $y = \sin^{-1} x$, $y = \cos^{-1} x$, and $y = \tan^{-1} x$ are as shown below.







Remark: We may use the fact that the derivative of the exponential function $y = f(x) = e^x$ is $f(x)$ itself to find the derivative of the inverse function $x = g(y) = \log y$. Since

$$f'(x) = \frac{dy}{dx} = y,$$

we see that

$$g'(y) = \frac{dx}{dy} = \frac{1}{y},$$

Thus for $x > 0$ we find that

$$\frac{d}{dx} \log x = \frac{1}{x}.$$

That is, the derivative of the natural logarithm is the reciprocal function.

Remark: The derivative of the logarithm to the base b follows immediately:

$$\frac{d}{dx} \log_b x = \frac{d}{dx} \frac{\log x}{\log b} = \frac{1}{x \log b}.$$

- Consider

$$F(x) = \log|x| = \begin{cases} \log x & x > 0, \\ \log(-x) & x < 0. \end{cases}$$

Then

$$F'(x) = \begin{cases} \dfrac{1}{x} & x > 0, \\ \\ \dfrac{1}{-x}(-1) & x < 0 \end{cases}$$
$$= \frac{1}{x} \quad \text{for all } x \neq 0.$$

That is, for $x \neq 0$ we find that

$$\frac{d}{dx}\log|x| = \frac{1}{x}.$$

Problem 3.21: Suppose that f and its inverse g are twice differentiable functions on $\mathbb{R}$. Let $a \in \mathbb{R}$ and denote $b = f(a)$.

(a) Implicitly differentiate both sides of the identity $g(f(x)) = x$ with respect to x.
By the Chain Rule,

$$g'(f(x))f'(x) = 1.$$

(b) Using part(a), prove that $f'(a) \neq 0$.
If $f'(a) = 0$, we would obtain a contradiction:

$$0 = g'(f(a))f'(a) = 1.$$

(c) Using parts (a) and (b), find a formula expressing $g'(b)$ in terms of $f'(a)$.

$$g'(b) = \frac{1}{f'(a)}.$$

(d) Show that

$$g''(b) = -\frac{f''(a)}{[f'(a)]^3}.$$

On differentiating the expression in part (a), we find that

$$g''(f(x))[f'(x)]^2 + g'(f(x))f''(x) = 0.$$

On setting $x = a$ and using part(c), we find that

$$g''(f(a))[f'(a)]^2 + \frac{f''(a)}{f'(a)} = 0,$$

from which the desired result immediately follows.

3.6 Logarithmic Differentiation

Because they can be used to transform multiplication problems into addition problems, logarithms are frequently exploited in calculus to facilitate the calculation of derivatives of complicated products or quotients. For example, if we need to calculate the derivative of a *positive* function $f(x)$, the following procedure may simplify the task:

1. Take the logarithm of both sides of $y = f(x)$.
2. Differentiate each side implicitly with respect to x.
3. Solve for dy/dx.

- Differentiate $y = x^{\sqrt{x}}$ for $x > 0$.

  We have
$$\log y = \log x^{\sqrt{x}} = \sqrt{x}\log x.$$
Thus
$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{1}{2\sqrt{x}}\log x + \sqrt{x}\left(\frac{1}{x}\right).\\ \Rightarrow \frac{dy}{dx} &= y\left(\frac{\log x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)\\ &= x^{\sqrt{x}}\left(\frac{\log x + 2}{2\sqrt{x}}\right).\end{aligned}$$

Problem 3.22: Show that the same result follows on differentiating $y = e^{\sqrt{x}\log x}$ directly.

- For $x > 0$ differentiate
$$y = -\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}.$$
  Since
$$\log(-y) = \frac{3}{4}\log x + \frac{1}{2}\log(x^2+1) - 5\log(3x+2),$$
  we find
$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{3}{4}\left(\frac{1}{x}\right) + \frac{1}{2}\left(\frac{1}{x^2+1}\right)(2x) - \frac{5}{3x+2}(3)\\ \Rightarrow \frac{dy}{dx} &= -\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right).\end{aligned}$$

Remark: We can use logarithmic differentiation to show that if $y = f(x) = x^n$ for some real number n, then $f'(x) = nx^{n-1}$. First, we take the absolute value of y to ensure that the argument of the logarithm is non-negative:

$$\log|y| = \log|x|^n = n\log|x|.$$

We then implicitly differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{n}{x},$$

from which we find that

$$\frac{dy}{dx} = \frac{ny}{x} = \frac{nx^n}{x} = nx^{n-1}.$$

Problem 3.23: Alternatively, show directly from the definition $x^n = e^{n\log x}$ that the rule $dx^n/dx = nx^{n-1}$ is valid for any real n.

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{n\log x} = e^{n\log x}n\frac{1}{x} = nx^{n-1}.$$

Remark: Recall that

$$\frac{1}{y} = \frac{d}{dy}\log(y) = \lim_{h\to 0}\frac{\log(y+h)-\log(y)}{h}.$$

In particular, at $y = 1/x$ we find

$$x = \lim_{h\to 0}\frac{\log\left(\frac{1}{x}+h\right)+\log x}{h} = \lim_{h\to 0}\log(1+xh)^{\frac{1}{h}} = \log\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

We thus obtain another expression for e^x:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

Remark: In particular at $x = 1$ we obtain a limit expression for the number e:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

3.7 Rates of Change: Physics

Problem 3.24: The position at time t of a particle in meters is given by the equation

$$x = f(t) = t^3 - 6t^2 + 9t.$$

Find (a) the velocity at time t.

$$v(t) = \frac{dx}{dt} = f'(t) = 3t^2 - 12t + 9.$$

(b) When is the particle at rest? Setting

$$0 = v(t) = 3(t^2 - 4t + 3) = 3(t-3)(t-1),$$

we see that the particle is at rest when $t = 1$ or $t = 3$.

(c) When is the particle moving forward?

The particle is moving forward when $v(t) > 0$; that is when $t - 3$ and $t - 1$ have the same sign. This happens when $t < 1$ and $t > 3$. For $t \in (1, 3)$ we see that $v(t) < 0$, so the particle moves backwards.

(d) Find the total distance travelled during the first five seconds.

Because the particle retraces it path for $t \in (1, 3)$, we must calculate these distance travelled during $[0, 1]$, $[1, 3]$, and $[3, 5]$ separately. From $t = 0$ to $t = 1$, the distance travelled is $|f(1) - f(0)| = |4 - 0| = 4m$. From $t = 1$ to $t = 3$, the distance travelled is $|f(3) - f(1)| = |0 - 4| = 4m$. From $t = 3$ to $t = 5$, the distance travelled is $|f(5) - f(3)| = |20 - 0| = 20m$. The total distance travelled is therefore $28m$.

(e) Determine the acceleration $a = dv/dt$ of the particle as a function of t.

$$a(t) = 6t - 12.$$

3.8 Related Rates

The Chain Rule is useful for solving problems with two variables that are related to one another. In this, the rate of change of one variable may be related to the rate of change of the other.

Problem 3.25: A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 3m higher than the bow of the boat. If the rope is pulled in at a rate of 2m/s, how fast is the boat approaching the dock when it is 4m from the dock?

Let r denote the length of the rope, from bow to pulley, and x the (horizontal) distance between the bow and the dock. Then $r(x) = \sqrt{x^2 + 3^2}$ so that $dr/dx = x/\sqrt{x^2 + 3^2}$. Thus

$$\frac{dx}{dt} = \frac{dx}{dr} \cdot \frac{dr}{dt} = \frac{\sqrt{4^2 + 3^2}}{4} \times 2 = \frac{5}{2} \text{ m/s}.$$

Problem 3.26: A stone thrown into a pond produces a circular ripple which expands from the point of impact. When the radius is 8m it is observed that the radius is increasing at a rate of 1.5m/s. How fast is the area increasing at that instant?

Problem 3.27: Water is leaking out of a tank shaped like an inverted cone (pointed end at the bottom) at a rate of 10 $\mathrm{m}^3/\mathrm{min}$. The tank has a height of 6m and a diameter at the top of 4m. How fast is the water level dropping when the height of the water in the tank is 2m?

3.9 Taylor Polynomials

See eclass notes.

3.10 Partial Derivatives

See eclass notes.

3.11 Hyperbolic Functions

Hyperbolic functions are combinations of e^x and e^{-x}:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \cosh x = \frac{e^x + e^{-x}}{2}, \tanh x = \frac{\sinh x}{\cosh x},$$

$$\mathrm{csch} x = \frac{1}{\sinh x}, \quad \mathrm{sech}\, x = \frac{1}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}.$$

Recall that the points $(x, y) = (\cos t, \sin t)$ generate a circle, as t is varied from 0 to 2π, since $x^2 + y^2 \cos^2 t + \sin^2 t = 1$. In contrast, the points $(x, y) = (\cosh t, \sinh t)$ generate a hyperbola, as t is varied over all real values, since $x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$ (hence the name hyperbolic functions). That is,

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1.$$

Note that

$$\frac{d}{dx} \sinh x = \frac{e^x + e^{-x}}{2} = \cosh x,$$

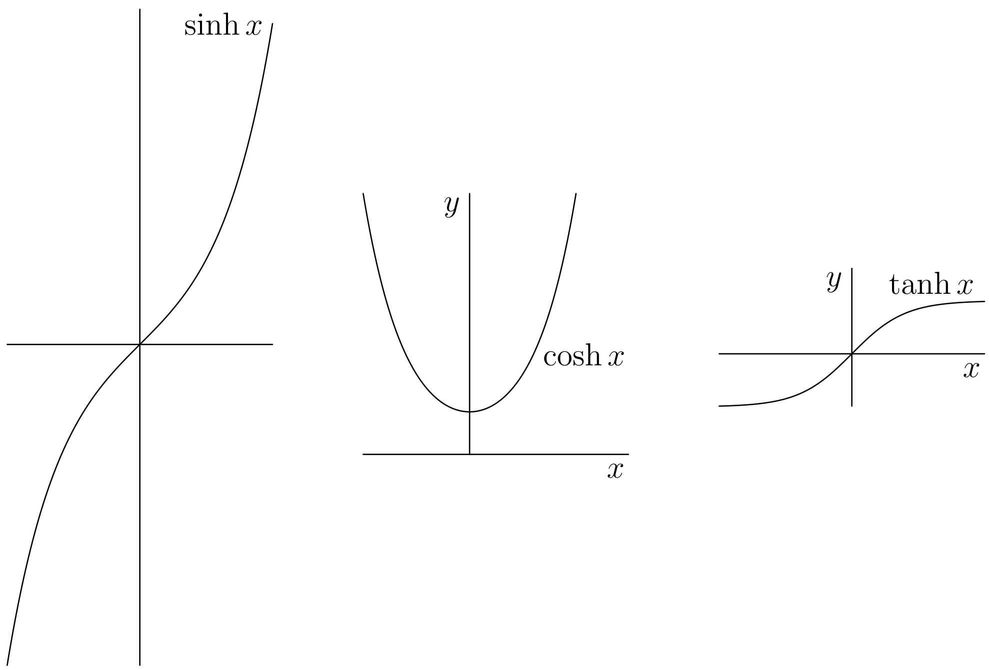
but

$$\frac{d}{dx} \cosh x = \frac{e^x - e^{-x}}{2} = \sinh x$$

(without any minus sign). Also,

$$\frac{d}{dx}\tanh x = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}.$$

Note that $\sinh x$ and $\tanh x$ are strictly monotonic, whereas $\cosh x$ is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$.



