Constraints on the spectral distribution of energy and enstrophy dissipation in forced two-dimensional turbulence

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Abstract

We study two-dimensional turbulence in a doubly periodic domain driven by a monoscale-like forcing and damped by various dissipation mechanisms of the form $\nu_\mu(-\Delta)^\mu$. By "monoscale-like" we mean that the forcing is applied over a finite range of wavenumbers $k_{\min} \leq k \leq k_{\max}$, and that the ratio of enstrophy injection $\eta \geq 0$ to energy injection $\varepsilon \geq 0$ is bounded by $k_{\min}^2 \varepsilon \leq \eta \leq k_{\max}^2 \varepsilon$. Such a forcing is frequently considered in theoretical and numerical studies of two-dimensional turbulence. It is shown that for $\mu \geq 0$ the asymptotic behaviour satisfies

$$\|u\|_1^2 \leq k_{\max}^2 \|u\|^2 ,$$

where $\|u\|^2$ and $\|u\|_1^2$ are the energy and enstrophy, respectively. If the condition of monoscale-like forcing holds only in a time-mean sense, then the inequality holds in the time mean. It is also shown that for Navier-Stokes turbulence ($\mu = 1$), the time-mean enstrophy dissipation rate is bounded from above by $2\nu_1 k_{\max}^2$. These results place strong constraints on the spectral distribution of energy and enstrophy and of their dissipation, and thereby on the existence of energy and enstrophy cascades, in such systems. In particular, the classical dual cascade picture is shown to be invalid for forced two-dimensional Navier–Stokes turbulence ($\mu = 1$) when it is forced in this manner. Inclusion of Ekman drag ($\mu = 0$) along with molecular viscosity permits a dual cascade, but is incompatible with the log-modified -3 power law for the energy spectrum in the enstrophy-cascading inertial range. In order to achieve the latter, it is necessary to invoke an inverse viscosity ($\mu < 0$). These constraints on permissible power laws apply for *any* spectrally localized forcing, not just for monoscale-like forcing.

Key words:
Two-dimensional turbulence, dual cascade, energy spectra, forced-dissipative equilibrium
PACS: 47.52.+j, 05.45.Jn, 47.17.+e, 47.27.Ak

1 Introduction

It has long been recognized (see for example Fjørtoft [9]) that the simultaneous existence of two quadratic inviscid invariants, energy and enstrophy, drastically changes the picture of two-dimensional (2D) Navier–Stokes turbulence in comparison with that of its 3D counterpart. In the 1960's it was suggested by Kraichnan [11], Leith [14], and Batchelor [2] (hereafter referred to as KLB) that these two constraints give rise to the realization of two distinct inertial ranges in wavenumber space for forced 2D turbulence. More precisely, the KLB theory predicts that for a 2D Navier–Stokes fluid driven by a spectrally localized forcing, energy is transferred to smaller wavenumbers while enstrophy is transferred to larger wavenumbers. For a system of finite size the energy is then predicted to cascade to the long-wavelength end of the spectrum up to the largest scale available (the system's linear length scale), while the enstrophy is predicted to cascade to the other end of the spectrum, down to a dissipation length scale. This prediction is well confirmed numerically in transient evolution from spectrally localized initial conditions. An immediate corollary of the dual cascade and the hypothesis of a scaling symmetry is that the energy spectrum scales as $k^{-5/3}$ in the energy-cascading range and as k^{-3} (with a possible logarithmic correction) in the enstrophy-cascading range, where k is the wavenumber. The latter scaling is consistent with the hypothesis that the dissipation of enstrophy is confined to the small scales; however, the former scaling is inconsistent with the hypothesis that the dissipation of energy is confined to the large scales. Hence, these scaling laws are incompatible with a persistent dual cascade in finite systems. This suggests that there is a problem with the KLB theory applied to finite systems. (We refer here only to the forced-dissipative equilibrium behaviour, not to transient evolution from spectrally localized initial conditions.)

For the case of a time-independent force, Constantin, Foias and Temam [7,8] have proven the existence of a compact global attractor for forced 2D Navier–Stokes turbulence in a bounded domain. Constantin, Foias and Manley [6] have furthermore shown that for the special case of a doubly periodic domain and forcing of a single length scale, the KLB scaling laws cannot be achieved on the global attractor. Although this strong result appears to prove the unrealizability of the KLB theory for forced 2D Navier–Stokes turbulence in a finite domain, as argued heuristically in the previous paragraph, the question arises whether it is an artifact of the special choice of a constant monoscale forcing.

In this paper we extend the results of Constantin, Foias and Manley [6] in several directions. First, for their choice of a constant monoscale forcing we

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derive an asymptotic bound on the enstrophy in terms of the energy and show that the time-mean enstrophy dissipation occurs around the forcing scale, precluding the existence of an enstrophy-cascading inertial range (whatever the energy spectrum). This result applies for any dissipation operator of the form $\nu_\mu(-\Delta)^\mu$ with $\mu \geq 0$, not just molecular viscosity. Second, we show that the result applies for any "monoscale-like" forcing, by which we mean forcing over a range of wavenumbers $k_{\min} \leq k \leq k_{\max}$ where the ratio of enstrophy injection $\eta \geq 0$ to energy injection $\varepsilon \geq 0$ is bounded by $k_{\min}^2 \varepsilon \leq \eta \leq k_{\max}^2 \varepsilon$ for all possible velocity fields u. This is a classical (though not exclusive) scenario for the KLB theory (e.g. Kraichnan [11], p.1421b; Pouquet et al. [21], p.314; and Lesieur [15], p.291), and is furthermore a common set-up in numerical simulations of forced 2D turbulence (e.g. Lilly [16]; Basdevant et al. [1]; Shepherd [22]). [1] If the condition of monoscale-like forcing holds only in a time-mean sense, then the time-mean results still go through and the asymptotic bound is replaced by a bound involving the time-mean energy and enstrophy.

The question then arises whether the KLB theory and the dual cascade picture can be recovered, for monoscale-like forcing, with other dissipation mechanisms. It is shown that the introduction of Ekman drag ($\mu = 0$)—a commonly used numerical device and one with some physical justification for geophysical applications [20]—in addition to regular viscosity permits the existence of an energy dissipation range at large scales and an enstrophy dissipation range at small scales, thus allowing a dual cascade. This is consistent with numerical results, e.g. Boffetta et al. [3], which demonstrate an inverse energy cascade and a $-5/3$ power law under these conditions. However, the log-modified -3 power law of Kraichnan [12] for the energy spectrum in the enstrophy-cascading range is shown to be unachievable; the spectrum must be algebraically steeper than -3. This result is consistent with numerical simulations of forced 2D turbulence under these conditions, which find steeper spectra. (e.g. Maltrud and Vallis [18] find the spectral slope of the enstrophy-cascading range to be between -3 and -4: the majority of their simulations yields values between -3.3 and -3.6. They also recover the scaling $k^{-5/3}$ in the energy-cascading range.)

