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STABILITY OF STATIONARY SOLUTIONS OF THE FORCED NAVIER-STOKES EQUATIONS ON THE TWO-TORUS

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Abstract. We study the linear and nonlinear stability of stationary solutions of the forced two-dimensional Navier-Stokes equations on the domain $[0, 2\pi] \times [0, 2\pi/\alpha]$, where $\alpha \in (0, 1]$, with doubly periodic boundary conditions. For the linear problem we employ the classical energy–enstrophy argument to derive some fundamental properties of unstable eigenmodes. From this it is shown that forces of pure x_2-modes having wavelengths greater than 2π do not give rise to linear instability of the corresponding primary stationary solutions. For the nonlinear problem, we prove the equivalence of nonlinear stability with respect to the energy and enstrophy norms. This equivalence is then applied to derive optimal conditions for nonlinear stability, including both the high- and low-Reynolds-number limits.

1. **Introduction.** We consider 2D incompressible fluid flow in a doubly periodic rectangular domain $T^2 = [0, 2\pi] \times [0, 2\pi/\alpha]$, where $\alpha \in (0, 1]$. The fluid is assumed to be driven by a monoscale forcing and damped by various dissipation mechanisms including Ekman drag (a linear mechanical friction), hypoviscosity, molecular viscosity, and hyperviscosity. The 2D Navier-Stokes equations which govern the fluid motion are written in an abstract form in a function space H as

$$\frac{du}{dt} + B(u, u) + A^\eta u = f, \;\; u(t = 0) = u_0. \tag{1}$$

A detailed description of the functional analysis setting for (1) is given in [1, 9, 10]. We recall that H is the L^2-space of periodic, non-divergent functions representing the velocity u with vanishing average in T^2. $B(u, u) = P((u \cdot \nabla)u)$ where P is the orthogonal projection in L^2 onto H, and $A = -P\Delta = -\Delta P$. The number η will be called the degree of viscosity. When $\eta = 1$ we have the usual molecular viscosity, while $\eta = 0$ corresponds to Ekman drag. The cases $\eta > 1$ and $\eta < 1$ correspond to hyperviscosity and hypoviscosity, respectively. Note that the generalized viscosity coefficient is taken to be unity.

The eigenfunctions of A which form an orthonormal basis of H are given by (see [6, 7])

$$e_k = \frac{\sqrt{\alpha}}{\sqrt{2}\,\pi|k|} k' \cos k \cdot x, \tag{2}$$

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$$e'_k = \frac{\sqrt{\alpha}}{\sqrt{2}\,\pi|k|} k' \sin k \cdot x, \tag{3}$$

where $k = (k_1, \alpha k_2)^T$, k_1 and k_2 are integers satisfying either $k_1 > 0$ or $k_1 = 0$ and $k_2 > 0$, and $k' = (\alpha k_2, -k_1)^T$. For convenience this set of wavevectors k will be denoted by K. We denote by H^γ the domain of definition of $A^{\gamma/2}$ for real γ. The (degenerate) eigenvalues of A are $|k|^2$ and the eigenspace corresponding to $|k|^2$ is denoted by $H(|k|)$. We will occasionally refer to $|k|$ as a wavenumber and $2\pi/|k|$ as the length scale (or wavelength) associated with the wavenumber $|k|$. In this study f is assumed to be a monoscale body force; i.e., $f \in H(|s|)$ for some wavenumber $|s|$. Further restrictions on f for the particular cases considered in this article will be stated in due course.

The Fourier representation of u is given by

$$u = \sum_{k\in K} (u_k e_k + u'_k e'_k). \tag{4}$$

To facilitate the linear analysis we allow u_k and u'_k to take complex values. The scalar product and the norm in H^γ are given respectively by

$$(u, v)_\gamma = \int_{T^2} u^* \cdot A^\gamma v \, dx = \sum_{k\in K} |k|^{2\gamma} (u_k^* v_k + u'^*_k v'_k), \tag{5}$$

$$\|u\|_\gamma = (u, u)_\gamma^{1/2} = \left(\sum_{k\in K} |k|^{2\gamma} (|u_k|^2 + |u'_k|^2) \right)^{1/2}. \tag{6}$$

The cases where $\gamma = 0, 1$ are special and the corresponding H-norm (superscript and subscript '0' are omitted in this case) and H^1-norm are known in the literature as the energy and enstrophy norm, respectively. A geometric constraint, referred to as the Poincaré inequality, is

$$\|u\|^2_{\gamma+\beta} \geq \lambda^\gamma_{\min} \|u\|^2_\beta \tag{7}$$

for non-negative γ, where $\lambda^{1/2}_{\min}$ (which generally depends on u) is the minimum wavenumber in the Fourier representation of u. In order to apply to arbitrary u one must take $\lambda_{\min} = \alpha^2$, being the first eigenvalue of A. The inequality reverses direction for non-positive γ.