Just as the inverse of e^x is $\log x$, the inverse of $\sinh x$ also involves $\log x$. Letting $y = \sinh^{-1} x$, we see that

$$x = \sinh y = \frac{e^y - e^{-y}}{2},$$

so that $e^y - e^{-y} - 2x = 0$. To solve for y, it is convenient to make the substitution $z = e^y$:

$$\begin{aligned} z - \frac{1}{z} - 2x &= 0 \\ \Rightarrow z^2 - 2xz - 1 &= 0. \end{aligned}$$

Thus

$$z = \frac{2x \pm \sqrt{(2x)^2 + 4}}{2},$$

so that $e^y = x \pm \sqrt{x^2 + 1}$. But since $e^y > 0$ for all $y \in \mathbb{R}$, only the positive square root is relevant. That is, for all real x,

$$\sinh^{-1} x = \log(x + \sqrt{x^2 + 1}).$$

Problem 3.28: Prove that the two solutions for $\cosh^{-1} x$ are given by $\log(x \pm \sqrt{x^2-1})$. Show directly that $\log(x+\sqrt{x^2-1}) = -\log(x-\sqrt{x^2-1})$.

Problem 3.29: Show that

$$\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

Problem 3.30: Show that

$$\frac{d}{dx}\sinh^{-1} x = \frac{d}{dx}\log\left(x+\sqrt{x^2+1}\right) = \frac{1}{\sqrt{x^2+1}}.$$

Also verify this result directly from the fact that

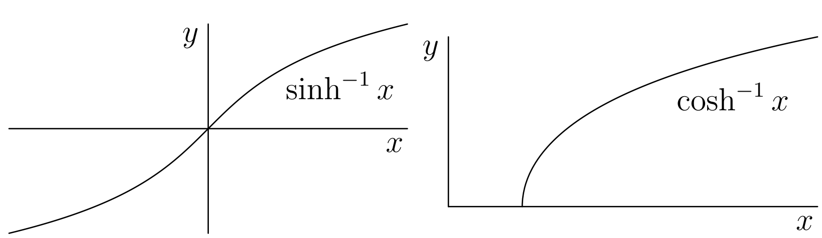
$$\frac{d}{dy}\sinh y = \cosh y.$$

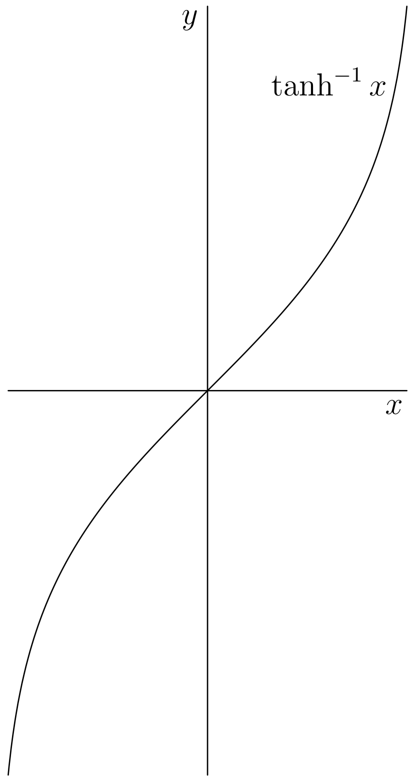
- Thus

$$\int_0^1 \frac{dx}{\sqrt{1+x^2}} = \left[\sinh^{-1} x\right]_0^1 = \left[\log(x+\sqrt{x^2+1})\right]_0^1 = \log\left(1+\sqrt{2}\right).$$

- To find $\frac{d}{dx}\cosh^{-1} x$, we can use the relation $\cosh^2 y - \sinh^2 y = 1$:

$$\begin{aligned}
y &= \cosh^{-1} x \\
\Rightarrow x &= \cosh y \\
\Rightarrow \frac{dx}{dy} &= \sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2-1} \\
\Rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{x^2-1}}.
\end{aligned}$$





Problem 3.31: Prove that

(a)
$$\cosh^2 t = \frac{\cosh 2t + 1}{2}$$

(b)
$$\sinh^2 t = \frac{\cosh 2t - 1}{2}$$

(c)
$$2 \sinh t \cosh t = \sinh 2t$$

Chapter 4

Applications of Differentiation

4.1 Maxima and Minima

Definition: f has a *global maximum* (*global minimum*) at c if

$$f(x) \le f(c) \quad (f(x) \ge f(c))$$

for all x in the domain of f. A global maximum or global minimum is sometimes called an *absolute maximum* or *absolute minimum*.

Definition: A function f has an *interior local maximum* (*interior local minimum*) at an interior point c of its domain if for some $\delta > 0$,

$$\begin{aligned} x \in (c-\delta, c+\delta) \Rightarrow f(x) &\le f(c) \\ (f(x) &\ge f(c)). \end{aligned}$$

Definition: An *extremum* is either a maximum or a minimum.

Remark: A global extremum is always a *local extremum* (but not necessarily an *interior local extremum*).

Remark: The following theorem guarantees that a continuous function always has a global maximum over a closed interval.

Theorem 4.1 (Extreme Value Theorem): *If f is continuous on $[a,b]$ then it achieves both a global maximum and minimum value on $[a,b]$. That is, there exists numbers c and d in $[a,b]$ such that*

$$f(c) \le f(x) \le f(d) \quad \textit{for all } x \in [a,b].$$

Theorem 4.2 (Fermat's Theorem): *Suppose*

(i) *f has an interior local extremum at c,*

(ii) *$f'(c)$ exists.*

Then $f'(c) = 0$.

Proof: Without loss of generality we can consider the case where f has an interior local maximum, i.e. there exists $\delta > 0$ such that

$$
\begin{aligned}
x \in (c-\delta, c+\delta) &\Rightarrow f(x) \le f(c) \\
&\Rightarrow \frac{f(x)-f(c)}{x-c} \begin{cases} \ge 0 & \text{if } x \in (c-\delta, c), \\ \le 0 & \text{if } x \in (c, c+\delta) \end{cases} \\
&\Rightarrow f'_L(c) \doteq \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \ge 0, \\
&\quad\; f'_R(c) \doteq \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \le 0 \\
&\quad \Rightarrow f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = 0.
\end{aligned}
$$

Remark: Theorem 4.2 establishes that the condition $f'(c) = 0$ is *necessary* for a differentiable function to have an interior local extremum. However, this condition alone is not *sufficient* to ensure that a differentiable function has an extremum at c; consider the behaviour of the function $f(x) = x^3$ near the point $c = 0$.

Remark: If a function is continuous on a closed interval, we know from Theorem 4.1 that it must achieve global maximum and minimum values somewhere in the interval. We know from Theorem 4.2 that if these extrema occur in the interior of the interval, the derivative of the function must either vanish there or else not exist. However, it is possible that the global maximum or minimum occurs at one of the endpoints of the interval; at these points, it is not at all necessary that the derivative vanish, even if it exists. It is also possible that an extremum occurs at a point where the derivative doesn't exist. For example, consider the fact that $f(x) = |x|$ has a minimum at $x = 0$.

Extrema can occur either at

(i) an end point,

(ii) a point where f' does not exist,

(iii) a point where $f' = 0$.

- Find the maxima and minima of

$$f(x) = 2x^3 - x^2 + 1 \text{ on } [0, 1].$$

Since f is continuous on $[0, 1]$ we know that it has a global maximum and minimum value on $[0, 1]$. Note that $f'(x) = 6x^2 - 2x = 2x(3x - 1) = 0$ in $(0, 1)$ only at the point $x = 1/3$. Theorem 4.2 implies that the only possible global interior extremum (which is of course also a local interior extremum) is at the point $x = 1/3$. By comparing the function values $f(1/3) = 26/27$ with the endpoint function values $f(0) = 1$ and $f(1) = 2$ we see that f has an (interior) global minimum value of $26/27$ at $x = 1/3$ and an (endpoint) global maximum value of 2 at $x = 1$. Hence $26/27 \leq f(x) \leq 2$ for all $x \in [0, 1]$.

4.2 The Mean Value Theorem

Theorem 4.3 (Rolle's Theorem): *Suppose*

(i) *f is continuous on $[a, b]$,*

(ii) *f' exists on (a, b),*

(iii) *$f(a) = f(b)$.*

Then there exists a number $c \in (a, b)$ for which $f'(c) = 0$.

Proof:

Case I: $f(x) = f(a) = f(b)$ for all $x \in [a, b]$ (i.e. f is constant on $[a, b]$)
$\Rightarrow f'(c) = 0$ for all $c \in (a, b)$.

Case II: $f(x_0) > f(a) = f(b)$ for some $x_0 \in (a, b)$. Theorem 4.1 $\Rightarrow f$ achieves its maximum value $f(c)$ for some $c \in [a, b]$. But

$$f(c) \geq f(x_0) > f(a) = f(b) \Rightarrow c \in (a, b).$$

$\therefore f$ has an interior local maximum at c.
Theorem 4.2 $\Rightarrow f'(c) = 0$.

Case III (Exercise): $f(x_0) < f(a) = f(b)$ for some $x_0 \in (a, b)$.

- $f(x) = x^3 - x + 1$.

  $f(0) = 1$, $f(1) = 1 \Rightarrow$ there exists $c \in (0, 1)$ such that $f'(c) = 0$.
  In this case we can actually find the point c. Since $f'(x) = 3x^2 - 1$, we can solve the equation $0 = f'(c) = 3c^2 - 1$ to deduce $c = \dfrac{1}{\sqrt{3}} \in (0, 1)$.

- Recall that $\sin n\pi = 0$, for all $n \in \mathbb{N}$. Rolle's Theorem tells us that $\frac{d}{dx} \sin x = \cos x$ must *vanish* (become zero) at some point $x \in (n\pi, (n+1)\pi)$. Indeed, we know that

$$\cos\left[\left(n + \frac{1}{2}\right)\pi\right] = \cos\left(\frac{2n+1}{2}\pi\right) = 0 \quad \text{for all } n \in \mathbb{N}.$$

- We can use Rolle's Theorem to show that the equation

$$f(x) = x^3 - 3x^2 + k = 0$$

  never has 2 distinct roots in $[0, 1]$, no matter what value we choose for the real number k. Suppose that there existed two numbers a and b in $[0, 1]$, with $a \neq b$ and $f(a) = f(b) = 0$. Then Rolle's Theorem $\Rightarrow$ there exists $c \in (a, b) \subset (0, 1)$ such that $f'(c) = 0$. But $f'(x) = 3x^2 - 6x = 3x(x - 2)$ has no roots in $(0, 1)$; this is a contradiction.

Q. What happens when the condition $f(a) = f(b)$ is dropped from Rolle's Theorem? Can we still deduce something similar?

A. Yes, the next theorem addresses precisely this situation.

Theorem 4.4 (Mean Value Theorem [MVT]): *Suppose*

(i) *f is continuous on $[a, b]$,*

(ii) *f' exists on (a, b).*

Then there exists a number $c \in (a, b)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remark: Notice that when $f(a) = f(b)$, the Mean Value Theorem reduces to Rolle's Theorem.

Proof: Consider the function

$$\varphi(x) = f(x) - M(x - a),$$

where M is a constant. Notice that $\varphi(a) = f(a)$. We choose M so that $\varphi(b) = f(a)$ as well:

$$M = \frac{f(b) - f(a)}{b - a}.$$

Then φ satisfies all three conditions of Rolle's Theorem:

(i) φ is continuous on $[a, b]$,

(ii) φ' exists on (a, b),

(iii) $\varphi(a) = \varphi(b)$.