It is finally shown that allowing an inverse viscosity ($\mu < 0$) together with regular viscosity does permit KLB scaling. It is notable that the numerical simulations of Borue [4], and the more recent high-resolution numerical sim-

[1] Constantin, Foias and Manley [6] showed that the KLB scaling is potentially realizable (not to say it actually occurs) if the forcing simultaneously injects energy at higher wavenumbers and removes energy at lower (though possibly nearby) wavenumbers. In this case the ratio of enstrophy to energy injection can be much larger than the square of the characteristic forcing wavenumbers and could actually lie in the dissipation range. Such a forcing is not monoscale-like by our definition.

ulations of Lindborg and Alvelius [17], which both claim to exhibit a -3 power-law enstrophy-cascading range, employ an inverse viscosity. The rather surprising implication of our results is that verifying the KLB theory in the case of monoscale-like forcing does not only depend on achieving sufficiently high Reynolds numbers (as is commonly believed), but also depends on the nature of the rather ad hoc dissipation operator employed at the large scales.

The alternative, considered by Constantin et al. [6], is to abandon the notion of monoscale-like forcing. This interesting possibility is not considered here.

The remainder of this paper is organized as follows. Section 2 describes the 2D Navier–Stokes equations and their mathematical setting, together with some basic inequalities. The concept of monoscale-like forcing is discussed further, and some examples given. Section 3 reviews some classical ideas and arguments in 2D turbulence. After those preliminaries, section 4 reports an asymptotic analysis, the results of which include a dynamical constraint and an upper bound on the energy dissipation rate. Section 5 extends the results of section 4 in an attempt to bound the time-mean enstrophy dissipation rate and explore possible scaling laws for the energy spectrum. These results in fact apply for *any* spectrally localized forcing, not just for monoscale-like forcing. The paper ends with some concluding remarks in the final section.

2 Governing equations and basic inequalities

We consider 2D incompressible fluid motion confined within a doubly periodic rectangular domain Ω. The fluid is assumed to be driven by a monoscale-like forcing (to be discussed further below) and damped by a variety of possible dissipation mechanisms including Ekman drag, hypoviscosity, molecular viscosity, hyperviscosity, and inverse viscosity. The 2D Navier–Stokes equations which govern the fluid motion are written in abstract form in a function space H as

$$\frac{\mathrm{d}u}{\mathrm{d}t} + B(u,u) + \sum_{\mu} \nu_\mu A^\mu u = f, \tag{1}$$
$$u(t=0) = u_0,$$

where $\nu_\mu > 0$ is a generalized viscosity coefficient, $A \equiv -\Delta$, and f is the forcing. The number μ will be called the degree of viscosity. When $\mu = 1$ we have the usual molecular viscosity, while $\mu = 0$ corresponds to Ekman drag. The cases in which $\mu > 1$ and $0 < \mu < 1$ correspond to hyperviscosity and hypoviscosity, respectively. We also consider inverse viscosity, i.e., negative values of μ. The summation is taken over all dissipation channels involved.

A detailed description of the functional analysis setting for (1) is given in Constantin and Foias [5] or Temam [25,26]. We recall that H is the L^2-space of periodic, non-divergent functions with vanishing average in Ω. $B(u,u) = P((u \cdot \nabla)u)$ where P is the orthogonal projection in L^2 onto H. We denote by H^α the domain of definition of $A^{\alpha/2}$ for real α. The (degenerate) positive eigenvalues of A are denoted by λ_k with the index k being the wavenumber, and the eigenspace corresponding to λ_k is denoted by $H(\lambda_k)$. We will occasionally refer to $\lambda_k^{-1/2}$ (or simply k^{-1}) as the length scale associated with the wavenumber k.

The scalar product and the norm in H^α are given respectively by

$$(u,v)_\alpha = \int_\Omega u \cdot A^\alpha v \, \mathrm{d}x, \tag{2}$$

$$\|u\|_\alpha = (u,u)_\alpha^{1/2}. \tag{3}$$

The cases where $\alpha = 0, 1$ are special and the corresponding H-norm (the superscript and subscript '0' are omitted in this case) and H^1-norm are known in the literature as the energy and enstrophy norm, respectively. A geometric constraint, referred to as the Poincaré inequality, is

$$\|u\|_{\alpha+\beta}^2 \geq \lambda_1^\alpha \|u\|_\beta^2 \tag{4}$$

for non-negative α, where λ_1 is the first (smallest) eigenvalue of A in H. The inequality reverses direction for non-positive α.

The bilinear operator $B(\cdot,\cdot)$ satisfies

$$(Au, B(v,v)) + (Av, B(v,u)) + (Av, B(u,v)) = 0 \tag{5}$$

for $u, v \in H^2$, and

$$(u, B(v,w)) = -(w, B(v,u)) \tag{6}$$

for $u, w \in H^1$ and $v \in H$. These identities arise by virtue of the non-divergent and periodic properties of the velocity field. In particular, we have

$$(u, B(u,u)) = 0, \tag{7}$$

$$(Au, B(u,u)) = 0. \tag{8}$$

We shall collectively refer to the above identities as the orthogonality properties of the nonlinear term. In the absence of forcing and dissipation, they give rise to conservation of energy and enstrophy. These conservation laws form the foundation for the idea of the inverse cascade of energy and the direct cascade of enstrophy, and the associated concepts of inertial ranges, as mentioned earlier and reviewed in the next section.

A well-known mathematical fact is the existence of a bounded finite-dimensional global attractor for the traditional 2D Navier–Stokes equations in a finite domain (as considered here). The research performed on this subject constitutes a rich literature. The book of Temam [26] provides a good treatment of the subject and a complete list of references. Some articles suitable for quick reference are Constantin et al. [7,8] and Ziane [27]. We shall not attempt to demonstrate the existence of such an attractor for the 2D Navier–Stokes system (driven by a time-independent forcing) with various degrees of viscosity (or combinations of them, as presently considered) but take its existence to be given.

