Extensive Chaos and Complexity of Two-Dimensional Turbulence

by

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Abstract

This thesis concerns how the dynamics of 2D fluid systems may become complex due to various physical parameters such as the forcing intensity and scale, the dissipation mechanism, and the system size. These complexity-determining factors are found to act collectively to give rise to the dynamical complexities. The fluid models in this study are shallow layers of viscous incompressible fluids on rectangular domains governed by the familiar 2D Navier-Stokes equations with and without a hypoviscosity term (linear Ekman drag is added to model large-scale dissipation).

Three main analyses are carried out. One is the asymptotic analysis of the 2D Navier-Stokes equations driven by a monoscale forcing. The results obtained include a dynamical constraint and possible scaling laws for the energy spectrum. These are then generalized to systems with a general viscosity and extended to systems with Ekman drag. The second analysis concerns the estimation of the attractor dimension of the systems under consideration. We employ the technique developed by Constantin-Foias-Temam for the calculation of attractor dimension of dissipative dynamical systems. The present investigation focuses on the optimal estimation of the dimensionality and its extensivity. It is found, in general, that the estimates do not depend on the physical parameters in a fixed functional form, but rather take different expressions in different regions of parameter space. In particular, the attractor dimension of the Navier-Stokes equations (with or without Ekman drag) is shown to grow linearly with the domain area (for a sufficiently large domain) if the kinematic viscosity and the forcing density and its scale are held fixed. We also show that slightly super-extensive behaviour prevails for a wide range of the parameters. The third investigation concerns the stability problem (both linear and nonlinear) of simple laminar stationary flows. The analysis examines how the flows may become unstable and explores the properties of some unstable eigenmodes. The familiar Fourier expansion method is used for this study. It is found that (in)stabilities depend on the forcing scale in a peculiar and nontrivial way.

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To my parents: Mr. Tran Van Len and Mrs. Nguyen Thi Quoi

Mammy and Daddy,

Thank you for having lifted me to that spot of the hill and for having fought it so hard to be with me until this day.

Contents

Chapter 1

Introduction

1.1 Characterization of chaotic systems

It is observed that the evolution of many nonlinear dissipative dynamical systems is never-repeating regardless of whether or not the driving forces are time-dependent. Non-repeating behaviour means that a system does not approach a stationary or periodic state asymptotically in time. Instead, it tends towards a strange attractor on which the motion is chaotic, i.e., not multiply periodic and unpredictable in time by virtue of being sensitive to the initial data.

Typically, a strange attractor arises when the phase space flow shrinks in some directions and stretches in others, while at the same time maintains a net contracting rate, so that volume elements shrink in time. To sustain chaotic behaviour, there must be at least one direction in which a volume element stretches. This condition is commonly taken as a definition of chaos and is mathematically expressible in terms of a positive Lyapunov exponent. In addition, the directions of shrinking and stretching continuously change; therefore a volume element not only gets stretched but also folded and twisted. In the long run, it becomes a multisheeted structure. A careful study in a low-dimensional system reveals the existence of an attractor that is (locally) a Cantor-like set in some directions [38]. Such a set is called a fractal set in the sense of Mandelbrot [40].

Since the attractor of a chaotic system is its permanent regime of evolution (more precisely it is the global attractor, i.e., the object which attracts all bounded sets in the phase space that describes all of the system's possible dynamics), one may characterize the system dynamics by its attractor characteristics. The following partially describe some properties which shall be studied in this thesis in the context of 2D fluid systems: (i) the spectral distribution of energy, (ii) the attractor dimension and its extensivity, (iii) the stability of a stationary flow (a trivial attractor as it can be called, so that this topic can be categorically put on an equal footing with the others). Among these, the second theme is least known, so that it deserves more attention in this introductory chapter. It is for that purpose that most of the remainder of this chapter is devoted.

Generally speaking, one may think of the dimension of an attractor as indicative of its "capacity". Or better still, one may think of the dimension as giving, in some way, the amount of information necessary to specify the position of a point on an attractor to within a given accuracy. In the world of chaos, however, one needs to describe the "capacity" of an attractor by a generalized dimension because the usual dimension is inadequate for fractal sets. There are several generalized dimensions that can be used to characterize a chaotic attractor. In this thesis only the Hausdorff dimension will be considered. A quick review of this subject can be found in many papers, eg., Farmer et al. [14]. For an extensive treatment of dimension in many contexts, see Mandelbrot [40, 41, 42]. Another characteristic of an attractor is its temporal aspect. Since an attractor is a dynamical object one may wish to know how a trajectory wanders within this fractal set, or how nearby trajectories diverge from one another as they evolve in time. Certainly, a geometrical picture of an attractor, regardless of its resolution, does not convey this information. One then resorts to the so-called Lyapunov spectrum for an understanding of the temporal behaviour of a system. This subject is behind the Constantin-Foias-Temam (CFT) theorem which is reviewed in appendix D and employed in chapter 3, but it is well beyond the scope of this thesis.