Hence there exists $c \in (a, b)$ such that

$$0 = \varphi'(c) = f'(c) - M = f'(c) - \frac{f(a) - f(b)}{b - a}.$$

Q. We know that when $f(x)$ is constant that $f'(x) = 0$. Does the converse hold?

A. No, a function may have zero slope somewhere without being constant (e.g. $f(x) = x^2$ at $x = 0$). However, if $f'(x) = 0$ for all $x \in [a, b]$, where $a \neq b$, we may then make use of the following result.

Theorem 4.5 (Zero Derivative on an Interval): *Suppose $f'(x) = 0$ for every x in an interval I (of nonzero length). Then f is constant on I.*

Proof: Let x, y be any two elements of I, with $x < y$. Since f is differentiable at each point of I, we know by Theorem 3.1 that f is continuous on I. From the MVT, we see that

$$\frac{f(x) - f(y)}{x - y} = f'(c) = 0$$

for some $c \in (x, y) \subset I$. Hence $f(x) = f(y)$. Thus, f is constant on I.

Theorem 4.6 (Equal Derivatives): *Suppose $f'(x) = g'(x)$ for every x in an interval I (of nonzero length). Then $f(x) = g(x) + k$ for all $x \in I$, where k is a constant.*

Theorem 4.7 (Monotonic Test): *Suppose f is differentiable on an interval I. Then*

(i) *f is increasing on $I \iff f'(x) \geq 0$ on I;*

(ii) *f is decreasing on $I \iff f'(x) \leq 0$ on I.*

Proof:
"$\Rightarrow$" Without loss of generality let f be increasing on I. Then for each $x \in I$,

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \geq 0.$$

"$\Leftarrow$" Suppose $f' \geq 0$ on I. Let $x, y \in I$ with $x < y$. The MVT $\Rightarrow$ there exists $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) \geq 0$$
$$\Rightarrow f(y) - f(x) \geq 0.$$

Hence f is increasing on I.

Remark: Theorem 4.7 only provides sufficient, not necessary, conditions for a function to be increasing (since it might not be differentiable).

- Consider the function $f(x) = \lfloor x \rfloor$, which returns the greatest integer less than or equal to x. Note that f is increasing (on $\mathbb{R}$) but $f'(x)$ does not exist at integer values of x.

Q. If we replace "increasing" with "strictly increasing" in Theorem 4.7 (i), can we then change "$\geq$" to "$>$"?

A. No, consider the strictly increasing function $f(x) = x^3$. We can only say $f'(x) = 3x^2 \geq 0$ since $f'(0) = 0$.

Problem 4.1: Prove that if f is continuous on $[a, b]$ and $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on $[a, b]$.

4.3 First Derivative Test

We have seen that points where the derivative of a function vanishes may or may not be extrema. How do we decide which ones are extrema and, of those, which are maxima and which are minima? One answer is provided by the First Derivative Test.

Definition: A point where the derivative of f is zero or does not exist is called a *critical* point.

Theorem 4.2 $\Rightarrow$ Local interior maxima and minima occur at critical points.

Remark: Not all critical points are extrema: consider $f(x) = x^3$ at $x = 0$.

Q. How do we decide which critical points c correspond to maxima, to minima, or neither?

A. If f is differentiable near c, look at the first derivative.

Theorem 4.8 (First Derivative Test): *Let c be a critical point of a continuous function f. If*

(i) *$f'(x)$ changes from negative to positive at c, then f has a local minimum at c;*

(ii) *$f'(x)$ changes from positive to negative at c, then f has a local maximum at c;*

(iii) *$f'(x)$ is positive on both sides of c or negative on both sides of c then f does not have a local extremum at c.*

Proof: (i) This follows directly from the fact that f is then decreasing to the left of c and increasing to the right of c.
(ii)-(iii) Exercises.

Problem 4.2: Give examples of differentiable functions which have the behaviours described in each of the cases above.

4.4 Second Derivative Test

In cases where the second derivative of f can be easily computed, the following test provides simple conditions for classifying critical points.

Theorem 4.9 (Second Derivative Test): *Suppose f is twice differentiable at a critical point c (this implies $f'(c) = 0$). If*

(i) $f''(c) > 0$, *then f has a local minimum at c;*

(ii) $f''(c) < 0$, *then f has a local maximum at c.*

Proof:

(i) $f''(c) > 0 \Rightarrow \lim\limits_{x\to c} \dfrac{f'(x) - f'(c)}{x - c} > 0 \Rightarrow \lim\limits_{x\to c} \dfrac{f'(x)}{x - c} > 0$

$\Rightarrow$ there exists $\delta > 0$ such that $f'(x) \begin{cases} < 0 & \text{for all } x \in (c-\delta, c), \\ > 0 & \text{for all } x \in (c, c+\delta) \end{cases}$

$\Rightarrow f$ has a local minimum at c by the First Derivative Test.

(ii) Exercise.

Remark: If $f''(c) = 0$, then anything is possible.

- $f(x) = x^3$,
  $f'(x) = 3x^2 = 0$ at $x = 0$,
  $f''(x) = 6x = 0$ at $x = 0$,
  f has neither a maximum nor minimum at $x = 0$.

- $f(x) = x^4$,
  $f'(x) = 4x^3 = 0$ at $x = 0$,
  $f''(x) = 12x^2 = 0$ at $x = 0$,
  f has a minimum at $x = 0$.

- $f(x) = -x^4$ has a maximum at $x = 0$.

Remark: The First Derivative Test can sometimes be helpful in cases where the Second Derivative Test fails, e.g. in showing that $f(x) = x^4$ has a minimum at $x = 0$.

Remark: The Second Derivative Test establishes only the local behaviour of a function, whereas the First Derivative Test can sometimes be used to establish that an extremum is global:

$$f(x) = x^2, \qquad f'(x) = 2x \begin{cases} < 0 & \text{for all } x < 0, \\ > 0 & \text{for all } x > 0. \end{cases}$$

Since f is decreasing for $x < 0$ and increasing for $x > 0$, we see that f has a global minimum at $x = 0$.

4.5 Convex and Concave Functions

Definition: A function is *convex* (sometimes called *concave up*) on an interval I if the secant line segment joining $(a, f(a))$ and $(b, f(b))$ lies on or above the graph of f for all $a, b \in I$.

Definition: A function f is *concave* (sometimes called *concave down*) on an interval I if $-f$ is convex on I.

Definition: An *inflection point* is a point on the graph of a function f at which the behaviour of f changes from convex to concave. For example, since $f(x) = x^3$ is concave on $(-\infty, 0]$ and convex on $[0, \infty)$, the point $(0, 0)$ is an inflection point.

Remark: Since the equation of the line through $(a, f(a))$ and $(b, f(b))$ is

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a),$$

the definition of convex says

$$f(x) \leq f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \qquad \text{for all } x \in [a, b], \quad \text{for all } a, b \in I. \tag{4.1}$$

The *linear interpolation* of f between $[a, b]$ on the right-hand side of Eq. (4.1) may be rewritten as:

$$f(x) \leq \left(\frac{b - x}{b - a}\right) f(a) + \left(\frac{x - a}{b - a}\right) f(b) \qquad \text{for all } x \in [a, b], \quad \text{for all } a, b \in I \tag{4.2}$$

or as

$$f(x) \leq f(b) + \frac{f(b) - f(a)}{b - a}(x - b) \qquad \text{for all } x \in [a, b], \quad \text{for all } a, b \in I. \tag{4.3}$$

The convexity condition may also be expressed directly in terms of the slope of a secant:

$$\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a} \le \frac{f(b)-f(x)}{b-x} \qquad \text{for all } x\in(a,b), \quad \text{for all } a,b\in I. \tag{4.4}$$

The left-hand inequality follows directly from Eq. (4.1) and the right-hand inequality follows from Eq. (4.3).

Theorem 4.10 (First Convexity Test): *Suppose f is differentiable on an interval I. Then*

(i) *f is convex $\iff$ f' is increasing on I;*

(ii) *f is concave $\iff$ f' is decreasing on I.*

Proof: Without loss of generality we only need to consider the case where f is convex.

"$\Rightarrow$" Suppose f is convex. Let $a,b\in I$, with $a<b$, and define

$$m(x)=\frac{f(x)-f(a)}{x-a} \quad (x\neq a), \qquad M(x)=\frac{f(b)-f(x)}{b-x} \quad (x\neq b).$$

From Eq. (4.4) we know that

$$m(x)\le m(b)=M(a)\le M(x)$$

whenever $a<x<b$. Hence

$$f'(a)=\lim_{x\to a} m(x)=\lim_{x\to a^+} m(x)\le m(b)=M(a)\le \lim_{x\to b^-} M(x)=\lim_{x\to b} M(x)=f'(b).$$

Thus f' is increasing on I.

"$\Leftarrow$" Suppose f' is increasing on I. Let $a,b\in I$, with $a<b$ and $x\in(a,b)$. By the MVT,

$$\frac{f(x)-f(a)}{x-a}=f'(c_1), \qquad \frac{f(b)-f(x)}{b-x}=f'(c_2)$$

for some $c_1\in(a,x)$ and $c_2\in(x,b)$. Since f' is increasing and $c_1<c_2$, we know that $f'(c_1)\le f'(c_2)$. Hence

$$\frac{f(x)-f(a)}{x-a}\le\frac{f(b)-f(x)}{b-x}$$

$$\Rightarrow f(x)\left[\frac{1}{x-a}+\frac{1}{b-x}\right]\le\frac{f(b)}{b-x}+\frac{f(a)}{x-a}=\frac{f(b)(x-a)+f(a)(b-x)}{(b-x)(x-a)},$$

which reduces to Eq. (4.2), so f is convex.

Theorem 4.11 (Second Convexity Test): *Suppose f'' exists on an interval I. Then*

(i) *f is convex on $I \iff f''(x) \geq 0 \quad$ for all $x \in I$;*

(ii) *f is concave on $I \iff f''(x) \leq 0 \quad$ for all $x \in I$.*

Proof: Apply Theorem 4.7 to f'.

Theorem 4.12 (Tangent to a Convex Function): *If f is convex and differentiable on an interval I, the graph of f lies above the tangent line to the graph of f at every point of I.*

Proof: Let $a \in I$. The equation of the tangent line to the graph of f at the point $(a, f(a))$ is $y = f(a) + f'(a)(x-a)$. Given $x \in I$, the MVT implies that $f(x) - f(a) = f'(c)(x-a)$, for some c between a and x. Since f is convex on I, we also know, from Theorem 4.10, that f' is increasing on I:

$$x < a \Rightarrow c < a \Rightarrow f'(c) \leq f'(a),$$

$$x > a \Rightarrow c > a \Rightarrow f'(c) \geq f'(a).$$

In either case $f(x) - f(a) = f'(c)(x-a) \geq f'(a)(x-a)$. Hence

$$f(x) \geq f(a) + f'(a)(x-a) \qquad \text{for all } x \in I.$$

- Consider $f(x) = \dfrac{1}{1+x^2}$ on $\mathbb{R}$.