In this study f is assumed to be monoscale-like, as defined in the Introduction. This means that the ratio of enstrophy to energy injection, which we denote by λ, is bounded a priori within a certain range and in particular from above, for all u. The time-independent monoscale body force $f \in H(\lambda_s)$, for an eigenvalue λ_s, considered by Constantin et al. [6] is a special case with $\lambda = \lambda_s$, although it is a rather peculiar special case, for two reasons. First, the instantaneous energy and enstrophy injection are not constrained to be non-negative [2], although they clearly must be so in the time mean since the dissipation of both quantities is non-negative. This means that the condition $\lambda = \lambda_s$ does not necessarily extend, even approximately, to the case of a constant external forcing applied over a range of wavenumbers; such a forcing is not "monoscale-like" in the sense we require (and as is often assumed in the KLB theory). Second, there exists an exact stationary solution (given by (26)). Because of this, there appears to be a belief that the flow in this case collapses onto the stationary solution and is not turbulent. However, for sufficiently strong forcing the forced scales will be unstable to nonlinear interactions, and numerical simulations (J.C. Bowman, personal communication, 2001) indeed confirm that there is nothing pathological about the case of a constant monoscale forcing: it develops a full spectrum, albeit one constrained by the inequalities discussed here.

The general case of a monoscale-like forcing applied over a finite range of wavenumbers, as often imagined in the KLB theory, can be realized in several ways. One example (Shepherd [22]) is $f = \sum_K cP(\lambda_k)u/\left\|P(\lambda_k)u\right\|^2$ where $c > 0$, $P(\lambda_k)$ is the projection onto $H(\lambda_k)$, $\|\cdot\|$ is the energy norm, and the summation is over the restricted set of wavenumbers K. For this forcing the energy and enstrophy injection are both constant in time and λ is the mean

[2] For this reason, a monoscale body force is not, strictly speaking, a special case of monoscale-like forcing, even though our results apply to both cases.

of λ_k over K. Another example is $f = \sum_K cP(\lambda_k)u$. Although the energy and enstrophy injection are now variable in time, both are positive definite and λ lies within the range $[\lambda_{\min}, \lambda_{\max}]$, where $\lambda_{\min}$ and $\lambda_{\max}$ are, respectively, the minimum and maximum λ_k in K. This commonly used forcing is known as "instability" forcing, for obvious reasons; in the geophysical context it is used in 2D turbulence to mimic forcing of the barotropic component of the flow by baroclinic instability (Lilly [16]; Basdevant et al. [1]). In both these examples, f depends on u. Another commonly used forcing is white-noise forcing over K (Lilly [16]). This is harder to control a priori, but in practice gives positive enstrophy and energy injection with λ generally fluctuating within the range $[\lambda_{\min}, \lambda_{\max}]$. For this case, the condition of monoscale-like forcing would need to be verified a posteriori, and would apply at best only in a time-mean sense. The results derived here would then apply only in the time mean.

We now introduce some terminology and derive some preliminary estimates which are employed later. In the theory of turbulence the characteristic wavenumber $\overline{k}$ defined by $\overline{k} = \|u\|_{1/2}^2 / \|u\|^2$ is often studied. However, we find it more natural and convenient in the context of this work to define the parameter

$$\Lambda = \frac{\|u\|_1^2}{\|u\|^2} , \tag{9}$$

the square root of which has the dimension of a wavenumber. Moreover, let $\Lambda_{E\mu}$ and $\Lambda_{Z\mu}$ be defined (for $\mu \neq 0$) by

$$\Lambda_{E\mu} = \left(\frac{\|u\|_\mu^2}{\|u\|^2} \right)^{1/\mu} , \tag{10}$$

$$\Lambda_{Z\mu} = \left(\frac{\|u\|_{1+\mu}^2}{\|u\|_1^2} \right)^{1/\mu} , \tag{11}$$

so that $2\nu_\mu \Lambda_{E\mu}^\mu$ and $2\nu_\mu \Lambda_{Z\mu}^\mu$ are, respectively, the (instantaneous) energy dissipation and enstrophy dissipation rates due to $\nu_\mu A^\mu$. The values of $\Lambda_{E\mu}$ and $\Lambda_{Z\mu}$ therefore indicate where, in wavenumber space, viscosity primarily operates in the dissipation of energy and enstrophy, respectively, for the dissipation operator of degree μ. Note that $\Lambda_{E1} \equiv \Lambda$; we will use the latter symbol exclusively in what follows.

We now derive several fundamental inequalities relating these parameters. In the following, α, β, γ, and λ are real numbers, and ϕ is an appropriate function so that all norms involved are well-defined. First, we have the following interpolation-type inequality which can be shown by Hölder's inequality:

$$\|\phi\|_{\beta+\gamma}^{\alpha} \leq \|\phi\|_{\alpha+\gamma}^{\beta} \|\phi\|_{\gamma}^{\alpha-\beta}, \tag{12}$$

for $\alpha \geq \beta \geq 0$. Note that the inequality reverses direction for $\alpha \leq \beta \leq 0$. Second, let us define

$$G(\alpha, \beta, \lambda, \phi) = \sum_{\lambda_k} (\lambda^\alpha - \lambda_k^\alpha)(\lambda^\beta - \lambda_k^\beta) \|P(\lambda_k)\phi\|^2 \tag{13}$$

for $\lambda > 0$. It is obvious from (13) that $G(\alpha, \beta, \lambda, \phi)$ is positive (negative) if and only if $\alpha\beta > 0$ ($\alpha\beta < 0$) and $\phi \notin H(\lambda)$ when λ is an eigenvalue of A. This implies that when $\alpha\beta \neq 0$, $G(\alpha, \beta, \lambda, \phi) = 0$ if and only if λ is an eigenvalue of A and $\phi \in H(\lambda)$. This important feature of $G(\alpha, \beta, \lambda, \phi)$ is used in some arguments of section 4. By rearranging terms we obtain

$$G(\alpha, \beta, \lambda, \phi) = \|\phi\|_{\alpha+\beta}^2 - \lambda^\beta \|\phi\|_\alpha^2 - \lambda^\alpha (\|\phi\|_\beta^2 - \lambda^\beta \|\phi\|^2). \tag{14}$$

It is then easy to show that

$$G\left(\alpha, \beta, \frac{\|\phi\|_\alpha^{2/\alpha}}{\|\phi\|^{2/\alpha}}, \phi\right) = \frac{\|\phi\|^2 \|\phi\|_{\alpha+\beta}^2 - \|\phi\|_\alpha^2 \|\phi\|_\beta^2}{\|\phi\|^2}, \tag{15}$$

from which it follows that

$$\|\phi\| \; \|\phi\|_{\alpha+\beta} \geq \|\phi\|_\alpha \|\phi\|_\beta \quad \text{for } \alpha\beta \geq 0, \tag{16}$$
$$\|\phi\| \; \|\phi\|_{\alpha+\beta} \leq \|\phi\|_\alpha \|\phi\|_\beta \quad \text{for } \alpha\beta \leq 0. \tag{17}$$