Let $\beta, \beta' \in \Re$ and $\kappa > 0$. Let $\phi \in H^\gamma$ where $\gamma = \mathrm{Max}\{\beta, \beta', \beta + \beta'\}$. We define (see also [12])

$$G(\beta, \beta', \kappa, \phi) = \sum_{|k|} (\kappa^{2\beta} - |k|^{2\beta})(\kappa^{2\beta'} - |k|^{2\beta'}) \|P(|k|)\phi\|^2, \tag{8}$$

where $P(|k|)\phi$ is the projection of ϕ onto the eigenspace $H(|k|)$. Note that $G(\beta, \beta', \kappa, \phi)$ is positive (negative) if and only if $\beta\beta' > 0$ ($\beta\beta' < 0$) and if $\phi \notin H(\kappa)$ if κ^2 happens to be an eigenvalue of A. This means that when $\beta\beta' \neq 0$, $G(\beta, \beta', \kappa, \phi) = 0$ if and only if κ^2 is an eigenvalue of A and $\phi \in H(\kappa)$. By rearranging terms we obtain

$$G(\beta, \beta', \kappa, \phi) = \|\phi\|^2_{\beta+\beta'} - \kappa^{2\beta'} \|\phi\|^2_\beta - \kappa^{2\beta}(\|\phi\|^2_{\beta'} - \kappa^{2\beta'} \|\phi\|^2). \tag{9}$$

In particular,

$$G(1, \eta, |s|, \phi) = \|\phi\|^2_{1+\eta} - |s|^2 \|\phi\|^2_\eta - |s|^{2\eta}(\|\phi\|^2_1 - |s|^2 \|\phi\|^2). \tag{10}$$

The following identities are well-known for real $u,\ v,\ w$ and are used repeatedly in what follows. The bilinear operator $B(\cdot,\cdot)$ satisfies

$$(Au, B(v,v)) + (Av, B(v,u)) + (Av, B(u,v)) = 0 \tag{11}$$

for $u, v \in H^2$, and

$$(u, B(v,w)) = -(w, B(v,u)) \tag{12}$$

for $u, w \in H^1$ and $v \in H$. These identities arise by virtue of the non-divergent and periodic properties of the velocity field. In particular, we have

$$(u, B(v,u)) = 0, \tag{13}$$

$$(Au, B(u,u)) = 0. \tag{14}$$

We shall collectively refer to the above identities as the orthogonality properties of the nonlinear term. In the absence of forcing and viscosity, they give rise to the conservation of energy and enstrophy.

2. **Linear stability.** Equation (1) possesses the stationary solution

$$\bar{u} = |s|^{-2\eta} f, \tag{15}$$

which will be referred to as the primary stationary solution. A stationary solution other than $\bar{u}$ (if it exists) will be referred to as a secondary stationary solution. (The existence of such a solution for the traditional 2D Navier-Stokes system under suitable conditions is demonstrated in [4].) Let the solution of (1) be written in the form

$$u = \bar{u} + v. \tag{16}$$

Then the governing equation for the deviation v from $\bar{u}$ reads

$$\frac{dv}{dt} = -B(\bar{u}, v) - B(v, \bar{u}) - B(v,v) - A^{\eta} v. \tag{17}$$

Let the linear operator in the above eq. be denoted by $L(\bar{u})$, viz.,

$$L(\bar{u})v = -B(\bar{u}, v) - B(v, \bar{u}) - A^{\eta} v. \tag{18}$$

For the linear stability problem the focus is on the eigenvalue problem

$$L(\bar{u})w = \sigma w, \tag{19}$$

where a positive (negative) real part of σ corresponds to an unstable (stable) eigenmode.

We now consider the special case $f = f_0 e'_s$ which corresponds to the stationary solution $\bar{u} = \bar{u}_s e'_s = |s|^{-2\eta} f_0 e'_s$, where $f_0 > 0$. Let an eigenvector be written in the Fourier representation (4). Then with reference to Liu's lemma in the appendix, it is seen that the odd and even components of the disturbance are separable and both satisfy the eigenvalue problem (19). Hence, it is sufficient to consider either one, say the even part in which case $w = \sum_{k\in K} w_k e_k$. Substituting this into (19), using Liu's lemma with $l = s$, and noting that $s' \cdot s = 0$, we obtain the relation

$$(|k|^{2\eta} + \sigma) w_k - \frac{\bar{u}_s \sqrt{\alpha}\, k' \cdot s}{2\sqrt{2}\,\pi |k||s|} \left[\frac{|k-s|^2 - |s|^2}{|k-s|} w_{k-s} - \frac{|k+s|^2 - |s|^2}{|k+s|} w_{k+s} \right] = 0. \tag{20}$$

This equation gives a three-term recurrence relation between $w_{k_0+(n-1)s}$, w_{k_0+ns}, and $w_{k_0+(n+1)s}$ for a given k_0. (Note that when $k-s \notin K$ then this mode appears as $s-k \in K$ with $w_{k-s} = -w_{s-k}$ because $e_{k-s} = -e_{s-k}$; see Figure 1.) Equation (20) is derived (for slightly different forms of $\bar{u}$) and employed in studying the unstable

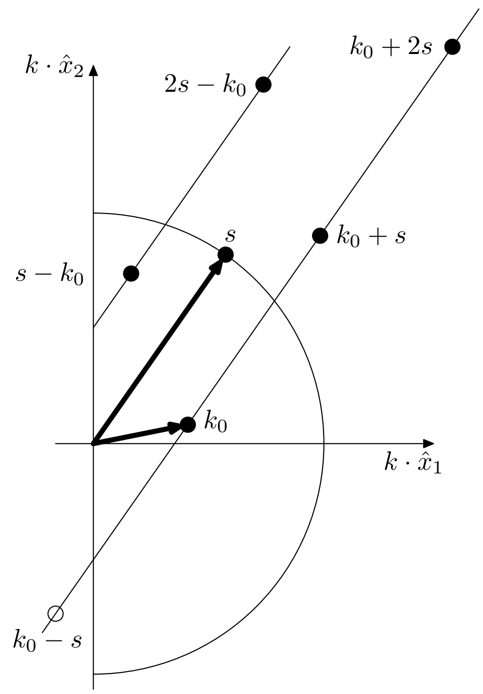

FIGURE 1. The wavevectors in the sequence $k_n = k_0 + ns$ of an unstable eigenmode. The domain K consists of the right half-plane, minus the lower half of the ordinate and the origin. When $k_n \notin K$ (for $n \leq -1$ in this figure) then $-k_n$ appears in the Fourier series of the unstable eigenmode. k_0 is designated as the member of the sequence with the smallest wavenumber. The semicircle has radius $|s|$. At most two of the members of this sequence can lie within the semicircle (k_0 and $-k_{-1}$ in this figure).

manifold, eigenvalue problem, and the bifurcation of $\bar{u}$ via the continued fraction method in [4, 5, 6, 7, 8].