  Observe that $f(0) = 1$ and $f(x) > 0$ for all $x \in \mathbb{R}$ and $\lim\limits_{x\to\pm\infty} f(x) = 0$. Note that f is even: $f(-x) = f(x)$. Also, $1 + x^2 \geq 1 \Rightarrow f(x) \leq 1 = f(0)$, so f has a maximum at $x = 0$. Alternatively, we can use either the First Derivative Test or the Second Derivative Test to establish this. We find

$$f'(x) = -\frac{2x}{(1+x^2)^2}$$

  and,

$$f''(x) = \frac{-2}{(1+x^2)^2} + \frac{2(2x)2x}{(1+x^2)^3} = \frac{-2 - 2x^2 + 8x^2}{(1+x^2)^3} = \frac{2(3x^2-1)}{(1+x^2)^3}.$$

  First Derivative Test: $\begin{cases} f'(x) > 0 \text{ on } (-\infty, 0) \Rightarrow f \text{ is increasing on } (-\infty, 0), \\ f'(x) < 0 \text{ on } (0, \infty) \Rightarrow f \text{ is decreasing on } (0, \infty) \end{cases}$
  $\Rightarrow f$ has a maximum at 0.

  Second Derivative Test: $f'(0) = 0, f''(0) = -2 < 0 \Rightarrow f$ has a maximum at 0.

$$\text{Convexity: } \begin{cases} f''(x) \geq 0 \text{ for } |x| \geq \dfrac{1}{\sqrt{3}}, \text{ i.e. } f \text{ is convex on } \left(-\infty, -\dfrac{1}{\sqrt{3}}\right] \cup \left[\dfrac{1}{\sqrt{3}}, \infty\right), \\[2ex] f''(x) \leq 0 \text{ for } |x| \leq \dfrac{1}{\sqrt{3}}, \text{ i.e. } f \text{ is concave on } \left[-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right], \\[2ex] f''(x) = 0 \text{ at } \pm\dfrac{1}{\sqrt{3}}; \text{ these correspond to inflection points.} \end{cases}$$

Problem 4.3: Consider the function $f(x) = (x+1)x^{2/3}$ on $[-1, 1]$.

(a) Find $f'(x)$.

On rewriting $f(x) = x^{5/3} + x^{2/3}$, we find

$$f'(x) = \frac{5}{3}x^{2/3} + \frac{2}{3}x^{-1/3} = \frac{x^{-1/3}}{3}(5x+2) \quad (x \neq 0).$$

(b) Determine on which intervals f is increasing and on which intervals f is decreasing.

Since

$$f'(x) \begin{cases} > 0, & -1 \leq x < -2/5, \\ = 0, & x = -2/5, \\ < 0, & -2/5 < x < 0, \\ \text{does not exist} & x = 0, \\ > 0, & 0 < x \leq 1, \end{cases}$$

we know that f is increasing on $[-1, -2/5]$ and $[0, 1]$. It is decreasing on $[-2/5, 0]$.

(c) Does f have any interior local extrema on $[-1, 1]$? If so, where do these occur? Which are maxima and which are minima?

Note that f has two critical points: $x = -2/5$ and $x = 0$. By the First Derivative Test, f has a local maximum at $x = -2/5$ and a local minimum at $x = 0$.

(d) What are the global minimum and maximum values of f and at what points do these occur?

On comparing the endpoint function values to the function values at the critical points, we conclude that f achieves its global minimum value of 0 at $x = -1$ and at $x = 0$. It has a global maximum value of 2 at $x = 1$.

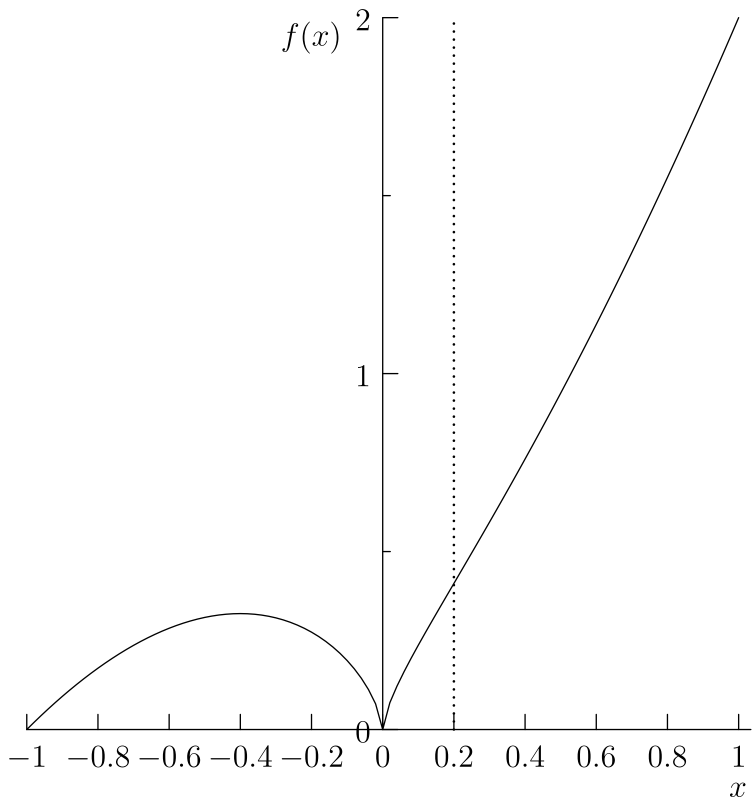
(e) Determine on which intervals f is convex and on which intervals f is concave.

Since $f''(x) = \frac{10}{9}x^{-1/3} - \frac{2}{9}x^{-4/3} = \frac{2}{9}x^{-4/3}(5x-1)$, we see that

$$f''(x) \begin{cases} < 0, & -1 \leq x < 0, \\ \text{does not exist}, & x = 0, \\ < 0, & 0 < x < 1/5, \\ = 0, & x = 1/5, \\ > 0, & 1/5 < x \leq 1. \end{cases}$$

Thus, f is concave on $[-1, 0]$ and $[0, 1/5]$ and convex on $[1/5, 1]$. Note that f is not concave on the interval $[-1, 1/5]$.

(f) Does f have any inflection points? If so, where?
Yes: f has an inflection point at $x = 1/5$.
(g) Sketch a graph of f using the above information.



4.6 L'Hôpital's Rule

Theorem 4.13 (Cauchy Mean Value Theorem): *Suppose*

(i) *f and g are continuous on $[a, b]$,*

(ii) *f' and g' exist on (a, b).*

Then there exists a number $c \in (a, b)$ for which

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

Proof: Consider

$$\phi(x) = [f(x) - f(a)][g(b) - g(a)] - [f(b) - f(a)][g(x) - g(a)].$$

Note that ϕ is continuous on $[a, b]$ and differentiable on (a, b). Since $\phi(a) = \phi(b) = 0$, we know from Rolle's Theorem that $\phi'(c) = 0$ for some $c \in (a, b)$; from this we immediately deduce the desired result.

Theorem 4.14 (L'Hôpital's Rule for $\frac{0}{0}$): *Suppose f and g are differentiable on (a,b), $g'(x) \neq 0$ for all $x \in (a,b)$, $\lim\limits_{x\to b^-} f(x) = 0$, and $\lim\limits_{x\to b^-} g(x) = 0$. Then*

$$\lim_{x\to b^-} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x\to b^-} \frac{f(x)}{g(x)} = L.$$

This result also holds if

(i) $\lim\limits_{x\to b^-}$ *is replaced by* $\lim\limits_{x\to a^+}$*;*

(ii) $\lim\limits_{x\to b^-}$ *is replaced by* $\lim\limits_{x\to \infty}$ *and b is replaced by ∞;*

(iii) $\lim\limits_{x\to b^-}$ *is replaced by* $\lim\limits_{x\to -\infty}$ *and a is replaced by $-\infty$.*

Proof: Theorem 3.1 $\Rightarrow$ f and g are continuous on (a,b). Consider

$$F(x) = \begin{cases} f(x) & a < x < b, \\ 0 & x = b. \end{cases}$$

$$G(x) = \begin{cases} g(x) & a < x < b, \\ 0 & x = b. \end{cases}$$

Since $\lim_{x\to b^-} f(x) = 0$, and $\lim_{x\to b^-} g(x) = 0$, we know for any $x \in (a,b)$ that F and G are continuous on $[x,b]$ and differentiable on (x,b). We can also be sure that G is nonzero on (a,b): if $G(x) = 0 = G(b)$ for some $x \in (a,b)$, Rolle's Theorem would imply that G', and hence g', vanishes somewhere in (x,b).

Given $\epsilon > 0$, we know there exists a number δ with $0 < \delta < b - a$ such that

$$x \in (b-\delta, b) \Rightarrow \left| \frac{f'(x)}{g'(x)} - L \right| < \epsilon.$$

If $x \in (b-\delta, b)$, Theorem 4.13 then implies that there exists a point $c \in (x,b)$ such that

$$\frac{f(x)}{g(x)} = \frac{F(x)}{G(x)} = \frac{F(x)-F(b)}{G(x)-G(b)} = \frac{F'(c)}{G'(c)} = \frac{f'(c)}{g'(c)},$$

so that

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(c)}{g'(c)} - L \right| < \epsilon.$$

That is, $\lim\limits_{x\to b^-} \dfrac{f(x)}{g(x)} = L$.

- Using L'Hôpital's Rule, we find

$$\lim_{x\to 0} \frac{\tan x}{x} = 1 \Leftarrow \lim_{x\to 0} \frac{\sec^2 x}{1} = 1,$$

- $$\lim_{x\to 1}\frac{x^n-1}{x-1} = n \Leftarrow \lim_{x\to 1}\frac{nx^{n-1}}{1} = n.$$

Remark: L'Hôpital's Rule should only be used where it applies. For example, it should not be used for when the limit does not have the $\frac{0}{0}$ form. For example, $0 = \lim\limits_{x\to 1}\dfrac{x-1}{x} \neq \lim\limits_{x\to 1}\dfrac{1}{1} = 1$.

Theorem 4.15 (L'Hôpital's Rule for $\frac{\infty}{\infty}$): *Suppose f and g are differentiable on (a,b), $g'(x) \neq 0$ for all $x \in (a,b)$, and $\lim\limits_{x\to b^-} f(x) = \infty$, and $\lim\limits_{x\to b^-} g(x) = \infty$. Then*

$$\lim_{x\to b^-}\frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x\to b^-}\frac{f(x)}{g(x)} = L.$$

This result also holds if

(i) $\lim\limits_{x\to b^-}$ *is replaced by* $\lim\limits_{x\to a^+}$*;*

(ii) $\lim\limits_{x\to b^-}$ *is replaced by* $\lim\limits_{x\to\infty}$ *and b is replaced by ∞;*

(iii) $\lim\limits_{x\to b^-}$ *is replaced by* $\lim\limits_{x\to-\infty}$ *and a is replaced by $-\infty$.*

Proof: We only need to make minor modifications to the proof used to establish Theorem 4.14. Choose δ such that $f(x) > 0$ and $g(x) > 0$ on $(b-\delta, b)$ and redefine

$$F(x) = \begin{cases} \frac{1}{f(x)} & b-\delta < x < b, \\ 0 & x = b, \end{cases} \qquad G(x) = \begin{cases} \frac{1}{g(x)} & b-\delta < x < b, \\ 0 & x = b. \end{cases}$$

Problem 4.4: Determine which of the following limits exist as a finite number, which are ∞, which are $-\infty$, and which do not exist at all. Where possible, compute the limit.