The ratio of $\Lambda_{E\mu}$ to $\Lambda_{E\mu'}$ is given by

$$\frac{\Lambda_{E\mu}}{\Lambda_{E\mu'}} = \left[\frac{\|u\|_\mu^{1/\mu} \|u\|^{1/\mu' - 1/\mu}}{\|u\|_{\mu'}^{1/\mu'}}\right]^2, \tag{18}$$

which, by applying (12) with $\alpha = \mu$, $\beta = \mu'$, and $\gamma = 0$ implies

$$\Lambda_{E\mu} \geq \Lambda_{E\mu'}, \quad \text{for } \mu \geq \mu' > 0, \tag{19}$$
$$\Lambda_{E\mu} \leq \Lambda_{E\mu'}, \quad \text{for } \mu \leq \mu' < 0. \tag{20}$$

By applying (12) to the ratio of $\Lambda_{Z\mu}/\Lambda_{Z\mu'}$ we also obtain the same inequalities for $\Lambda_{Z\mu}$ and $\Lambda_{Z\mu'}$. The ratio of $\Lambda_{Z\mu}^\mu$ to $\Lambda_{E\mu}^\mu$ is given by

$$\frac{\Lambda_{Z\mu}^{\mu}}{\Lambda_{E\mu}^{\mu}} = 1 + \frac{G(\mu,1,\Lambda,u)\,\|u\|^2}{\|u\|_1^2\,\|u\|_\mu^2}. \tag{21}$$

Since $G(\mu,1,\Lambda,u)$ takes the sign of μ (see above), the above ratio indicates the simple fact that a viscosity dissipates more enstrophy than energy, while an inverse viscosity dissipates more energy than enstrophy. Note, however, that

$$\Lambda_{Z\mu} \geq \Lambda_{E\mu}, \quad \text{for all } \mu \neq 0. \tag{22}$$

We will compare Λ, $\Lambda_{E\mu}$ and $\Lambda_{Z\mu}$ with λ in subsequent sections.

3 The KLB theory

We now review the arguments leading to the KLB theory. In the absence of forcing and dissipation, the 2D Navier–Stokes system conserves energy and enstrophy due to (7) and (8). There is a consensus that an initial distribution of energy cascades towards larger scales, with a downscale cascade of energy practically forbidden. Arguments against the downscale cascade of energy were advanced by Taylor [24] and Lee [13]. A celebrated "proof" of the upscale cascade is due to Fjørtoft [9]. In his argument, Fjørtoft considered the change in energy for three different scales coupled nonlinearly. Because of conservation of enstrophy, energy must flow from the intermediate scale to the smaller and larger scales, or vice versa. For energy initially on the intermediate scale, it was argued that the larger scale acquires most of the transferred energy. In the present setting Fjørtoft's argument goes as follows. Let $l < k < m$ be the three wavenumbers (corresponding to three scales $m^{-1} < k^{-1} < l^{-1}$) that are involved in the energy transfer. Furthermore, let $\Delta E(\cdot)$ denote the change of energy in each scale. It is easy to see from the two conservation laws that

$$\begin{aligned} \Delta E(l) + \Delta E(k) + \Delta E(m) &= 0, \\ \lambda_l \Delta E(l) + \lambda_k \Delta E(k) + \lambda_m \Delta E(m) &= 0. \end{aligned}$$

Hence,

$$\begin{aligned} \Delta E(l) &= -\frac{\lambda_m - \lambda_k}{\lambda_m - \lambda_l}\Delta E(k), \\ \Delta E(m) &= -\frac{\lambda_k - \lambda_l}{\lambda_m - \lambda_l}\Delta E(k). \end{aligned}$$

It is argued in [9] that if one takes $m = 2k = 4l$, for example, then ($\lambda_m = 4\lambda_k = 16\lambda_l$)

$$\frac{\Delta E(l)}{\Delta E(m)} = 4.$$

Therefore, changes in the kinetic energy are distributed in the ratio 4:1 on the components with the double and half scale, respectively, if no other components are involved in the energy transformation.

There are a number of problems with this argument. First, the inviscid system is time reversible, so an upscale cascade of energy cannot be established by the above argument alone. Second, the three scales involved in the energy transformation must satisfy the triad requirement, [3] and not all choices of the interacting triads give $\Delta E(l)/\Delta E(m) > 1$ [19]. Nevertheless, Merilees and Warn [19] find that an initial spectral peak spreads out with most of the energy going upscale—a result very well confirmed numerically. In particular, they find that given an intermediate wavenumber k, roughly 70% of interacting triads exchange more energy with lower wavenumbers, while roughly 60% of interacting triads exchange more enstrophy with higher wavenumbers. Given a monoscale forcing at a wavenumber s, then, it is tempting to arrive at the picture of the dual cascade of energy to larger scales, and of enstrophy to smaller scales. However, in forced-dissipative turbulence with a full spectrum, the validity of this picture is not at all self-evident.

In particular, the KLB theory envisages an enstrophy dissipation range for $k > k_\nu$ with $k_\nu \gg \lambda^{1/2}$ (high Reynolds number), so that the forcing and dissipation scales are well separated. If molecular viscosity is the only dissipation mechanism, then the dissipation of enstrophy is at least k_ν^2 times the dissipation of energy. On the other hand, the forcing of enstrophy is only λ times the forcing of energy. This would suggest that forced-dissipative equilibrium is unrealizable for high-Reynolds-number 2D Navier–Stokes turbulence. Yet, Constantin et al. [7,8] have proven the existence of a global attractor for this system for the case of time-independent forcing. Unless that case is pathological, it follows that the assumption of an enstrophy dissipation range confined to $k > k_\nu \gg \lambda^{1/2}$ must be wrong. In order to deduce the directions of energy and enstrophy cascades, one needs to know the spectral distribution of energy and enstrophy dissipation, and these are not preordained.

[3] Although the geometry considered in [9] is a sphere, interactions in the form of triads are to be observed [23]. For an interacting triad of similar scales where one wishes to have the smallest and largest scales as far away from the intermediate scale as possible, the scale ratio of 1:2:3 is a better approximation than 1:2:4. For such a case, the ratio of the upscale cascade energy to the downscale cascade energy is approximately 5:3.