Let σ_r denote the real part of σ. We see that for $\sigma_r \geq 0$ (20) has no non-trivial solutions when $k' \cdot s = 0$. Thus, for unstable eigenmodes w must take the form

$$w(k_0) = \sum_n w_n e_{k_n}, \tag{21}$$

where n is an integer and where $k_n = k_0 + ns$ for some $k_0 \in K$ with $k_0' \cdot s \neq 0$. Without loss of generality we may designate $|k_0|$ as the smallest wavenumber of the sequence $\{|k_n|\}$. We observe that there exist at most two wavenumbers $\leq |s|$ in the sequence $\{|k_n|\}$ (see Figure 1). One of these smallest wavenumbers is, by definition, $|k_0|$ and the other is either $|k_1|$ or $|k_{-1}|$, depending on the angle between s and k_0. From the observation in this paragraph two important properties of the eigenvalue problem (19) follow.

First, the Fourier series of an unstable eigenmode is not terminating, i.e.,

$$w_n \neq 0, \ \forall n. \tag{22}$$

Indeed, suppose otherwise that there is an N such that $w_N = 0$. We then write $w = p + q$, where $p = \sum_{n<N} w_n e_{k_n}$ and $q = \sum_{n>N} w_n e_{k_n}$. Since only $|k_0|$ and either $|k_{-1}|$ or $|k_1|$ can be $\leq |s|$, either p or q consists of no Fourier modes of wavenumbers $< |s|$. Let us therefore assume, without loss of generality, that p satisfies this condition. Taking the scalar product of p with (19) in H and H^1, respectively, we obtain (noting the separability of p and q)

$$\begin{aligned}\sigma \|p\|^2 + \|p\|_\eta^2 &= (\bar{u}, B(p_r,p_r) + B(p_i,p_i)) \\ &\quad -i(p_r, B(\bar{u},p_i) + B(p_i,\bar{u})) + i(p_i, B(\bar{u},p_r) + B(p_r,\bar{u})),\end{aligned} \tag{23}$$

$$\begin{aligned}\sigma \|p\|_1^2 + \|p\|_{1+\eta}^2 &= (A\bar{u}, B(p_r,p_r) + B(p_i,p_i)) \\ &\quad -i(Ap_r, B(\bar{u},p_i) + B(p_i,\bar{u})) + i(Ap_i, B(\bar{u},p_r) + B(p_r,\bar{u})),\end{aligned} \tag{24}$$

where the orthogonality properties have been employed for the real (p_r) and imaginary parts (p_i) of p. Multiplying (23) by $|s|^2$ and subtracting (24), noting that the real parts on the right-hand sides of (23) and (24) cancel, we obtain

$$\sigma_r(|s|^2 \|p\|^2 - \|p\|_1^2) = \|p\|_{1+\eta}^2 - |s|^2 \|p\|_\eta^2 . \tag{25}$$

For $p \neq 0$ the right-hand side of (25) is positive due to the Poincaré inequality. Likewise, the difference in the brackets on the left-hand side of (25) is negative. So, $\sigma_r \geq 0$ requires that $p = 0$. From the recurrence relation it is easy to see that $p = 0 \Longrightarrow q = 0$. Hence, the Fourier series of an unstable eigenmode of $\bar{u}$ is non-terminating.

Second, there exist no nontrivial unstable solutions (21) to (19) for $|k_0| \geq |s|$. To see this we repeat the steps leading to (25) with p replaced by w and obtain

$$\sigma_r(|s|^2 \|w\|^2 - \|w\|_1^2) = \|w\|_{1+\eta}^2 - |s|^2 \|w\|_\eta^2 . \tag{26}$$

For $w \neq 0$ the right-hand side of (26) is positive due to the Poincaré inequality. Likewise, the difference in the brackets on the left-hand side of (26) is negative. Hence, for $\sigma_r \geq 0$ there is no non-trivial solution to (19) in this case. As a consequence of this, it is interesting to note that $\bar{u}$ possesses no unstable eigenmodes for the special case $s = (0, \alpha s_2)^T$, where $\alpha s_2 \leq 1$, as there exist no $k_0 \in K$ such that $|k_0| < |s|$ and $k_0' \cdot s \neq 0$.