(a)

$$\lim_{x\to 1}\frac{\log x}{x-1}$$

$$= \lim_{x\to 1}\frac{1/x}{1} = 1.$$

(b)

$$\lim_{x\to 1} \frac{e^x - 1 - x}{x^2}$$

$$= \lim_{x\to 1} \frac{e^x - 1}{2x} = \lim_{x\to 1} \frac{e^x}{2} = \frac{1}{2}.$$

(c)

$$\lim_{x\to\infty} \frac{e^x}{x^2}$$

$$= \lim_{x\to 1} \frac{e^x}{2x} = \lim_{x\to 1} \frac{e^x}{2} = \infty.$$

(d)

$$\lim_{x\to 0} x^x$$

$$= \lim_{x\to 0} e^{x \log x} = e^{\lim_{x\to 0} x \log x} = e^{\lim_{x\to 0}} \frac{\log x}{1/x} = e^{\lim_{x\to 0}} \frac{1/x}{-1/x^2} = e^{\lim_{x\to 0}} - x = e^0 = 1.$$

(e)

$$\lim_{x\to 1} \frac{\sin(x^{99}) - \sin(1)}{x - 1}$$

One could use L'Hôpital's Rule here, but it is even simpler to note that this is just the definition of the derivative of the function $f(x) = \sin(x^{99})$ at $x = 1$. Since

$$f'(x) = \cos(x^{99}) 99 x^{98},$$

the limit reduces to $f'(1) = 99\cos(1)$.

(f)

$$\lim_{x\to\pi/4} \frac{\tan x - 1}{x - \pi/4}$$

Letting $f(x) = \tan x$, we see that this is just the definition of $f'(\pi/4) = \sec^2(\pi/4) = 2$.

(g)

$$\lim_{x\to 0} \frac{\tan x - x}{x^3}$$

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^3 x \sin x}{6x} = \lim_{x\to 0} \frac{-6\sec^2 x \sin^2 x + 2\sec^2 x \cos x}{6} = \frac{1}{3},$$

on applying the 0/0 form of L'Hôpital's Rule three times. Alternatively, after the second application of L'Hôpital's Rule, one can use the fact that $\lim_{x\to 0} \sin x/x = 1$.

4.7 Slant Asymptotes

Definition: A function f is asymptotic to a *slant asymptote* $y = mx + b$ if

$$\lim_{x\to\infty} [f(x) - (mx + b)] = 0.$$

or

$$\lim_{x\to-\infty} [f(x) - (mx + b)] = 0$$

This means that the vertical distance between the function f and the line $y = mx+b$ approaches zero in the limit as $x \to \infty$ or $x \to -\infty$, respectively.

Remark: If

$$\lim_{x\to\infty} [f(x) - (mx + b)] = 0,$$

then

$$\lim_{x\to\infty} [\frac{f(x) - (mx + b)}{x}] = 0.$$

We also know that $\lim_{x\to\infty} \frac{b}{x} = 0$. On adding these equations we find that

$$\lim_{x\to\infty} [\frac{f(x)}{x} - m] = 0.$$

Hence

$$m = \lim_{x\to\infty} \frac{f(x)}{x}.$$

Once we know m then we can find

$$b = \lim_{x\to\infty} [f(x) - mx].$$

- The function $f(x) = x^3/(x^2+1)$, has no vertical asymptotes (since the denominator is never zero) and no horizontal asymptotes since $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. However, it does have a slant asymptote $y = mx + b$ where

$$m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2}{(x^2 + 1)} = 1$$

and

$$b = \lim_{x\to\infty} \frac{x^3}{(x^2 + 1) - x} = \lim_{x\to\infty} \frac{x^3 - x(x^2 + 1)}{(x^2 + 1)} = \lim_{x\to\infty} \frac{x}{(x^2 + 1)} = 0.$$

So f has a slant asymptote of $y = x$ as $x \to \infty$. Note that f also has a slant asymptote of $y = x$ as $x \to -\infty$.

Problem 4.5: Show that the function $f(x) = x + \frac{\sin x}{x}$ has a slant asymptote of $y = x$ as $x \to \pm\infty$.

4.8 Optimization Problems

- Determine the rectangle having the largest area that can be inscribed inside a right-angle triangle of side lengths a, b, and $\sqrt{a^2+b^2}$, if the sides of the rectangle are constrained to be parallel to the sides of length a and b.

  Let the vertices of the triangle be $(0,0)$, $(a,0)$, $(0,b)$ and those of the rectangle be $(0,0)$, $(0,x)$, (x,y), $(0,y)$, where $0 \le x \le a$. By similar triangles we see that

$$\frac{y}{a-x} = \frac{b}{a}.$$

  The area A of the rectangle is given by

$$A(x) = xy = \frac{b}{a}x(a-x) = bx - \frac{b}{a}x^2,$$

  so that

$$A'(x) = b - \frac{2b}{a}x = \frac{b}{a}(a-2x).$$

  Since A is continuous on the closed interval $[0,a]$ we know that A must achieve maximum and minimum values in $[0,a]$. Since $A'(x)$ exists everywhere in $(0,a)$, the only points we need to check are $x = a/2$, where $A'(x) = 0$, and the endpoints $x = 0$ and $x = a$; at least one of these must represent a maximum area and one must represent a minimum area. Since $A(a/2) = ab/4$ and $A(0) = A(a) = 0$ we see that the maximum area is $ab/4$ and the minimum area is 0. Thus, the largest rectangle that can be inscribed has side lengths $a/2$ and $b/2$.

Problem 4.6: A canoeist is at the southwest corner of a square lake of side 1 km. She would like to travel to the northeast corner of the lake by rowing to a point on the north shore at a speed of 3 km/h in a straight line at an angle θ measured relative to north. She then plans to walk east along the north shore at a speed of 6 km/h until she arrives at her destination.

At what angle θ should the canoeist row in order to arrive at her destination in the shortest possible time? What is this minimum time? Prove that your answer corresponds to a minimum.

The time to reach her destination is

$$T(\theta) = \frac{1}{3\cos\theta} + \frac{1-\tan\theta}{6}.$$

The continuous function T must achieve global minimum and maximum values on $[0, \frac{\pi}{4}]$. First, we look for critical points of this function on $(0, \frac{\pi}{4})$:

$$0 = T'(\theta) = \frac{\sin\theta}{3\cos^2\theta} - \frac{1}{6\cos^2\theta} \Rightarrow \sin\theta = \frac{1}{2}.$$

The only critical point in $(0, \frac{\pi}{4})$ is at $\theta = \frac{\pi}{6}$. By simply comparing values, we see that the endpoint value $T(0) = 1/2$ is an exterior global maximum, the endpoint value $T(\frac{\pi}{4}) = \sqrt{2}/3$ is an endpoint local maxima, and $T(\frac{\pi}{6}) = \frac{1+\sqrt{3}}{6}$ is the global minimum value. Thus the canoeist should row at an angle $\frac{\pi}{6}$ relative to north.

Problem 4.7: Maximize the total surface area (including the top and bottom) of a can with volume 1000cm^3 and the shape of a circular cylinder.

The total surface area of cylinderical can of radius r and height h is given by $2\pi r^2+2\pi rh$, where the volume is constrained to be $\pi r^2 h = 1000$. We can use the volume constraint to eliminate one variable, say h, from the problem:

$$h = \frac{1000}{\pi r^2},$$

allowing us to express the area A solely as a function of r:

$$A(r) = 2\pi r^2 + \frac{2000}{r}.$$

We need to maximize $A(r)$ on the interval $(0, \infty)$. On noting that $A'(r) = 4\pi r - 2000/r^2 = 4r(\pi - 500/r^3)$, we see that the only critical point of A occurs at $c = \sqrt[3]{\frac{500}{\pi}}$. Since $(0, \infty)$ is not a closed interval, we cannot use the Exteme Value Theorem. Instead, we note that the first derivative of A is negative for $r < c$ and positive for c. This means that A is decreasing on $(0, c]$ and increasing on $[c, \infty)$. Thus A has a global minimum at $r = c$.

4.9 Newton's Method

In 1823, Abel proved that no general algebraic solution (involving only arithmetic operations and radicals) exists for finding roots of fifth-degree polynomials. This result was generalized by Galois to all degrees above four.

If we need to find the roots to a polynomial of degree five or higher, or to a transcendental (non-algebraic) equation like

$$\cos x - x = 0,$$

then we must resort to a numerical method.

Newton's method (also called the *Newton-Raphson* method for finding a root to a function $f(x)$ with a continuous derivative begins with an initial guess x_1. The equation for the tangent line to the graph of f at $(x_1, f(x_1)$ is

$$y - f(x_1) = f'(x_1)(x - x_1).$$

This line intersects the x axis when $y = 0$, at a point $(x_2, 0)$:

$$0 - f(x_1) = f'(x_1)(x_2 - x_1).$$

On solving for x_2 we find

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Newton's method doesn't always work, but when it does, the point x_2 will be closer to the root of f than x_1. On repeating this procedure using x_2 as initial guess, we obtain our next guess x_3 for the root:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

Inductively, we define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If $\lim_{n\to\infty} x_n$ exists and equals c then we see that

$$0 = \lim_{n\to\infty} x_{n+1} - \lim_{n\to\infty} x_n + \lim_{n\to\infty} \frac{f(x_n)}{f'(x_n)} = c - c + \lim_{n\to\infty} \frac{f(x_n)}{\lim_{n\to\infty} f'(x_n)} = \frac{f(c)}{f'(c)}$$

since f and f' are continuous. Thus $f(c) = 0$ so we have determined that c is a root of f $f(c) = 0$.

- To find a root of $f(x) = \cos x - x$, first compute $f'(x) = -\sin x - 1$. The Newton iteration appears as

$$x_{n+1} = x_n + \frac{\cos x - x}{\sin x + 1}.$$

Starting with an initial guess $x_1 = 1$, we find

$$x_2 = 0.750363867840,$$

$$x_3 = 0.739112890911,$$

$$x_4 = 0.739085133385,$$

$$x_5 = 0.739085133215,$$

$$x_6 = 0.739085133215,$$

from which it appears that there is a root of f very close to $x = 0.739085133215$. Indeed we verify that $\cos x - x$ is about 2.7×10^{-13}.

Chapter 5

Integration

5.1 Areas

Suppose, given a function $f(x) \geq 0$ on $[a, b]$ that we wish to determine the area of the region bounded by the graph of $f(x)$, the x axis, and the lines $x = a$ and $x = b$. That is, we want to find the area of the region

$$\mathcal{S} = \{(x, y) : a \leq x \leq b,\ 0 \leq y \leq f(x)\}.$$

We could approximate the area as a sum of areas of rectangles as in Figure 5.1 or as in Figure 5.2, determining the height of each rectangle by the function value at the left or right endpoint of each subinterval, respectively.

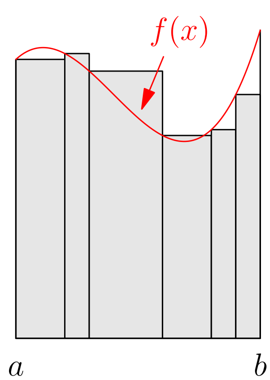


Figure 5.1: Left-endpoint approximation.

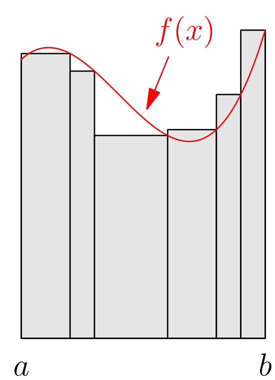


Figure 5.2: Right-endpoint approximation.