4 Asymptotic behaviour: Dynamical constraint

We first derive the governing equation for the evolution of Λ in the absence of a forcing term. Taking the scalar product of (1) with u and Au, respectively, we obtain the governing equations for the decay of the energy and enstrophy for $f = 0$ (note that the nonlinear term vanishes in both cases due to the orthogonality properties):

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + 2\sum \nu_\mu \|u\|_\mu^2 = 0, \tag{23}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_1^2 + 2\sum \nu_\mu \|u\|_{1+\mu}^2 = 0. \tag{24}$$

Note that the subscript μ has been dropped from the sum ($\sum$) as there is no risk of confusion. Taking the time derivative of the expression for Λ and substituting (23) and (24) we obtain

$$\begin{aligned}\frac{\mathrm{d}\Lambda}{\mathrm{d}t} &= 2\sum \nu_\mu \frac{\|u\|_1^2\|u\|_\mu^2 - \|u\|^2\|u\|_{1+\mu}^2}{\|u\|^4} \\ &= -2\sum \nu_\mu \frac{G(\mu,1,\Lambda,u)}{\|u\|^2}.\end{aligned} \tag{25}$$

A couple of remarks are in order:

Remark. If Ekman drag ($\mu = 0$) is the only dissipation mechanism involved, then $\mathrm{d}\Lambda/\mathrm{d}t = 0$ because $G(0,1,\Lambda,u) = 0$. In this special case both enstrophy and energy are dissipated at the same rate $2\nu_0$. Since Λ remains constant, the distribution of energy and enstrophy should then be dramatically different from the viscous ($\mu > 0$) cases.

Remark. A viscosity causes Λ to decrease monotonically since $G(\mu,1,\Lambda,u) \geq 0$ for all $\mu > 0$, while an inverse viscosity has the opposite effect. If a single viscosity mechanism is involved, the rate of decay of Λ is greater for a more spread out spectrum as compared with a sharper spectrum having the same energy and enstrophy. This can be seen by the fact that $G(\mu,1,\Lambda,u)$ is greater in the former case.

We next consider the time-independent monoscale forcing $f \in H(\lambda_s)$, where $\lambda_s > \lambda_1$, considered by Constantin et al. [6]. For the arguments to follow we note that (1) possesses a stationary solution given by

$$\bar{u} = \left(\sum \nu_\mu \lambda_s^\mu\right)^{-1} f. \tag{26}$$

This stationary solution is referred to as the primary stationary solution. A stationary solution other than $\bar{u}$ (if it exists) is identified as a secondary stationary solution. The existence of such a solution for the traditional 2D Navier–Stokes system under suitable conditions is demonstrated in [10]. The energy and enstrophy evolve according to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u\right\|^2+\sum\nu_\mu\left\|u\right\|_\mu^2=(u,f), \tag{27}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u\right\|_1^2+\sum\nu_\mu\left\|u\right\|_{1+\mu}^2=(Au,f). \tag{28}$$

Since $(Au,f)=\lambda_s(u,f)$ in this case we can multiply the energy equation (27) by λ_s and subtract it from the enstrophy equation (28) to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left\|u\right\|_1^2-\lambda_s\left\|u\right\|^2\right\}+\sum\nu_\mu\left\{\left\|u\right\|_{1+\mu}^2-\lambda_s\left\|u\right\|_\mu^2\right\}=0. \tag{29}$$

This equation can be rewritten in terms of the function $G(\mu,1,\lambda_s,u)$ as

$$\frac{1}{2}\frac{\mathrm{d}\xi}{\mathrm{d}t}+\sum\nu_\mu\lambda_s^\mu\xi=-\sum\nu_\mu G(\mu,1,\lambda_s,u), \tag{30}$$

where $\xi\equiv\ \left\|u\right\|_1^2-\lambda_s\left\|u\right\|^2$. Now we require that μ be non-negative, i.e. no inverse viscosity is allowed, so that the right-hand side of (30) is non-positive. This means that any positive value of ξ will be dissipated away, while a non-positive value will evolve but remain non-positive for all time. To see this explicitly, consider the formal solution of (30), for all $t\geq t_0$, given below:

$$\begin{aligned}\xi(t)=\exp\left\{-2\sum\nu_\mu\lambda_s^\mu t\right\}\times \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad & (31)\\ \left(\xi_0\exp\left\{2\sum\nu_\mu\lambda_s^\mu t_0\right\}-2\int\limits_{t_0}^{t}\exp\left\{2\sum\nu_\mu\lambda_s^\mu t'\right\}\sum\nu_\mu G(\mu,1,\lambda_s,u)\mathrm{d}t'\right),\end{aligned}$$

where $\xi_0=\xi(t=t_0)$. It is easy to see that if $\xi_0\leq 0$ then $\xi(t)\leq 0$ for all time. On the other hand, if $\xi_0>0$ then $\xi(t)$ becomes non-positive for all $t\geq T$ where T is the solution of

$$\xi_0\exp\left\{2\sum\nu_\mu\lambda_s^\mu t_0\right\}=2\int\limits_{t_0}^{T}\exp\left\{2\sum\nu_\mu\lambda_s^\mu t'\right\}\sum\nu_\mu G(\mu,1,\lambda_s,u)\mathrm{d}t'. \tag{32}$$

Equation (32) always has a finite solution for T unless $\sum\nu_\mu G(\mu,1,\lambda_s,u)$ decreases exponentially in time more rapidly than $\exp\{-2\sum\nu_\mu\lambda_s^\mu t\}$. Since the function $G(\mu, 1, \lambda_s, u)$ vanishes if and only if $u \in H(\lambda_s)$, this can occur only if u asymptotically approaches (with respect to the various norms) the eigenspace $H(\lambda_s)$. Now $H(\lambda_s)$ is part of the stable manifold of $\bar{u} = (\sum \nu_\mu \lambda_s^\mu)^{-1} f$; this can be seen from the fact that $B(u,u) = 0$ for all $u \in H(\lambda_k)$, $\forall k$, so $H(\lambda_s)$ is invariant and every trajectory on it exponentially converges to $\bar{u}$ at the rate $2 \sum \nu_\mu \lambda_s^\mu$. It follows that any trajectories which asymptotically approach $H(\lambda_s)$ also converge to $\bar{u}$. Therefore, the cases for which a finite solution for T is questionable turn out to be restricted to trajectories on the stable manifold of $\bar{u}$.

The argument in the last paragraph indicates that for a general trajectory not on the stable manifold of $\bar{u}$, ξ acquires a non-positive value in a finite time. Within the stable manifold of $\bar{u}$, only $\bar{u}$ lies on the global attractor and $\xi = 0$ for $u = \bar{u}$. But before drawing a final conclusion for ξ on the global attractor it is necessary to secure the non-positiveness of ξ on homoclinic trajectories of $\bar{u}$ which, if they exist, are part of the global attractor. Recall that a homoclinic trajectory (also known as a homoclinic orbit) of a stationary solution is one that emanates from the stationary solution and terminates on that same stationary solution. It is a special limit cycle [4] with infinite period. It can be seen from (31) that ξ does not acquire a positive value on any homoclinic orbit of $\bar{u}$ during its infinitely long journey, if such an orbit exists. It would, rather, start from $\bar{u}$ with $\xi = 0$, evolve with some negative value for ξ, and then ξ would increase to terminate at $\bar{u}$ with $\xi = 0$ again.