When there exists an unstable eigenmode w, (26) implies that the signs of $(|s|^2 \|w\|^2 - \|w\|_1^2)$ and $(\|w\|_{1+\eta}^2 - |s|^2 \|w\|_\eta^2)$ are the same. When $\eta = 0$, (26) leads directly to

$$|s|^2 \|w\|^2 - \|w\|_1^2 = 0. \tag{27}$$

When $\eta > 0$, (10) together with $G(1, \eta, |s|, w) > 0$ (since $w \notin H(|s|)$) rules out the possibility of a negative sign, leading to

$$|s|^2 \|w\|^2 - \|w\|_1^2 > 0 \text{ and } \|w\|_{1+\eta}^2 - |s|^2 \|w\|_\eta^2 > 0. \tag{28}$$

In conclusion, we have proven the following

Theorem 2.1. *(I) The primary stationary flow $\bar{u} = \bar{u}_s e_s'$ of the Navier-Stokes system, for $s = (0, \alpha s_2)^T$ with $\alpha s_2 \leq 1$, is linearly stable for arbitrary values of $\bar{u}_s$.*

(II) Unstable eigenvectors of the eigenvalue problem $L(\bar{u})w = \sigma w$ satisfy (27) for $\eta = 0$ or (28) for $\eta > 0$. Moreover, the real part of the eigenvalue, σ_r, is given by

$$\sigma_r = \frac{\|w\|_{1+\eta}^2 - |s|^2 \|w\|_\eta^2}{|s|^2 \|w\|^2 - \|w\|_1^2}, \tag{29}$$

for $\eta > 0$.

REMARK 1. *The arguments leading to* $w_n \neq 0$, $\forall n$ *do not make use of the dissipation term in an essential manner. In particular, if the right-hand side of (25) is identically zero (i.e., the dissipation term is absent), then* $\sigma_r > 0$ *still implies* $p = 0$, *and the subsequent arguments follow. Hence, for the inviscid case, the Fourier series of an unstable eigenmode of the basic flow* $\bar{u}$ *is non-terminating. Furthermore, the spectrum of such an unstable eigenmode satisfies*

$$|s|^2 \|w\|^2 - \|w\|_1^2 = 0. \tag{30}$$

Since the eigenmode cannot be entirely in $H(|s|)$, *for which (30) would hold trivially, this implies that an unstable disturbance must have a component with a wavenumber smaller than* $|s|$. *This recovers a well-known result in the inviscid linear problem (see, for example,* [2, 3]*).*

REMARK 2. *A basic flow of mode* $(0, \alpha s_2)^T$ *is an* x_1*-directed flow of cross-flow* (x_2) *wavelength* $2\pi/\alpha s_2$. *There are* N *such linearly stable basic flows where* N *is the largest integer* $\leq 1/\alpha$. *We note that* $\alpha s_2 \leq 1$ *corresponds to a wavelength in* x_2 *that is greater than or equal to the domain size in* x_1. *As noted in the previous remark, an unstable mode must have a component with a wavenumber smaller than* $|s|$ *(and* $k_0' \cdot s \neq 0$*). This constraint is consistent with the well-known arguments of* [2] *regarding energy transfers in two-dimensional Euler flows. Basic flows with* $\alpha s_2 \leq 1$ *cannot satisfy this constraint, which explains their stability: all disturbances with* $k_0' \cdot s \neq 0$ *are of smaller scale.*

3. **Nonlinear stability.** We now present the main result in the nonlinear stability analysis. In this section, the term *nonlinear stability* means asymptotic (global) stability.

Unlike the nonlinear stability problem for finite-dimensional systems in which all norms are equivalent, a nonlinear stability analysis for an infinite-dimensional system requires a specified norm. A stationary solution of an infinite-dimensional system that is shown to be nonlinearly stable with respect to a given norm may not necessarily be stable with respect to another norm. The present problem turns out to be an exceptional case in which the stability of $\bar{u}$ with respect to the energy norm implies the stability of $\bar{u}$ with respect to the enstrophy norm, and vice versa. In particular, we will prove the following

LEMMA 3.1. *The following conditions are equivalent:*
(I) $\bar{u}$ *is nonlinearly stable with respect to the energy norm.*
(II) $\bar{u}$ *is nonlinearly stable with respect to the enstrophy norm.*

Proof. (II) automatically implies (I) because of the Poincaré inequality. For the other direction the proof goes as follows. Taking the scalar product of (17) with v in H and H^1 and noting the orthogonality properties of $B(\cdot,\cdot)$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 - (\bar{u}, B(v,v)) + \|v\|_\eta^2 = 0, \tag{31}$$

$$\frac{1}{2}\frac{d}{dt}\|v\|_1^2 - (A\bar{u}, B(v,v)) + \|v\|_{1+\eta}^2 = 0. \tag{32}$$

It is easy to see that if $v \in H(|s|)$ at some time then v belongs to $H(|s|)$ for all time since $H(|s|)$ is invariant, because $B(v,v) = 0$ if $v \in H(|k|)$ for all $k \in K$. Moreover, $\|v\|_\gamma \to 0$, in $H(|s|)$, $\forall\gamma$. Hence, we assume $v \notin H(|s|)$ for the rest of this proof.