Definition: Let f be a function on $[a, b]$. Divide the interval $[a, b]$ into subintervals $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots n$. A *Riemann sum* of f is any sum of the form

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}),$$

where the *sample points* x_i^* are arbitrarily chosen from $[x_{i-1}, x_i]$. Here $f(x_i^*)$ refers to the height of a rectangle of width $x_i - x_{i-1}$ that approximates the contribution to the area coming from the ith subinterval $[x_{i-1}, x_i]$. On summing up these contributions from all n subintervals, we obtain an approximation to the total area under the function f from $x = a$ to $x = b$.

Definition: The *left Riemann sum*

$$\sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

is obtained by choosing $x_i^* = x_{i-1}$.

Definition: The *right Riemann sum*

$$\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

is obtained by choosing $x_i^* = x_i$.

Definition: A common choice for the subinterval endpoints is $x_i = a + i(b - a)/n$, which generates subintervals with a uniform width:

$$x_i - x_{i-1} = \frac{b - a}{n}.$$

- The right Riemann sum corresponding to $f(x) = x$ on $[0, 1]$ partitioned into n uniform subintervals $[x_{i-1}, x_i]$, where $x_i = i/n$, is

$$\sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{n+1}{2n}$$

since the subinterval widths $x_i - x_{i-1} = \dfrac{1 - 0}{n} = \dfrac{1}{n}$.

We now use the concept of a Riemann sum to obtain a precise definition for the notion of the *area* under a function.

Definition: The area of the region bounded by the graph of a positive function $f(x)$, the x axis, and the lines $x = a$ and $x = b$ is given by the *Riemann Integral*

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}),$$

whenever this limit exists and is independent of the choice of sample points x_i^* and subintervals $[x_{i-1}, x_i]$, provided the subinterval widths $x_i - x_{i-1}$ approach zero as $n \to \infty$. In this case we say that f is *integrable* on $[a, b]$ and define the *definite integral* of f on $[a, b]$ as

$$\int_a^b f = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}).$$

The next theorem tells us that the Riemann integral of any continuous function on a bounded interval $[a, b]$ always exists.

Theorem 5.1 (Integrability of Continuous Functions): *If f is continuous on $[a, b]$ then $\int_a^b f$ exists.*

Remark: By the definition of integrability, all uniform Riemann sums of a continuous function f on $[a, b]$ will converge to $\int_a^b f$ as the width of the subintervals approaches zero.

- The definite integral of the continous function $f(x) = x$ on $[0, 1]$ can be computed as the limit of its right Riemann sum:

$$\int_a^b f = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

  Notice that this result agrees with the area of a right triangle with two sides of length one.

- The definite integral of the continous function $f(x) = x$ on $[0, 1]$ can also be computed as the limit of its left Riemann sum:

$$\int_a^b f = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i-1}{n} \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} (i-1) = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=0}^{n-1} i$$
$$= \lim_{n\to\infty} \frac{1}{n^2} \left(0 + \sum_{i=1}^{n-1} i\right) = \lim_{n\to\infty} \frac{1}{n^2} \left(\frac{(n-1)n}{2}\right) = \lim_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2}.$$

  This result agrees with the definite integral computed using the right Riemann sum.

Definition: If $a \leq b$, define $\int_b^a f = -\int_a^b f$.

Remark: This implies that $\int_a^a f = 0$.

Theorem 5.2 (Piecewise Integration): *Let c be a real number. If $\int_a^c f$ and $\int_c^b f$ exists then $\int_a^b f$ exists and equals $\int_a^c f + \int_c^b f$.*

Remark: Using piecewise integration, we can integrate any function with a finite number of jump discontinuities over a closed interval.

Theorem 5.3 (Linearity of Integral Operator): *Suppose $\int_a^b f$ and $\int_a^b g$ exist. Then*

(i) $\int_a^b (f+g)$ exists and equals $\int_a^b f + \int_a^b g$,

(ii) $\int_a^b (cf)$ exists and equals $c\int_a^b f$ for any constant $c \in \mathbb{R}$.

Theorem 5.4 (Integral Bounds): *Suppose for $a < b$ that*

(i) $\int_a^b f$ exists,

(ii) $m \leq f(x) \leq M$ for $x \in [a,b]$.

Then

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

Theorem 5.5 (Preservation of Non-Negativity): *If $f(x) \geq 0$ for all $x \in [a,b]$, where $a < b$, and $\int_a^b f$ exists then $\int_a^b f \geq 0$.*

Proof: Set $m = 0$ in Theorem 5.4.

Theorem 5.6 (Absolute Integral Bounds): *If $|f(x)| \leq M$ for all x between a and b and $\int_a^b f$ exists then $\left|\int_a^b f\right| \leq M\,|b-a|$.*

Proof: Set $m = -M$ in Theorem 5.4.

Theorem 5.7 (Triangle Inequality for Integrals): *Let f be an integrable function on $[a,b]$. Then*

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Proof: Consider the integrable functions $f(x)$, $-|f(x)|$ and $|f(x)|$. For every $x \in [a,b]$,

$$-|f(x)| \leq f(x) \leq |f(x)|.$$

Thus

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|.$$

This means that $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Remark: When we write $\int f$ we understand that f is a function of some variable. Let's call it x. We want to partition a portion of the x axis into $\{x_0, x_1, \ldots, x_n\}$ and compute Riemann sums based on function values $f(x_i^*)$ and interval widths $x_i - x_{i-1}$. Similarly, when we write f', it is clear that we mean the derivative of f with respect to its argument, whatever that may be. However, if we want to differentiate the function $y = f(u)$, where $u = x^2$, it is important to know whether we are differentiating with respect to u or with respect to x. Likewise, suppose we wish to calculate the integral of f. It is equally important to know whether we are calculating the integral with respect to u or with respect to x, because the area under the graph of $y = f(u)$ with respect to u will in general differ from the area under the graph of $y = f(x^2)$ with respect to x. Since we can differentiate with respect to different variables, it is only reasonable that we should be able to integrate with respect to different variables as well. It will often be helpful to indicate explicitly with respect to which variable we are integrating, that is, which variable do we use to construct the differences $x_i - x_{i-1}$ in the Riemann sums.

Definition: We can specify the integration variable by writing $\int_0^1 f(x)\,dx$ instead of just $\int_0^1 f$. The notation $f(x)\,dx$ reminds us that the Riemann sums consists of function values multiplied by interval widths, $x_i - x_{i-1}$.

5.2 Fundamental Theorem of Calculus

Definition: A differentiable function F is called an *antiderivative* of f at an interior point x of its domain if $F'(x) = f(x)$.

Remark: If $F(x)$ is an antiderivative of f, then so is $F(x) + C$ for any constant C.

Theorem 5.8 (Families of Antiderivatives): *Let $F_0(x)$ be an antiderivative of f on an interval I. Then F is an antiderivative of f on I $\iff$ $F(x) = F_0(x) + C$ for some constant C.*

Proof:

"$\Leftarrow$" Let $F(x) = F_0(x) + C$. Then $F'(x) = F_0'(x) = f(x)$; that is, F is an antiderivative of f on I.

"$\Rightarrow$" Since

$$\frac{d}{dx}[F(x) - F_0(x)] = F'(x) - F_0'(x) = f(x) - f(x) = 0,$$

we see by Theorem 4.5 that $F(x) - F_0(x)$ is constant on I.

Theorem 5.9 (Antiderivatives at Points of Continuity): *Suppose*

(i) $\int_a^b f$ *exists;*

(ii) f *is continuous at* $c \in (a, b)$.

Then f *has the antiderivative* $F(x) = \int_a^x f$ *at* $x = c$.

Proof: Given $\epsilon > 0$, we know from the continuity of f at c that there exists a $\delta > 0$ such that

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

We use this bound in Theorem 5.6 to conclude for $|h| < \delta$ that

$$\left| \int_c^{c+h} [f(x) - f(c)]\, dx \right| \leq \epsilon \, |c + h - c| = \epsilon \, |h| \, .$$

Consider $F(x) = \int_a^x f$ for $x \in [a, b]$. Then for $0 < |h| < \delta$ we see that

$$\begin{aligned} \left| \frac{F(c+h) - F(c)}{h} - f(c) \right| &= \frac{1}{|h|} \left| \int_a^{c+h} f(x)\, dx - \int_a^c f(x)\, dx - f(c)h \right| \\ &= \frac{1}{|h|} \left| \int_c^{c+h} f(x)\, dx - f(c) \int_c^{c+h} 1\, dx \right| \\ &= \frac{1}{|h|} \left| \int_c^{c+h} [f(x) - f(c)]\, dx \right| \leq \epsilon. \end{aligned}$$

But this is just the statement that the limit

$$F'(c) = \lim_{h \to 0} \frac{F(c+h) - F(c)}{h}$$

exists and equals $f(c)$.

Remark: In particular, Theorem 5.9 says that, at any point $x \in (a, b)$ where an integrable function f is continuous,

$$\frac{d}{dx} \int_a^x f = f(x).$$

Thus we see that differentiation and integration are in a sense opposite processes. The actual situation is slightly complicated by the fact that antiderivatives are not unique, as we saw in Theorem 5.8. However, note that the arbitrary constant C in Theorem 5.8 disappears upon differentiation of the antiderivative.

Theorem 5.10 (Antiderivative of Continuous Functions): *If* f *is continuous on* $[a, b]$ *then* f *has an antiderivative on* $[a, b]$.

Proof: The antiderivative of f on $[a,b]$ is just the antiderivative $\int_a^x \bar{f}$ of the *continuous extension* $\bar{f}$ of f onto all of $\mathbb{R}$:

$$\bar{f}(x) = \begin{cases} f(a) & \text{if } x < a, \\ f(x) & \text{if } a \le x \le b, \\ f(b) & \text{if } x > b. \end{cases}$$

Theorem 5.11 (Fundamental Theorem of Calculus [FTC]): *Let f be integrable and have an antiderivative F on $[a,b]$. Then*

$$\int_a^b f = F(b) - F(a).$$

Proof: Parition $[a,b]$ into $\{x_0, x_1, \ldots, x_n\}$. Since F is differentiable on $[a,b]$, the MVT tells us that for each $i = 1, \ldots n$ there exists a $c_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)(x_i - x_{i-1}).$$

Consider the Riemann sum

$$\sum_{i=1}^n f(c_i)(x_i - x_{i-1}) = \sum_{i=1}^n [F(x_i) - F(x_{i-1})] = F(x_n) - F(x_0) = F(b) - F(a),$$

independent of n. Since f is integrable, the value of $\int_a^b f$ must equal the limit of this Riemann sum as $n \to \infty$. That is, $\int_a^b f = F(b) - F(a)$.

Remark: It is possible for a function to be integrable, but have no antiderivative. But by Theorem 5.9, we know that such a function cannot be continuous. An example is the function

$$f(x) = \begin{cases} -1 & \text{if } -1 \le x < 0, \\ \\ 1 & \text{if } 0 \le x \le 1. \end{cases}$$

Being piecewise continuous, f is integrable. However, the FTC implies that f cannot be the derivative of another function F. For if $f = F'$, then $\int_0^x f = F(x) - F(0)$, so that

$$\begin{aligned} F(x) = F(0) + \int_0^x f &= F(0) + \begin{cases} \int_0^x (-1) & \text{if } -1 \le x < 0, \\ \\ \int_0^x 1 & \text{if } 0 \le x \le 1, \end{cases} \\ &= F(0) + \begin{cases} -1(x-0) & \text{if } -1 \le x < 0, \\ \\ 1(x-0) & \text{if } 0 \le x \le 1, \end{cases} \\ &= F(0) + |x|, \end{aligned}$$

which we know is not differentiable at $x = 0$, regardless of what $F(0)$ is.