Therefore, it can be concluded that on the global attractor of the Navier–Stokes system, the energy and enstrophy satisfy

$$\|u\|_1^2 \leq \lambda_s \|u\|^2 . \tag{33}$$

This inequality will be referred to as the dynamical constraint. It is noted, for the sake of completeness, that the argument leading to (33) also implies that $\bar{u}$ is the only point on the global attractor where the equality occurs. Inequality (33), together with the Poincaré inequality, gives

$$\lambda_1 \|u\|^2 \leq \|u\|_1^2 \leq \lambda_s \|u\|^2 . \tag{34}$$

An immediate corollary of the dynamical constraint is a similar constraint on the enstrophy of a secondary stationary solution if one exists. In fact,

[4] Note that a regular limit cycle which would intersect $H(\lambda_s)$ does not exist because $H(\lambda_s)$ is part of the stable manifold of $\bar{u}$. Alternatively, the existence of such a cycle would imply from (31) that $\int_0^{T_0} \exp\{2 \sum \nu_\mu \lambda_s^\mu t'\} \sum \nu_\mu G(\mu, 1, \lambda_s, u) \mathrm{d}t' = 0$, where T_0 is the period of the cycle. But this would require the cycle to be entirely in $H(\lambda_s)$, which would further reduce the cycle to $\bar{u}$.

(33) becomes a strict inequality for any secondary stationary solution. More interpretation of (33) is given in the next section.

Remark. The addition of an inverse viscosity to a system which possesses a global attractor does not jeopardize its existence. However, the spectral distribution of energy and enstrophy on the attractor does not necessarily obey the dynamical constraint in this case.

Remark. The requirement of a time-independent f is merely for the sake of securing the existence of a global attractor. Equation (33) is a constraint on the long-time behaviour of the dynamics whether or not the monoscale forcing $f \in H(\lambda_s)$ is time-independent.

We now extend the above result to the case of a more general monoscale-like forcing as defined in the Introduction. It is easy to see that when λ is constant the above analysis applies with λ in place of λ_s. (In general, there will not exist a stationary solution as in the case of a monoscale forcing.) For variable λ, we have the a priori inequalities (by hypothesis)

$$\lambda_{\min}(u, f) \le (Au, f) \le \lambda_{\max}(u, f), \quad (u, f) \ge 0, \tag{35}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are, respectively, the minimum and maximum λ_k of the restricted set K of wavenumbers through which energy and enstrophy are injected. Note that with variable λ, we find it necessary to require the condition $(u, f) \ge 0$ at all times. Then (27), (28), and (35) lead to (30) with the equality replaced by $\le$ and with λ_s replaced by $\lambda_{\max}$. It follows that $\xi(t)$ is bounded from above by zero, for sufficiently large time, and thus the dynamical constraint (33) must hold on the attractor (if it exists) with λ_s replaced by $\lambda_{\max}$.

Finally, if (35) only holds in a time-mean sense — note that the time-mean energy injection $\langle (u, f) \rangle \ge 0$ necessarily since the energy dissipation is non-negative — then we may take the time mean of (30) (with λ_s replaced by $\lambda_{\max}$ and the equality replaced by $\le$) to obtain $\langle \xi \rangle \le 0$, equivalently

$$\langle \|u\|_1^2 \rangle \le \lambda_{\max} \langle \|u\|^2 \rangle, \tag{36}$$

and the dynamical constraint holds in the time mean.

5 Dissipation rates and spectral distributions

In this section we examine the time-mean dissipation rates of energy and enstrophy and possible spectral slopes of energy for three combinations of generalized viscosity: (i) molecular viscosity alone, (ii) molecular viscosity and Ekman drag, and (iii) molecular viscosity and inverse viscosity. Since we are assessing the validity of the KLB theory, which applies to statistically stationary turbulence, we may assume that all time-mean quantities exist.

For simplicity we first treat the case of monoscale forcing as in the last section, leaving the question of the applicability of the results to the more general monoscale-like forcing until the end of the section. For the first two cases, the results from the last section indicate that both energy and its dissipation are confined to scales no smaller than the injection scale. In particular, the dynamical constraint gives

$$\bar{\Lambda} \equiv \frac{\langle\|u\|_1^2\rangle}{\langle\|u\|^2\rangle} \leq \lambda_s. \tag{37}$$

This means that in both cases the spectrum adjusts so that the dissipation of energy takes place on scales no smaller than the forcing scale. Moreover, the energy dissipation rates are bounded from above by $2\nu_1\lambda_s$ for case (i) and by $2\nu_1\lambda_s + 2\nu_0$ for case (ii).

We know that viscosity acts on smaller scales for enstrophy than it does for energy. The question is where, in wavenumber space, the enstrophy dissipation takes place. To answer this question we resort to (29). Since both energy and enstrophy are bounded from above on the global attractor, taking the time mean of (29) leads to

$$\sum \nu_\mu \left\{\langle\|u\|_{1+\mu}^2\rangle - \lambda_s\langle\|u\|_\mu^2\rangle\right\} = 0. \tag{38}$$

This equation is referred to as the balance equation. Attempts to constrain the time-mean enstrophy dissipation rate and the scaling law of the energy spectrum constitute the remainder of this section.

Case (i): The balance equation reads in this case

$$\langle\|u\|_2^2\rangle = \lambda_s\langle\|u\|_1^2\rangle, \tag{39}$$

so

$$\bar{\Lambda}_{Z1} \equiv \frac{\langle\|u\|_2^2\rangle}{\langle\|u\|_1^2\rangle} = \lambda_s. \tag{40}$$

This well-known result corresponds to (6) of Constantin et al. [6]. It implies that the time-mean enstrophy dissipation rate is given by $2\nu_1\lambda_s$, and that the enstrophy dissipation is concentrated around the forcing scale. It follows that there can be no direct enstrophy cascade. Moreover, this constraint gives rise to a very steep spectral slope for $k > s$. Let $E(k)$ be defined by

$$E(k) = \langle \|P(\lambda_k)u\|^2 \rangle$$

so that the total energy $\mathcal{E}$ is given by

$$\mathcal{E} = \int_{\sqrt{\lambda_1}}^{\infty} E(k)\mathrm{d}k,$$

where the discrete wavenumber has been changed to a continuous one for the sake of convenience. Now suppose that $E(k) \propto k^{-\delta}$ over a range $[k_0, k_1]$ where $s < k_0 \ll k_1$; the dissipation of enstrophy over that range, D_Z, is then given by

$$D_Z \propto \int_{k_0}^{k_1} k^{4-\delta}\mathrm{d}k \;=\; \frac{1}{5-\delta}\left(k_1^{5-\delta} - k_0^{5-\delta}\right) \quad [\delta \neq 5]. \tag{41}$$

Equation (40) requires that D_Z be dominated by the large scales. As a result, the satisfactory values of δ are confined to $\delta > 5$. ($\delta = 5$ is not acceptable.)