1.2 Routes to complexity

Suppose that one is dealing with a continuum model, e.g. a model described by partial differential equations, which has a priori an infinite number of degrees of freedom. There seem to be two closely related types of complexities which give rise to what is dubbed "spatio-temporal chaos". They are, roughly speaking: spatial disorder and temporally erratic behaviour. On the one hand, the degree of spatial complexity may be thought of as being the minimum number of independent spatial modes needed to describe each and every snapshot of the dynamics adequately, or equivalently the attractor dimension. On the other hand, temporally erratic behaviour can be associated with how the amplitudes of the modes change in time as the system evolves (i.e., how energy transfers back and forth among the active modes). This thesis deals with both types of complexities but the focus is more on the former and the word complexity will be loosely used in that sense.

Dynamical complexities can be best envisaged from the bifurcation point of view. Suppose that the dynamics of a system reduces to a single stable fixed point (which is designated as the origin) for suitably chosen parameters. As the parameters are varied the origin becomes unstable and bifurcations occur. The simplest bifurcation type is the pitchfork bifurcation which gives birth to a one-dimensional unstable manifold through the origin and two stable fixed points. The attractor then consists of the fixed points and the two heteroclinic orbits connecting them. As the parameters are further varied, the two newly born fixed points may become unstable and hence more complexities entail. These two simple stages of bifurcation already give rise to very complicated attractors in low-dimensional phase spaces (the Lorenz attractor in 3-dimensional phase space, for example).

Dynamical complexities are due to various inter-related factors. Some of these factors are: (i) the forces that drive the system away from equilibria; (ii) the dissipation mechanisms that act to balance the driving forces and to keep the dynamics bounded; (iii) the volume of the physical system which determines the number of free modes which may be excited and then survive in the limit of large time. The main concern in this thesis is how these factors may give rise to dynamical complexities in 2D fluid systems. Traditionally, the Reynolds number — which can be viewed as a (non-dimensional) combination of these factors if a given forcing can be somehow related to a typical fluid speed and if molecular viscosity is the sole dissipation mechanism — is a measure of dynamical complexity. A similar interpretation can be attributed to the Grashof number, which arises from studies on the number of degrees of freedom of turbulence.

It is almost always the case that forces in continuum physics, and fluid dynamics in particular, are continuously distributed rather than localized in space. This makes the forcing length scale an important factor. Intuitively, a system should respond differently to driving forces of different scales. There are two fundamental types of responses from a fluid to this factor. One is the response from viscosity to give rise to total dissipation rates for the energy and enstrophy. Surprisingly curious effects of this will be seen in the next chapter. The other is the response from the system's free modes through nonlinear interactions, as demonstrated in chapter 4.

Since a system's free modes are determined by the system size, the previous paragraph implies that there are collective effects among the forcing, viscosity, and system size which give rise to dynamical complexities. The intimate connection between these factors makes the notion of their independence ambiguous. Take, for example, the notion of a fixed forcing in a system of varying size. If the forcing density is held fixed while its scale is allowed to vary with the same proportion as the system size, then it looks fixed from a certain perspective. From the dynamical point of view, however, a reasonable definition of a fixed forcing is that of a fixed density and scale. In this thesis a monoscale forcing is considered in the majority of cases, so that the notion of a fixed forcing can be made unambiguous by specifying its amplitude and scale.

An example of a natural fluid system which would not have a forcing of fixed amplitude and scale if its size were to be increased is the earth's atmosphere. There are two main driving forces for this system: the heat contrasts between oceans and continents, and the differential solar energy flux arriving at the top of the earth's atmosphere. If the earth radius, R_e, were to be increased (with the same increasing proportion for the oceans and continents) then the length scale of both sources would increase accordingly.

1.3 Extensive chaos

It is natural to ask how complexity, as measured by the fractal dimension or the Hausdorff dimension of the attractor set of a given dynamical system, scales with the physical parameters. Do the parameters act independently? Or are there collective effects among them? The results in this thesis, as mentioned in the last section, suggest a negative answer to the former or equivalently a positive answer to the latter since there are optimal stability conditions and optimal attractor-dimension estimates which take various functional forms for different regions of parameter space. Nevertheless, there have been attempts, pending definite answers to the questions above, to understand chaos by looking at a particular physical parameter and somehow ignoring the rest by assuming that there exists a control parameter (possibly a combination of all other parameters in some functional form) which may be kept constant while the said parameter is varied. Some investigations in that direction, where size was the only variable, have been reported in [11, 12, 21] in which the idea of extensive chaos was introduced and explored. It has been argued that the dimension of a certain class of physical systems, namely large and homogeneous ones, scales linearly with the volume of the systems. The dynamics of this class is dubbed "extensive chaos". The physical interpretation of this idea, according to its originators, is that large homogeneous systems become complex with increasing size in a simple way by replicating weakly-interacting and statistically similar subsystems of some characteristic size. This characteristic size is identified as the dimension correlation length [10, 21], the concept of which will be briefly touched upon in chapter 3. In other words, an extensively chaotic system is a system such that there are no interesting collective effects with growing size, as might be expected if small dynamical units somehow bind themselves into a larger effective unit with fewer total degrees of freedom.