Since $A\bar{u} = |s|^2\bar{u}$ we can multiply the energy equation (31) by $|s|^2$ and subtract the enstrophy equation (32) to obtain

$$\frac{1}{2}\frac{d}{dt}\left\{|s|^2\|v\|^2 - \|v\|_1^2\right\} + \left\{|s|^2\|v\|_\eta^2 - \|v\|_{1+\eta}^2\right\} = 0. \tag{33}$$

Using (10), this equation can be rewritten in terms of the function $G(1,\eta,|s|,v)$ as

$$\frac{1}{2}\frac{d}{dt}\left\{|s|^2\|v\|^2 - \|v\|_1^2\right\} + |s|^{2\eta}\left\{|s|^2\|v\|^2 - \|v\|_1^2\right\} = G(1,\eta,|s|,v). \tag{34}$$

Since $G(1,\eta,|s|,v) \geq 0$ for $\eta \geq 0$, (34) implies

$$\liminf_{t\to\infty}\left\{|s|^2\|v\|^2 - \|v\|_1^2\right\} \geq 0. \tag{35}$$

Hence, (I) implies (II) and the proof of the lemma is complete. □

Remark 3. *For a secondary stationary solution, denoted by $\bar{u}'$ if it exists, we have $|s|\,\|\bar{u}'\| = \|\bar{u}'_1\|$ for $\eta = 0$ and $|s|\,\|\bar{u}'\| > \|\bar{u}'\|_1$ for $\eta > 0$ (see [11, 12]). However, a disturbance from $\bar{u}'$ does not necessarily obey the same equality or inequality as $\bar{u}'$ does, as $t \to \infty$. Hence (I) does not necessarily imply (II).*

We now derive a sufficient condition for the stability of $\bar{u}$ in the special case $s = (0, \alpha s_2)^T$. We also consider only the case of molecular viscosity, $\eta = 1$, for the remainder of the paper. We have from (31) that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 = (\bar{u}, B(v,v)) - \|v\|_1^2 = \bar{u}_s(e'_s, B(v,v)) - \|v\|_1^2. \tag{36}$$

We evaluate the trilinear form in (36) by expanding $B(v,v)$ as in the appendix. Since, for $s = (0,\alpha s_2)^T$, there exist no triads in K satisfying $k + l = s$, and because of the orthogonality of the basis functions, the trilinear form $(e'_s, B(v,v))$ reduces to

$$(e'_s, B(v,v)) = \sum_{k-l=s} \frac{\sqrt{\alpha}\,k'\cdot l(|k|^2 - |l|^2)}{2\sqrt{2}\,\pi|k||l||s|}(v_k v_l + v'_k v'_l). \tag{37}$$

The factors in the coefficients of $v_k v_l$ and $v'_k v'_l$ are given by

$$k'\cdot l = k_1\alpha s_2 = l_1\alpha s_2$$

$$|k|^2 - |l|^2 = |l+s|^2 - |l|^2 = \alpha^2 s_2^2 + 2\alpha^2 s_2 l_2$$

$$|k||l||s| = \alpha s_2(l_1^2 + (\alpha s_2 + \alpha l_2)^2)^{1/2}(l_1^2 + \alpha^2 l_2^2)^{1/2}.$$

Therefore,

$$\frac{k'\cdot l(|k|^2 - |l|^2)}{|k||l||s|} = \frac{l_1\alpha s_2(\alpha s_2 + 2\alpha l_2)}{(l_1^2 + (\alpha s_2 + \alpha l_2)^2)^{1/2}(l_1^2 + \alpha^2 l_2^2)^{1/2}}. \tag{38}$$

Upon substitution of the above identity the trilinear form then reads

$$(e'_s, B(v,v)) = \sum_{l_1=1}^{\infty}\sum_{l_2=-\infty}^{\infty} \frac{\sqrt{\alpha}\,l_1\alpha s_2(\alpha s_2 + 2\alpha l_2)(v_l v_{l+s} + v'_l v'_{l+s})}{2\sqrt{2}\,\pi(l_1^2 + (\alpha s_2 + \alpha l_2)^2)^{1/2}(l_1^2 + \alpha^2 l_2^2)^{1/2}}. \tag{39}$$

Note that each v_l (v'_l), for $l \neq (0, \alpha l_2)^T$, appears exactly twice in the above sum. Meanwhile, the dissipation term can be majorized according to

$$\|v\|_1^2 = \sum_k |k|^2(v_k^2 + {v'}_k^2) \geq \sum_k |k||k+s|(|v_k v_{k+s}| + |v'_k v'_{k+s}|). \tag{40}$$

Substituting (39) and (40) into the energy equation (36) then yields (note the change of the dummy indices l_1 and l_2 to k_1 and k_2)

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq \sum_{k_1=1}^{\infty}\sum_{k_2=-\infty}^{\infty}\Bigg(\frac{Uk_1\alpha s_2|\alpha s_2+2\alpha k_2|}{2(k_1^2+(\alpha s_2+\alpha k_2)^2)^{1/2}(k_1^2+\alpha^2k_2^2)^{1/2}} - (k_1^2+(\alpha s_2+\alpha k_2)^2)^{1/2}(k_1^2+\alpha^2k_2^2)^{1/2}\Bigg)(|v_k v_{k+s}|+|v'_k v'_{k+s}|), \tag{41}$$

where $U = \bar{u}_s\sqrt{\alpha}/(\sqrt{2}\,\pi)$ is the amplitude of $\bar{u}$. Nonlinear stability of $\bar{u}$ follows provided the right-hand side of (41) is non-positive. This condition is satisfied when

$$\frac{Uk_1\alpha s_2|\alpha s_2+2\alpha k_2|}{2(k_1^2+(\alpha s_2+\alpha k_2)^2)(k_1^2+\alpha^2k_2^2)} \leq 1, \tag{42}$$

for all positive integers k_1 and all integers k_2. Since the left-hand side of (42) is greatest when $k_1 = 1$ for each k_2, the stability condition reduces to

$$\frac{U\alpha s_2|\alpha s_2+2\alpha k_2|}{2(1+(\alpha s_2+\alpha k_2)^2)(1+\alpha^2k_2^2)} \leq 1. \tag{43}$$