Remark. A scaling law of the energy spectrum $k^{-\delta}$ with $\delta > 5$ is much steeper than the classical k^{-3} inertial range power law (see Kraichnan [11,12]). Hence, the result of this section suggests that the latter scaling law is unrealizable in 2D Navier–Stokes turbulence for a constant monoscale forcing. This result was derived by Constantin et al. [6].

Remark. For a viscosity of degree $\mu \geq 0$ instead of molecular viscosity, which includes Ekman drag ($\mu = 0$), the satisfactory values of δ are confined to $\delta > 3 + 2\mu$.

Remark. Although an inverse energy cascade cannot be excluded, we can rule out the classical $k^{-5/3}$ power law [11] for any such range by a similar argument. Suppose that $E(k) \propto k^{-\gamma}$ over a range $[k_0, k_1]$ where $k_0 \ll k_1 < s$; the dissipation of energy over that range, D_E, is then given by

$$D_E \propto \int_{k_0}^{k_1} k^{2-\gamma}\mathrm{d}k \;=\; \frac{1}{3-\gamma}\left(k_1^{3-\gamma} - k_0^{3-\gamma}\right) \quad [\gamma \neq 3]. \tag{42}$$

An inverse cascade of energy requires that D_E be dominated by the small-k end of the range, which requires $\gamma > 3$. ($\gamma = 3$ is not acceptable.) Therefore, the $k^{-5/3}$ power law for 2D Navier–Stokes turbulence is incompatible with an inverse energy cascading range. This result is clearly independent of the nature of the forcing.

Case (ii): The balance equation reads in this case

$$\nu_1 \left\{ \langle \|u\|_2^2 \rangle - \lambda_s \langle \|u\|_1^2 \rangle \right\} + \nu_0 \left\{ \langle \|u\|_1^2 \rangle - \lambda_s \langle \|u\|^2 \rangle \right\} = 0, \tag{43}$$

so

$$\nu_1 \bar{\Lambda}_{Z1} + \nu_0 = \nu_1 \lambda_s + \frac{\nu_0 \lambda_s}{\bar{\Lambda}} \geq \nu_1 \lambda_s + \nu_0. \tag{44}$$

This implies that the time-mean enstrophy dissipation rate is bounded from below by $2\nu_1\lambda_s + 2\nu_0$. Equivalently, we have $\bar{\Lambda}_{Z1} \geq \lambda_s$. In other words, Ekman drag allows the spectrum to adjust toward the small scales (as compared with case (i) for which $\bar{\Lambda}_{Z1} = \lambda_s$), so that the enstrophy dissipation can occur at scales smaller than the forcing scale. To see the extent of such an adjustment we rewrite the equality in (44) as

$$\frac{\bar{\Lambda}_{Z1}}{\lambda_s} = 1 + \frac{\nu_0}{\nu_1 \lambda_s} \left(\frac{\lambda_s}{\bar{\Lambda}} - 1 \right). \tag{45}$$

The ratio $\bar{\Lambda}_{Z1}/\lambda_s > 1$ because $\lambda_s/\bar{\Lambda} > 1$ (strictly speaking we need to exclude the primary stationary solution $\bar{u}$, the only point on the global attractor for which $\lambda_s/\bar{\Lambda} = 1$). The question is whether $\bar{\Lambda}_{Z1}/\lambda_s$ can be much greater than unity for a system with $\nu_0 \gg \nu_1 \lambda_s$, where the large-scale dissipation is dominated by Ekman drag. This is not obvious since we expect that $\lambda_s/\bar{\Lambda} \to 1$ as $\nu_0/(\nu_1\lambda_s) \to \infty$ because the dynamical constraint is $\bar{\Lambda} = \lambda_s$ when Ekman drag alone is responsible for dissipation. However, we do not know exactly how this limit is approached as $\nu_0/(\nu_1\lambda_s) \to \infty$. Nevertheless, it appears that in principle a direct enstrophy cascade could be achieved in this system.

We now examine implications for scaling of the energy spectrum. For an energy cascading range $k_0 \ll k_1 < s$, the energy dissipation is given, with the same notation as before, by

$$D_E \propto \int_{k_0}^{k_1} \left(\nu_0 k^{-\gamma} + \nu_1 k^{2-\gamma} \right) \mathrm{d}k$$

$$= \frac{\nu_0}{1-\gamma}\left(k_1^{1-\gamma} - k_0^{1-\gamma}\right) + \frac{\nu_1}{3-\gamma}\left(k_1^{3-\gamma} - k_0^{3-\gamma}\right) \quad [\gamma \neq 1, 3]. \tag{46}$$

We have already assumed that $\nu_0 \gg \nu_1 \lambda_s$ so that Ekman drag dominates on the large scales, and therefore the energy dissipation occurs primarily on the large scales provided $\gamma > 1$. On the other hand, we want negligible dissipation of enstrophy on the large scales, and this requires $\gamma < 3$. Thus $1 < \gamma < 3$, which includes the $\gamma = 5/3$ power law of the KLB theory.

For an enstrophy-cascading range $s < k_0 \ll k_1$, the enstrophy dissipation is given by

$$\begin{aligned} D_Z &\propto \int_{k_0}^{k_1} \left(\nu_0 k^{2-\delta} + \nu_1 k^{4-\delta}\right) \mathrm{d}k \\ &= \frac{\nu_0}{3-\delta}\left(k_1^{3-\delta} - k_0^{3-\delta}\right) + \frac{\nu_1}{5-\delta}\left(k_1^{5-\delta} - k_0^{5-\delta}\right) \quad [\delta \neq 3, 5]. \end{aligned} \tag{47}$$

It is seen that enstrophy dissipation will occur primarily on the small scales provided $\delta < 5$. ($\delta = 5$ is not acceptable.) On the other hand, we want negligible dissipation of energy on the small scales, and this requires $\delta > 3$. ($\delta = 3$ is not acceptable.) Thus $3 < \delta < 5$, and the $\delta = 3$ power law of the KLB theory (with or without the logarithmic correction of [12]) is not permitted. In fact, it is in any case excluded (as in case (i)) by the dynamical constraint (37).