Remark: It is also possible for an integrable function f to be discontinuous at a point but still have an antiderivative F. Consider $f = F'$, where

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Although f is discontinuous at 0, is still integrable on any finite interval.

Theorem 5.12 (FTC for Continuous Functions): *Let f be continuous on $[a, b]$ and let F be any antiderivative of f on $[a, b]$. Then*

$$\int_a^b f = F(b) - F(a).$$

Proof: This follows directly from Theorem 5.1 and the FTC.

Remark: The FTC says that a *definite integral* $\int_a^b f$ is equal to the value of any antiderivative F of f at b minus the value of the **same** function F at a. That is, $\int_a^b f = [F(x)]_a^b$, where the notation $[F(x)]_a^b$ or $F(x)|_a^b$ is shorthand for the difference $F(b) - F(a)$.

- Let $f(x) = x$. Then

$$\int_0^1 f = \left[\frac{x^2}{2} + c\right]_0^1 = \frac{1}{2} + c - (0 + c) = \frac{1}{2}.$$

Remark: We need a convenient notation for an antiderivative.

Definition: If an integrable function f has antiderivative F, we write $F = \int f$ and say F is the *indefinite integral* of f.

$$\int f = F \text{ means } f = F'.$$

For example,

$$\int x\,dx = \frac{x^2}{2} + C \quad \text{means} \quad x = \frac{d}{dx}\left(\frac{x^2}{2} + C\right).$$

Remark: Remember that the definite integral $\int_a^b f(x)\,dx$ is a number, whereas the indefinite integral $\int f(x)\,dx$ represents a *family* of functions that differ from each other by a constant.

- Since
$$\frac{d}{dx}\frac{x^{p+1}}{p+1} = x^p,$$
we know that
$$\int_a^b x^p\,dx = \left.\frac{x^{p+1}}{p+1}\right|_a^b = \frac{b^{p+1}-a^{p+1}}{p+1} \qquad \text{if } p \neq -1.$$

- Also,
$$\int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = -[\cos x]_0^\pi = -[-1-1] = 2.$$

- But
$$\int_0^{2\pi} \sin x\,dx = [-\cos x]_0^{2\pi} = [-1-(-1)] = 0.$$

- $$\int_0^1 e^x\,dx = [e^x]_0^1 = e-1.$$

- $$\int_1^2 \frac{1}{x}\,dx = [\log x]_1^2 = \log 2 - \log 1 = \log 2.$$

- Consider
$$F(x) = \log|x| = \begin{cases} \log x & x>0, \\ \log(-x) & x<0. \end{cases}$$
Then
$$F'(x) = \begin{cases} \dfrac{1}{x} & x>0, \\[2ex] \dfrac{1}{-x}(-1) & x<0 \end{cases}$$
$$= \frac{1}{x} \ \text{ for all } x \neq 0.$$
Therefore, we see that
$$\int \frac{1}{x}\,dx = \log|x| + C \qquad \left(\textbf{not } \frac{x^0}{0}\right),$$
where C is an arbitrary constant. Thus
$$\int x^n\,dx = \begin{cases} \frac{x^{n+1}}{n+1} + C & \text{if } n \neq -1, \\[2ex] \log|x| + C & \text{if } n = -1. \end{cases}$$

- $$\int_{-2}^{-1} \frac{1}{x}\,dx = [\log|x|]_{-2}^{-1} = \log 1 - \log 2 = -\log 2.$$

- Consider the inverse trigonometric function $y = \sin^{-1} x$ for $x \in [-1, 1]$. Recall that

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \text{ for } (-1, 1)$$

and

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \text{ for } x \in (-\infty, \infty).$$

These results yield two important antiderivatives:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

and

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C.$$

- The function

$$F(x) = \int_0^x \frac{1}{\cos t}\,dt$$

is differentiable for $x \in [0, \frac{\pi}{2})$.

We don't yet know F, but we do know its derivative. Thm 5.9 $\Rightarrow$

$$F'(x) = \frac{1}{\cos x}.$$

Furthermore, suppose

$$G(x) = \int_0^{x^2} \frac{1}{\cos t}\,dt = F(x^2)$$

for $x \in \left[0, \sqrt{\frac{\pi}{2}}\right)$. Then

$$G'(x) = F'(x^2)\,2x = \frac{2x}{\cos(x^2)} \quad \text{by the Chain Rule.}$$

Problem 5.1: Suppose f is a continuous function and g and b are differentiable functions on $[a,b]$. Prove that

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))b'(x) - f(a(x))a'(x).$$

Let F be an antiderivative for f. Theorem FTC states that

$$\int_{a(x)}^{b(x)} f(t)\,dt = F(b(x)) - F(a(x)).$$

Hence, using the Chain Rule,

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = F'(b(x))b'(x) - F'(a(x))a'(x) = f(b(x))b'(x) - f(a(x))a'(x).$$

5.3 Substitution Rule

Q. What is $\int \tan x\,dx = \displaystyle\int \frac{\sin x}{\cos x}\,dx$?

A. On differentiating $F(x) = -\log|\cos x| + C$, we see that $\int \tan x\,dx = F(x)$.

Q. Are there systematic ways of finding such antiderivatives?

A. Yes, the following theorem *Substitution Rule*) is often helpful.

Theorem 5.13 (Substitution Rule): *Suppose g' is continuous on $[a,b]$ and f is continuous on $g([a,b])$. Then*

$$\int_{x=a}^{x=b} f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du} = \int_{u=g(a)}^{u=g(b)} f(u)\,du.$$

Proof: Theorem 5.9 $\Rightarrow$ f has an antiderivative F:

$$F'(u) = f(u) \quad \text{for all } u \in g([a,b]).$$

Consider $H(x) = F(g(x))$. Then

$$\begin{aligned} H'(x) &= F'(g(x))g'(x) \\ &= f(g(x))g'(x), \end{aligned}$$

that is, H is an antiderivative of $(f\circ g)g'$. Letting $u = g(x)$, we may then write

$$\int f(g(x))g'(x)\,dx = H(x) = F(g(x)) = F(u) = \int f(u)\,du$$

and, using the FTC,

$$\int_a^b f(g(x))g'(x) = [H(x)]_a^b = [F(g(x))]_a^b = F(g(b)) - F(g(a)) = \int_{u=g(a)}^{u=g(b)} f(u)\,du.$$

- Suppose we wish to calculate $\int_0^1 (x^2+2)^{99}\,2x\,dx$. One could expand out this polynomial and integrate term by term, but a much easier way to evaluate this integral is to make the substitution $u = g(x) = x^2+2$. To help us remember the factor $\frac{du}{dx} = g'(x) = 2x$ we formally write $du = g'(x)\,dx = 2x\,dx$,
$$\int_{x=0}^{x=1} (x^2+2)^{99} 2x\,dx = \int_{u=2}^{u=3} u^{99}\,du = \frac{u^{100}}{100}\bigg|_2^3 = \frac{3^{100}-2^{100}}{100}.$$
- To compute $\int x\sqrt{x^2+1}\,dx$, it is helpful to substitute $u = x^2+1 \Rightarrow du = 2x\,dx$.
$$\begin{aligned}\int x\sqrt{x^2+1}\,dx = \int u^{1/2}\,\frac{du}{2} &= \frac{1}{\cancel{2}}\frac{\cancel{2}}{3}u^{3/2}+C \qquad \longleftarrow \text{(don't leave in this form)}\\ &= \frac{1}{3}(x^2+1)^{3/2}+C.\end{aligned}$$
Check:
$$\frac{d}{dx}\left[\frac{1}{3}(x^2+1)^{3/2}+C\right] = \frac{1}{\cancel{3}}\frac{\cancel{3}}{\cancel{2}}(x^2+1)^{1/2}\,\cancel{2}\,x = x\sqrt{x^2+1}.$$
- The subsitution $u = 2x+1$ reduces the integral $\int_0^4 \sqrt{2x+1}\,dx$ to
$$\int_1^9 \sqrt{u}\,\frac{du}{2} = \left[\frac{1}{3}u^{3/2}\right]_1^9 = \frac{1}{3}\big(3^3-1\big) = \frac{26}{3}.$$
- The subsitution $u = \log x$ reduces the integral
$$\int_1^e \frac{\log x}{x}\,dx = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}.$$
- The change of variables $u = e^t \Rightarrow du = e^t\,dt$ allows us to evaluate
$$\int \frac{e^t}{e^t+1}\,dt = \int \frac{du}{u+1} = \log|u+1|+C = \log(e^t+1)+C.$$
- The substitution $u = \frac{x}{a} \Rightarrow x = au \Rightarrow dx = adu$, where a is a constant, allows us to evaluate any integral of the form
$$\begin{aligned}\int \frac{1}{x^2+a^2}\,dx &= \int \frac{1}{a^2\big(\frac{x^2}{a^2}+1\big)}\,dx\\ &= \frac{1}{a^2}\int \frac{1}{u^2+1}\,a\,du\\ &= \frac{1}{a}\int \frac{1}{u^2+1}\,du\\ &= \frac{1}{a}\tan^{-1}u+C\\ &= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C.\end{aligned}$$

Problem 5.2: Find

$$\int \cos t \sqrt{\sin t}\, dt.$$

$$= \frac{2}{3} \sin^{3/2} t + C.$$

Problem 5.3: Let α be a real number. Find

$$\int x^{-\alpha} e^{-\alpha x} \left(\frac{1}{x} + 1 \right) dx.$$

We use the substitution $u = \log x + x$ to rewrite

$$\int e^{-\alpha(\log x + x)} \left(\frac{1}{x} + 1 \right) dx = \int e^{-\alpha u}\, du = \begin{cases} \frac{-e^{-\alpha u}}{\alpha} + C & \text{if } \alpha \neq 0, \\ u + C & \text{if } \alpha = 0. \end{cases}$$

$$= \begin{cases} -\frac{x^{-\alpha} e^{-\alpha x}}{\alpha} + C & \text{if } \alpha \neq 0, \\ \log x + x + C & \text{if } \alpha = 0. \end{cases}$$

Problem 5.4: Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^{\pi/2} f(\sin x)\, dx = \int_0^{\pi/2} f(\cos x)\, dx.$$

This follows on using the substitution $u = \pi/2 - x$:

$$\int_0^{\pi/2} f(\sin x)\, dx = \int_0^{\pi/2} f\left(\cos\left(\frac{\pi}{2} - x \right) \right) dx = -\int_{\pi/2}^{0} f(\cos u)\, du = \int_0^{\pi/2} f(\cos u)\, du.$$

Problem 5.5:

(a) Show that any function $f : \mathbb{R} \to \mathbb{R}$ can be decomposed as a sum of an even function f_e and an odd function f_o. Hint: Construct explicit expressions for f_e and f_o in terms of $f(x)$ and $f(-x)$ and show that they are even and odd functions, respectively.

Let

$$f_e(x) = \frac{f(x) + f(-x)}{2},$$

$$f_o(x) = \frac{f(x) - f(-x)}{2},$$

Then $f(x) = f_e(x) + f_o(x)$.