The problem of determining the nonlinear stability condition for $\bar{u}$ now reduces to determining the greatest value of the left-hand side of (42) for all integers k_2 and setting it ≤ 1. For that purpose, let us determine the maximum of the function

$$g(x) = \frac{|c+2x|}{(1+x^2)(1+(c+x)^2)},$$

where c is a positive parameter and x is a continuous real variable. It can be seen that $g(x)$ is symmetric about the line $x = -c/2$; therefore, the translation $x \to x + c/2$ helps reduce the problem to finding the maximum of

$$h(x) = \frac{2x}{(1+(x-c/2)^2)(1+(x+c/2)^2)},\ x > 0.$$

Differentiating $h(x)$ with respect to x, setting the derivative to zero, and solving the resulting equation for x in terms of c one obtains the single real solution

$$x = \left(\frac{c^2-4+2(c^4+4c^2+16)^{1/2}}{12}\right)^{1/2},$$

which corresponds to a maximum of $h(x)$. Therefore, $g(x)$ peaks at

$$x = -\frac{c}{2} + \left(\frac{c^2-4+2(c^4+4c^2+16)^{1/2}}{12}\right)^{1/2},$$

with the maximum value

$$\bar{g}(c) = \frac{\left(c^2+\frac{2}{3}((c^4+4c^2+16)^{1/2}-c^2-2)\right)^{1/2}}{1+\left(c^2+\frac{1}{3}((c^4+4c^2+16)^{1/2}-c^2-2)\right)+\frac{1}{36}((c^4+4c^2+16)^{1/2}-c^2-2)^2}.$$

With this result $\bar{u}$ is nonlinearly stable when

$$U\alpha s_2\bar{g}(\alpha s_2) \leq 2. \tag{44}$$

This is a sufficient condition for the nonlinear stability of $\bar{u}$.

There is a feature of $\bar{g}(\alpha s_2)$ in (44) which leads to an interesting difference between the conditions for stability for $\alpha s_2 \ll 1$ and for $\alpha s_2 \gg 1$. In the former case, $\bar{g}(\alpha s_2) \sim 3\sqrt{3}/8$ so (44) reduces to

$$U \alpha s_2 \leq \frac{16}{3\sqrt{3}}. \tag{45}$$

In the latter case, $\bar{g}(\alpha s_2)$ becomes

$$\bar{g}(\alpha s_2) \sim \frac{1}{\alpha s_2},$$

and the nonlinear stability of $\bar{u}$ therefore prevails when

$$U \leq 2. \tag{46}$$

The above analysis concerns only the stability of $\bar{u}$ with respect to the energy norm. A similar approach is now applied to derive a sufficient condition for the stability of $\bar{u}$ with respect to the enstrophy norm. Because of the equivalence of the two normed stability conditions a comparison between them can be made and an optimal condition deduced. Now,

$$\frac{1}{2}\frac{d}{dt}\|v\|_1^2 = (A\bar{u}, B(v,v)) - \|v\|_2^2 = |s|^2 \bar{u}_s (e'_s, B(v,v)) - \|v\|_2^2 . \tag{47}$$

The global stability of $\bar{u}$ is established if the right-hand side of (47) is negative. The trilinear form has already been estimated above. Meanwhile, the dissipation term can be majorized according to

$$\|v\|_2^2 = \sum_k |k|^4 (v_k^2 + {v'}_k^2) \;\geq\; \sum_k |k|^2 |k+s|^2)(|v_k v_{k+s}| + |v'_k v'_{k+s}|). \tag{48}$$

Substituting (39) and (48) into the enstrophy equation (47) then yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|_1^2 \;\leq\; \sum_{k_1=1}^{\infty}\sum_{k_2=-\infty}^{\infty} \Bigg(& \frac{U k_1 \alpha^3 s_2^3 \,|\alpha s_2 + 2\alpha k_2|}{2(k_1^2 + (\alpha s_2 + \alpha k_2)^2)^{1/2}(k_1^2 + \alpha^2 k_2^2)^{1/2}} \\ & - (k_1^2 + (\alpha s_2 + \alpha k_2)^2)(k_1^2 + \alpha^2 k_2^2)\Bigg) (|v_k v_{k+s}| + |v'_k v'_{k+s}|). \end{aligned} \tag{49}$$

Nonlinear stability of $\bar{u}$ follows provided the right-hand side of (49) is non-positive. This condition is satisfied when

$$\frac{U \alpha^3 s_2^3\, k_1 |\alpha s_2 + 2\alpha k_2|}{2(k_1^2 + (\alpha s_2 + \alpha k_2)^2)^{3/2}(k_1^2 + \alpha^2 k_2^2)^{3/2}} \leq 1, \tag{50}$$

for all positive integers k_1 and all integers k_2. Since the left-hand side of (50) is greatest when $k_1 = 1$ for each k_2, the stability condition reduces to

$$\frac{U \alpha^3 s_2^3 \,|\alpha s_2 + 2\alpha k_2|}{2(1 + (\alpha s_2 + \alpha k_2)^2)^{3/2}(1 + \alpha^2 k_2^2)^{3/2}} \leq 1. \tag{51}$$