Case (iii): Similar to case (ii) we have

$$\frac{\bar{\Lambda}_{Z1}}{\lambda_s} = 1 + \frac{\nu_\mu \bar{\Lambda}^\mu_{E\mu}}{\nu_1 \lambda_s}\left(\frac{\lambda_s}{\bar{\Lambda}} - \frac{\bar{\Lambda}^\mu_{Z\mu}}{\bar{\Lambda}^\mu_{E\mu}}\right), \tag{48}$$

where

$$\bar{\Lambda}^\mu_{E\mu} \equiv \frac{\langle \|u\|^2_\mu \rangle}{\langle \|u\|^2 \rangle}, \quad \bar{\Lambda}^\mu_{Z\mu} \equiv \frac{\langle \|u\|^2_{1+\mu} \rangle}{\langle \|u\|^2_1 \rangle}. \tag{49}$$

Unlike case (ii) where only one ratio on the right-hand side of (45), namely $\lambda_s/\bar{\Lambda}$, depends on the actual spectrum, all three ratios on the right-hand side of (48) are spectrum dependent. Among them we expect $\bar{\Lambda}^\mu_{Z\mu}/\bar{\Lambda}^\mu_{E\mu}$ to be smaller than unity for negative μ (see section 2). Now the ratio $\lambda_s/\bar{\Lambda}$ is not always greater than unity, and depends on the outcome of the competition between molecular viscosity and inverse viscosity. Nevertheless, for suitable values of ν_μ and μ a dual cascade certainly appears to be possible. In this case, energy cascades upscale and is dissipated by inverse viscosity, while enstrophy cascades downscale and is dissipated by molecular viscosity. Of course, inverse viscosity has no physical basis and is employed purely for numerical reasons.

We now extend the results of this section to the more general case of a monoscale-like forcing. It is only necessary to do so for cases (i) and (ii) since those are where we have derived restrictions. It is easy to see that everything goes through when λ is constant. For variable λ, or if the condition of monoscale-like forcing holds only in the time mean, the balance equation (38) may be replaced by

$$\sum \nu_\mu \left\{ \langle \|u\|^2_{1+\mu} \rangle - \lambda_{\max} \langle \|u\|^2_\mu \rangle \right\} \leq 0. \tag{50}$$

The arguments concerning admissible spectral slopes go through without change since they do not refer at all to the forcing mechanism; in the definitions of the inertial subranges the forcing wavenumber s is replaced with either $\lambda_{\min}^{1/2}$ or $\lambda_{\max}^{1/2}$, as appropriate. This leaves only the prohibition of a direct enstrophy cascade in case (i). There one obtains $\bar{\Lambda}_{Z1} \leq \lambda_{\max}$ in place of (40), which still precludes a direct enstrophy cascade.

6 Concluding remarks

In this paper we have derived various constraints on the asymptotic behaviour of the 2D Navier–Stokes equations in a finite domain driven by a monoscale-like forcing and damped by a general class of dissipation operators. By monoscale-like forcing we mean that the ratio of enstrophy to energy injection lies within the range of the square of the forcing wavenumbers, for all possible velocity fields u, which is a common (though not exclusive) scenario in the KLB theory (e.g. Kraichnan [11], p.1421b; Pouquet et al. [21], p.314; and Lesieur [15], p.291). Note that this crucial property is *not* guaranteed for a constant external forcing, except in the special case of a monoscale forcing. The results obtained include constraints on the time-mean enstrophy dissipation range, an upper bound on the system's enstrophy in terms of the energy and the forcing scale, and bounds on possible scaling laws for the energy spectrum. If the condition of monoscale-like forcing holds only in the time mean, then the upper bound on the enstrophy holds in the time mean rather than asymptotically. The validity of the dual cascade picture is examined by determining where, in wavenumber space, dissipation of energy and enstrophy occurs.

For 2D Navier–Stokes turbulence, with molecular viscosity the only form of dissipation, no direct enstrophy cascade is permitted; rather, enstrophy dissipation is required to occur in the vicinity of the forcing scale. Although a reverse energy cascade is permitted (not to say that it occurs), it is shown to be incompatible with the $-5/3$ power-law scaling of KLB theory for the energy spectrum in such a range (as noted previously by Constantin et al. [6]).

Ekman drag has often been employed in numerical simulations of 2D turbulence to allow a second dissipation channel. Furthermore, Ekman drag is a reasonable representation of a frictional planetary boundary layer in the geophysical context. We show rigorously how the introduction of Ekman drag together with viscosity breaks the strong constraints on the spectral distribution of enstrophy dissipation, and thereby allows the possibility of a dual cascade. However, the rigorous bound on the enstrophy precludes the existence of the -3 power-law scaling of KLB theory (with or without a logarithmic correction) for the energy spectrum in an enstrophy-cascading range. Instead, the power law is shown to be between -3 and -5. This is consistent with the published results of numerical simulations.

It is shown that the use of an inverse viscosity together with viscosity allows the possibility of KLB power-law scaling in the enstrophy-cascading range. Indeed, the numerical simulations of Borue [4], and the more recent high-resolution numerical simulations of Lindborg and Alvelius [17], which both claim to exhibit this scaling, employ an inverse viscosity. Our results suggest that this is not a coincidence.

It should be emphasized that our derived constraints on power laws for the energy spectrum apply for any spectrally localized forcing, not just for monoscale-like forcing.

The results derived in this paper are believed to be particular to 2D turbulence, because they all rely on the conservation of both energy and enstrophy in nonlinear interactions, as expressed through (27) and (28) and subsequent key relations such as (29), (38), and (50). There is no analogue of (28) in 3D turbulence.

Constantin et al. [6] have shown that for the special case of a spectrally-localized constant external forcing, the realization of a direct enstrophy cascade in forced 2D Navier–Stokes turbulence in a finite domain requires a combination of energy input and energy removal by the forcing. The generalization of this result to our case is that a direct enstrophy cascade within forced 2D Navier–Stokes turbulence requires the abandonment of the popular concept of monoscale-like forcing (as defined here). Of course, the $-5/3$ power-law scaling remains incompatible with an upscale energy cascade, irrespective of the forcing mechanism.

A general result of this study is that the choice of forcing mechanisms and dissipation operators has implications for the spectral distribution of energy and enstrophy dissipation, and thus for the possible existence of energy and enstrophy cascades. Furthermore, the choice of dissipation operators has implications for permissible scalings of the energy spectrum. In choosing forcing mechanisms and dissipation operators for numerical reasons, one should be mindful of these constraints.

Acknowledgements

The work reported here represents part of CVT's Ph.D. thesis at the University of Toronto, which was supported by University of Toronto Open Fellowships and Department of Physics Burton Fellowships. TGS would like to acknowledge support from the Natural Sciences and Engineering Research Council and the Meteorological Service of Canada. Both authors are grateful to John Bowman, Joe Tribbia, Jeff Weiss, and particularly Ricardo Rosa for helpful comments on the manuscript.

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