(b) Show using Theorem 5.2 and an appropriate substitution that if f_e is an even integrable function on $[-a, a]$, then

$$\int_{-a}^{a} f_e = 2\int_0^a f_e.$$

$$\begin{aligned}\int_{-a}^{a} f_e(x)\,dx &= \int_{-a}^{0} f_e(x)\,dx + \int_0^a f_e(x)\,dx = -\int_a^0 f_e(-x)\,dx + \int_0^a f_e(x)\,dx \\ &= \int_0^a f_e(x)\,dx + \int_0^a f_e(x)\,dx = 2\int_0^a f_e.\end{aligned}$$

(c) Show that if f_o is an odd integrable function on $[-a, a]$ that

$$\int_{-a}^{a} f_o = 0.$$

$$\begin{aligned}\int_{-a}^{a} f_o(x)\,dx &= \int_{-a}^{0} f_o(x)\,dx + \int_0^a f_o(x)\,dx = -\int_a^0 f_o(-x)\,dx + \int_0^a f_o(x)\,dx \\ &= -\int_0^a f_o(x)\,dx + \int_0^a f_o(x)\,dx = 0.\end{aligned}$$

(d) Deduce that

$$\int_0^a f + \int_{-a}^{0} f = 2\int_0^a f_e$$

and

$$\int_0^a f - \int_{-a}^{0} f = 2\int_0^a f_o.$$

We find

$$\int_0^a f + \int_{-a}^0 f = \int_0^a (f_e + f_o) + \int_{-a}^0 (f_e + f_o) = \int_{-a}^a f_e + \int_{-a}^a f_o = 2\int_0^a f_e$$

and

$$\int_0^a f - \int_{-a}^0 f = \int_0^a (f_e + f_o) - \int_{-a}^0 (f_e + f_o) = \int_0^a f_e - \int_0^a f_e + \int_0^a f_o + \int_0^a f_o = 2\int_0^a f_o.$$

- Since the integrand is even, we can simplify

$$\int_{-2}^{2} (x^4+1)\,dx = 2\int_0^2 (x^4+1)\,dx = 2\left[\frac{x^5}{5} + x\right]_0^2 = 2\left(\frac{2^5}{5} + 2\right) = \frac{2^6}{5} + 4 = \frac{84}{5}.$$

- Since the integrand is odd, we can simplify

$$\int_{-\pi/2}^{\pi/2} \csc x \, dx = 0.$$

Problem 5.6: (a) Let f be an odd function with antiderivative F. Prove that F is an even function. Note: we do not assume that f is continuous or even integrable.

We are given that $f(-x) = -f(x)$ and $F'(x) = f(x)$. Hence

$$\frac{d}{dx}[F(x) - F(-x)] = f(x) + f(-x) = 0,$$

so that

$$F(x) - F(-x) = C$$

for some constant C. Evaluating this result at $x = 0$, we see that $C = 0$. Hence $F(x) = F(-x)$, that is, F is even.

(b) If f is an even function with antiderivative F, can one always find an antiderivative G of f that is odd? Are all antiderivatives of f odd? Prove or provide a counterexample for each of these statements.

We are given that $f(-x) = f(x)$ and $F'(x) = f(x)$. Hence

$$\frac{d}{dx}[F(x) + F(-x)] = f(x) - f(-x) = 0,$$

so that

$$F(x) + F(-x) = C,$$

where C is a constant. Let $G(x) = F(x) - C/2$. Then G is an antiderivative of f and $G(-x) = F(-x) - C/2 = -F(x) + C/2 = -G(x)$, so G is odd. However, not all antiderivatives of f are odd. Consider the even function $f(x) = 1$. The antiderivative $x + 1$ is not an odd function, although the antiderivative $G(x) = x$ is.

5.4 Numerical Approximation of Integrals

There are many continuous functions such as

$$\frac{e^x}{x}, \frac{\sin x}{x}, \text{ and } e^{x^2},$$

for which the antiderivative cannot be expressed in terms of the elementary functions introduced so far. For applications where one needs only the value of a definite integral, one possibility is to approximate the integral numerically.

To illustrate the numerical evaluation of definite integrals, it is helpful to consider an integral for which we know the exact answer, such as $\int_0^1 f\,dx$, where $f(x) = x^2$. For the partition $\{0, \frac{1}{2}, 1\}$ of $[0,1]$ we can find a lower bound

$$L = 0\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{8} = 0.125$$

and an upper bound

$$U = \frac{1}{4}\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = \frac{5}{8} = 0.625$$

on the integral.

That is,

$$L \leq \int_0^1 x^2\,dx \leq U$$

but neither L nor U provides us with a very good approximation to the integral. Notice that the average of L and U, namely $(L+U)/2 = 3/8 = 0.375$, is much closer to the exact value (1/3) of the definite integral and that since f is increasing, L is identical to the left Riemann sum $S_L = \sum_{i=1}^n f(x_{i-1})(x_i - x_{i-1})$ and U is the right Riemann sum $S_R = \sum_{i=1}^n f(x_i)(x_i - x_{i-1})$. This suggests that it may be better to approximate the integral by using the *Trapezoidal Rule*

$$T_n \doteq \frac{S_L + S_R}{2} = \sum_{i=1}^n \frac{f(x_{i-1}) + f(x_i)}{2}(x_i - x_{i-1}),$$

Remark: For a uniform partition with fixed a, b, and f, T depends only on the number of subintervals.

Q. How accurately does T_n, for a uniform partition of $[a,b]$ into n subintervals, approximate $\int_a^b f$? How does the error depend on n?

A. First, we look at a special case of this question where there is only one subinterval.

Theorem 5.14 (Linear Interpolation Error): *Let f be a twice-differentiable function on $[0,h]$ satisfying $|f''(x)| \leq M$ for all $x \in [0,h]$. Let*

$$L(x) = f(0) + \frac{f(h) - f(0)}{h}x.$$

Then

$$\int_0^h |L(x) - f(x)|\,dx \leq \frac{Mh^3}{12}.$$

Proof: Let $x \in (0,h)$ and

$$\varphi(t) = L(t) - f(t) - Ct(t-h),$$

where C is chosen so that $\varphi(x) = 0$. Then

$$\varphi(0) = L(0) - f(0) = 0,$$

$$\varphi(h) = L(h) - f(h) = 0.$$

From Rolle's Theorem, we then know that there exists $x_1 \in (0,x)$ and $x_2 \in (x,h)$ such that

$$\varphi'(x_1) = \varphi'(x_2) = 0.$$

Again by Rolle's Theorem, we know that there exists $c \in (x_1, x_2)$ such that

$$0 = \varphi''(c) = -f''(c) - 2C,$$

noting that L is linear. Therefore $C = -f''(c)/2$ and since $\varphi(x) = 0$,

$$L(x) - f(x) = \frac{-1}{2} f''(c) x(x-h),$$

where $c \in (0,h)$ depends on x. That is, for every $x \in [0,h]$ we have

$$|L(x) - f(x)| \le \frac{1}{2} M x(h-x),$$

so

$$\int_0^h |L(x) - f(x)|\, dx \le \frac{M}{2} \int_0^h x(h-x)\, dx = \frac{M}{2} \left[\frac{x^2 h}{2} - \frac{x^3}{3} \right]_0^h = \frac{M h^3}{12}.$$

Theorem 5.15 (Trapezoidal Rule Error): *Consider a uniform partition of $[a,b]$ into n subintervals of width $h = (b-a)/n$, and f be a twice-differentiable function on $[a,b]$ satisfying $|f''(x)| \le M$ for all $x \in [a,b]$. Then the error $E_n^T \doteq T_n - \int_a^b f$ of the uniform Trapezoidal Rule*

$$T_n = h \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}$$

satisfies

$$\left| E_n^T \right| \le \frac{n M h^3}{12} = \frac{M(b-a)^3}{12 n^2}.$$

Proof: We need to add up the contribution to the error from each subinterval. If we temporarily relabel the endpoints of each subinterval 0 and h, we may apply Theorem 5.14 to obtain a contribution, $\left| \int_0^h L - \int_0^h f \right| \le \int_0^h |L - f| \le M h^3/12$, from each of the n subintervals.

Remark: We can rewrite the Trapezoidal Rule as

$$T_n = \frac{h}{2}[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)].$$

- We can use the Trapezoidal Rule to approximate $\int_1^2 \frac{1}{x}\,dx$ with $n = 5$ subintervals of width $h = 1/5$:

$$\begin{aligned}\int_1^2 \frac{1}{x}\,dx \approx T_n &= \frac{1}{10}\left[\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right]\\ &\approx 0.6956.\end{aligned}$$

The exact value of the integral is $\log 2 = 0.6931\ldots$.

Remark: Typically, a more accurate method than the Trapezoidal Rule is the *Midpoint Rule*

$$M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)(x_i - x_{i-1}),$$

which has the additional advantage of requiring one less function evaluation.

Problem 5.7: Show that the Midpoint Rule has an error $E_n^M \doteq M_n - \int_a^b f$ satisfying

$$\left|E_n^M\right| \le \frac{M(b-a)^3}{24n^2}.$$

Notice that this bound is a factor of 2 smaller than the error bound for the Trapezoidal Rule.

- Let us use the Midpoint Rule to approximate $\int_1^2 \frac{1}{x}\,dx$ with $n = 5$ subintervals of width $h = 1/5$:

$$\begin{aligned}\int_1^2 \frac{1}{x}\,dx \approx M_n &= \frac{1}{5}\left[\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right]\\ &\approx 0.6919,\end{aligned}$$

which is indeed closer than T_n to the exact value of $\log 2$ (by roughly a factor of 2).

Remark: Even better are the higher-order methods, such as *Simpson's Rule*, which fits parabolas rather than line segments to the data values $f(x_0), f(x_1), \ldots, f(x_n)$, where n is even. This approximation is given by

$$S_n = \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)],$$

with an error $E_n^S \doteq S_n - \int_a^b f$ satisfying

$$\left|E_n^S\right| \le \frac{K(b-a)^5}{180n^4} \qquad \text{if } \left|f^{(4)}(x)\right| \le K \text{ for all } x \in [a, b].$$

Problem 5.8: Consider the function $f(x) = 1/(1+x^2)$ on $[0,1]$. Let P be a uniform partition on $[0,1]$ with 2 subintervals of equal width.

(a) Compute the left Riemann sum S_L.
Since the partition is uniform,

$$S_L = \frac{1}{2}\left(\frac{4}{5} + \frac{1}{2}\right) = \frac{13}{20}.$$

(b) Compute the right Riemann sum S_R.

$$S_R = \frac{1}{2}\left(1 + \frac{4}{5}\right) = \frac{9}{10}.$$

(c) Use your results in part (a) and (b) to find lower and upper bounds for π.
We see that

$$\frac{13}{20} = S_L \le \frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2}\,dx \le S_R = \frac{9}{10}.$$

Thus $\dfrac{13}{5} \le \pi \le \dfrac{18}{5}$.

(d) Use the Trapezoidal Rule to find a numerical estimate for π.
We find that π is approximately

$$4\left(\frac{1}{2}\right)\left(\frac{1+\frac{4}{5}}{2} + \frac{\frac{4}{5}+\frac{1}{2}}{2}\right) = 2\left(\frac{9}{10} + \frac{13}{20}\right) = \frac{31}{10}.$$

(e) Obtain a better rational estimate for π by using the Midpoint Rule.
We find that π is approximately

$$4\left(\frac{1}{2}\right)\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right) = 2\left(\frac{16}{17} + \frac{16}{25}\right) = 32\left(\frac{1}{17} + \frac{1}{25}\right) = 32\left(\frac{42}{425}\right) = \frac{1344}{425}.$$

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