Similar to the previous calculation for the energy norm, the present problem reduces to determining the maximum of the function

$$g_1(x) = \frac{|c + 2x|}{(1 + x^2)^{3/2}(1 + (c+x)^2)^{3/2}},$$

or equivalently the maximum of

$$h_1(x) = \frac{2x}{(1 + (x - c/2)^2)^{3/2}(1 + (x + c/2)^2)^{3/2}}, \;\; x > 0,$$

where c is a positive parameter and x is a continuous real variable. It turns out that $g_1(x)$ peaks at

$$x = -\frac{c}{2} + \left(\frac{2c^2 - 8 + (9c^4 - 12c^2 + 144)^{1/2}}{20}\right)^{1/2},$$

with the maximum value

$$\bar{g}_1(c) = \frac{20\left(c^2 - 4 + \frac{1}{2}(9c^4 - 12c^2 + 144)^{1/2}\right)^{1/2}}{\left(2 + 7c^2 + (9c^4 - 12c^2 + 144)^{1/2} + \frac{1}{40}(3c^2 + 8 - (9c^4 - 12c^2 + 144)^{1/2})^2\right)^{3/2}}.$$

With this result $\bar{u}$ is nonlinearly stable when

$$U\alpha^3 s_2^3\, \bar{g}_1(\alpha s_2) \le 2. \tag{52}$$

When $\alpha s_2 \ll 1$, $\bar{g}_1(\alpha s_2) \sim 25\sqrt{5}/108$ so (52) reduces to

$$U\alpha^3 s_2^3 \le \frac{25\sqrt{5}}{54}. \tag{53}$$

For $\alpha s_2 \gg 1$, $\bar{g}(\alpha s_2)$ becomes

$$\bar{g}(\alpha s_2) \sim \frac{1}{\alpha^2 s_2^2},$$

and the nonlinear stability of $\bar{u}$ therefore prevails when

$$U\alpha s_2 \le 2. \tag{54}$$

A comparison between (45) and (53) indicates that the latter is optimal, while a comparison between (46) and its counterpart (54) favours the former.

It is not hard to see that $\bar{g}(\alpha s_2)$ and $\bar{g}_1(\alpha s_2)$ are of the same order when $\alpha s_2 \sim 1$. In that case either (44) or (52) may be taken as a sufficient condition for the stability of $\bar{u}$. The linearly marginal case ($\alpha s_2 = 1$) is particularly interesting, so we will derive an explicit criterion for its nonlinear stability. Evaluating $\bar{g}(\alpha s_2)$ and $\bar{g}_1(\alpha s_2)$ at $\alpha s_2 = 1$ we obtain $\bar{g}(1) > \bar{g}_1(1) = 0.3589\cdots$. Therefore, (52) is optimal and we have the following stability criterion for this case:

$$U \le 5.572\cdots. \tag{55}$$

In summary, we have the following optimal conditions for nonlinear stability of $\bar{u}$

THEOREM 3.1. *Consider (1) for $\eta = 1$ and $f = f_0 e'_s$, where $s = (0, \alpha s_2)^T$. The corresponding primary stationary solution $\bar{u}$ is asymptotically (globally) stable when*

$$U \le 2 \text{ for } \alpha s_2 \gg 1, \tag{56}$$

$$U \le 5.572\cdots \text{ for } \alpha s_2 = 1, \tag{57}$$

and

$$U\alpha^3 s_2^3 \le \frac{25\sqrt{5}}{54} \text{ for } \alpha s_2 \ll 1, \tag{58}$$

where U is the amplitude of $\bar{u}$.

The remarks below provide some physical interpretations of the conditions for nonlinear stability.

REMARK 4. *Since the basic flow length scale is $2\pi/(\alpha s_2)$ and the viscosity is unity, the Reynolds number is given by $Re = 2\pi U/(\alpha s_2)$. In the regime $\alpha s_2 \ll 1$, which is stable to linear dynamics, (58) gives nonlinear stability for*

$$Re \leq \frac{25\sqrt{5}\,\pi}{27\alpha^4 s_2^4} \quad \left(\gg \frac{25\sqrt{5}\,\pi}{27}\right). \tag{59}$$

Hence flows with $Re \gg 1$ are stable for sufficiently small αs_2.

REMARK 5. *In the regime $\alpha s_2 \gg 1$ where linear stability is not established, from (56) we get*

$$Re \leq \frac{4\pi}{\alpha s_2} \ (\ll 4\pi) \tag{60}$$

so Re must be very small. This is definitely the viscous limit.

Appendix. The bilinear form $B(v, v)$. We shall give an explicit form of $B(v, v)$ which is employed in this paper. The lemma below is taken from [6, 7] (note that $\alpha = 1$ in these articles).

LEMMA 3.2. *For every k and l in the definition of e_k and e'_k $(k \neq l)$*

$$B(e_k, e_l) + B(e_l, e_k) = \frac{\sqrt{\alpha}\,k' \cdot l(|k|^2 - |l|^2)}{2\sqrt{2}\,\pi|k||l|}\left(\frac{e'_{k+l}}{|k+l|} + \frac{e'_{k-l}}{|k-l|}\right),$$

$$B(e'_k, e'_l) + B(e'_l, e'_k) = -\frac{\sqrt{\alpha}\,k' \cdot l(|k|^2 - |l|^2)}{2\sqrt{2}\,\pi|k||l|}\left(\frac{e'_{k+l}}{|k+l|} - \frac{e'_{k-l}}{|k-l|}\right),$$

$$B(e_k, e'_l) + B(e'_l, e_k) = -\frac{\sqrt{\alpha}\,k' \cdot l(|k|^2 - |l|^2)}{2\sqrt{2}\,\pi|k||l|}\left(\frac{e_{k+l}}{|k+l|} - \frac{e_{k-l}}{|k-l|}\right).$$

Proof. It suffices to give a proof for one of the equations. The others are proven in exactly the same manner. It is found by direct calculation that

$$\begin{aligned}(e_k \cdot \nabla)e_l &= -\frac{\alpha(k' \cdot l)l'}{2\pi^2|k||l|}\cos k \cdot x \ \sin l \cdot x,\\ (e_l \cdot \nabla)e_k &= -\frac{\alpha(l' \cdot k)k'}{2\pi^2|k||l|}\cos l \cdot x \ \sin k \cdot x.\end{aligned} \tag{A-1}$$

Adding the equations of (A-1) and noting that $k' \cdot l = -l' \cdot k$ one finds

$$\begin{aligned}(e_k \cdot \nabla)e_l + (e_l \cdot \nabla)e_k &= -\frac{\alpha(k' \cdot l)}{2\pi^2|k||l|} \times \\ &\quad \big(l' \cos k \cdot x \ \sin l \cdot x - k' \cos l \cdot x \ \sin k \cdot x\big) \\ &= -\frac{\alpha(k' \cdot l)}{4\pi^2|k||l|} \times \\ &\quad \big((l' - k')\sin(k+l) \cdot x - (l' + k')\sin(k-l) \cdot x\big).\end{aligned} \tag{A-2}$$

The right hand side of (A-2) is identically zero for $l = k$. When $l \neq k$ it can be projected onto H to yield

$$\begin{aligned}B(e_k, e_l) + B(e_l, e_k) &= -\frac{\sqrt{\alpha}\,k' \cdot l}{2\sqrt{2}\,\pi|k||l|} \times \\ &\quad \left(\frac{(l' - k') \cdot (k' + l')}{|k+l|}e'_{k+l} - \frac{(l' + k') \cdot (k' - l')}{|k-l|}e'_{k-l}\right) \\ &= \frac{\sqrt{\alpha}\,k' \cdot l(|k|^2 - |l|^2)}{2\sqrt{2}\,\pi|k||l|}\left(\frac{e'_{k+l}}{|k+l|} + \frac{e'_{k-l}}{|k-l|}\right),\end{aligned} \tag{A-3}$$

where the projection P of the vector sum involved onto H has been performed term by term via

$$P(m\sin l\cdot x) \quad = \quad \frac{\sqrt{2}\,\pi m\cdot l'}{\sqrt{\alpha}\,|l|}e'_l. \tag{A-4}$$

The first equation of the lemma is thus proven. □

It is now easy to see that

$$\begin{aligned} B(v,v) \quad = \quad & \sum_{k,l\in K}\frac{\sqrt{\alpha}\,k'\cdot l(|k|^2-|l|^2)}{2\sqrt{2}\,\pi|k||l|}\left(\frac{e'_{k+l}}{|k+l|}+\frac{e'_{k-l}}{|k-l|}\right)v_kv_l \\ & -\sum_{k,l\in K}\frac{\sqrt{\alpha}\,k'\cdot l(|k|^2-|l|^2)}{2\sqrt{2}\,\pi|k||l|}\left(\frac{e'_{k+l}}{|k+l|}-\frac{e'_{k-l}}{|k-l|}\right)v'_kv'_l \\ & -\sum_{k,l\in K}\frac{\sqrt{\alpha}\,k'\cdot l(|k|^2-|l|^2)}{2\sqrt{2}\,\pi|k||l|}\left(\frac{e_{k+l}}{|k+l|}-\frac{e_{k-l}}{|k-l|}\right)v_kv'_l. \end{aligned} \tag{A-5}$$

REFERENCES

[1] P. Constantin and C. Foias, The Navier-Stokes Equations, Univ. of Chicago Press, Chicago, 1988.

[2] R. Fjørtoft, On the changes in the spectral distribution of kinetic energy for a two-dimensional, non-divergent flow, Tellus, 5 (1953), pp. 225–237.

[3] A.E. Gill, The stability of planetary waves on an infinite beta-plane, Geophys. Fluid Dyn., 6 (1974), pp. 29–47.

[4] V.I. Iudovich, Example of the generation of a secondary stationary or periodic flow when there is loss of stability of the laminar flow of a viscous incompressible fluid, J. Appl. Math. Mech. 29 (1965), pp. 527–544.

[5] Y. (C.) Li, On 2D Euler equations. I. On the energy-Casimir stability and the spectra for linearized 2D Euler equations, J. Math. Phys., 41 (2000), pp. 728–758.

[6] V.X.S. Liu, A sharp lower bound for the Hausdorff dimension of the global attractors of the 2D Navier-Stokes equations, Comm. Math. Phys., 158 (1993), pp. 327–339.

[7] V.X.S. Liu, Instability for the Navier-Stokes equations on the 2-dimensional torus and a lower bound for the Hausdorff dimension of their global attractors, Comm. Math. Phys., 147 (1992), pp. 217–230.

[8] L.D. Meshalkin and I.G. Sinai, Investigation of the stability of a stationary solution of a system of equations for the plane movement of an incompressible viscous liquid, J. Appl. Math. Mech., 25 (1961), pp. 1140–1143.

[9] R. Temam, Navier-Stokes Equations and Nonlinear Functional Analysis, SIAM, Philadelphia, 1983.

[10] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, 2nd ed., Springer–Verlag, New York, 1997.

[11] C.V. Tran, Extensive Chaos and Complexity of Two-Dimensional Turbulence, Ph.D. thesis, Department of Physics, University of Toronto, 2001.

[12] C.V. Tran and T.G. Shepherd, Constraints on the spectral distribution of energy and enstrophy dissipation in forced two-dimensional turbulence, Physica D, 165 (2002), III–IV, pp. 199–212.

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