Let us examine the idea of extensive chaos with the following classical argument. In a dissipative system there exists a length scale, l_D, below which all modes are damped and only modes above the dissipation scale are active. Intuitively, the attractor dimension is approximately equal to the number of active degrees of freedom. If other parameters can somehow be controlled so that l_D remains fixed while the volume of the system of typical length L is increased, then the number of modes above l_D grows as $(L/l_D)^d$ where d is the dimension of the physical space. This conclusion is best appreciated by the following geometrical demonstration. Let us consider a two-dimensional domain Ω, e.g. a square of side L, and a Hilbert space of doubly-periodic functions on Ω, $H(\Omega)$, spanned by $\{\cos\frac{2\pi}{L}(k_1x_1+k_2x_2), \sin\frac{2\pi}{L}(k_1x_1+k_2x_2)\}$ where k_1 and k_2 are integers. A basic mode of wavevector $(k_1,k_2)^T$ above the length scale l_D satisfies

$$(k_1^2+k_2^2)^{1/2} \leq L/l_D. \tag{1.1}$$

Modes satisfying the above inequality can be geometrically identified as lattice sites, denoted by filled circles, inside the circle of radius L/l_D in Figure 1.1. It is easy to see that the number of dots inside the circle is approximately the area of the circle (for a large enough radius). Hence the number of modes above the length l_D is proportional to $(L/l_D)^2$ which is proportional to the domain area. Hence, extensive behaviour might be expected to be common among dynamical systems. More accurately, those systems whose l_D's are independent of the system size ought to be extensive. Thus, the problem of proving that a system is extensively chaotic might be restricted to showing that its dissipation length is size-independent. Although expected to be

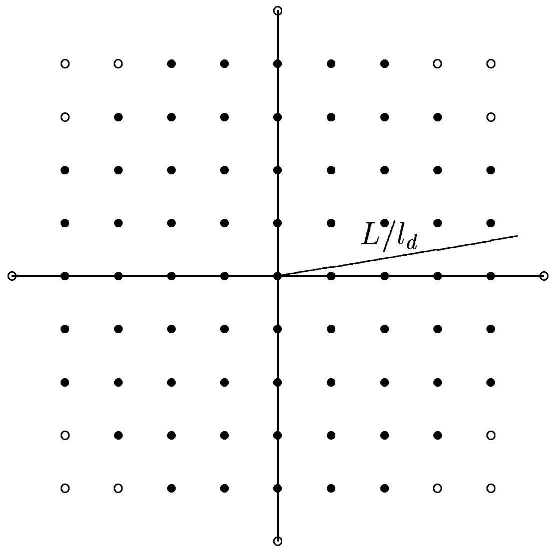

Figure 1.1: *Lattice sites inside the circle of radius L/l_D. The number of sites is proportional to $(L/l_D)^2$ for $L \gg l_D$*

common, extensive behaviour has only been partly validated for a few cases. On the theoretical front, for example, extensive behaviour has been established for the 2D Navier-Stokes equations (for a special class of the driving force, see chapter 3) and the one-dimensional complex Ginzburg-Landau equation [1]. On the experimental side, it has been shown numerically that the Kuramoto-Sivashinsky model [43] and the Miller-Huse model [51] behave extensively.

The interpretation that an extensive system consists of weakly-interacting and statistically similar subsystems of some characteristic size clearly implies that the dynamics of scales larger than the subsystem scale is not sustainable. It must be so or else the subsystems would not be weakly-interacting. In addition, the driving force should have a length scale smaller than the subsystem scale so that the partition of the mother system into statistically similar subsystems makes sense physically. The idea of extensive chaos, so interpreted, is arguably plausible for the 3D Navier-Stokes equations because of the downscale energy cascade [64, 31] which could entail scale-breaking in large systems. In contrast, in two-dimensional turbulence the inverse energy cascade [3, 27, 30] should lead to large-scale correlations which would make extensivity questionable. Indeed, meteorological two-point correlation functions such as that of the geopotential do not fall off substantially over the entire earth [5]. Thus, the possible extensivity of the 2D Navier-Stokes equations is an interesting question.

1.4 2D fluid systems and their number of degrees of freedom

Most of our knowledge about attractor sets of dynamical systems comes from studies of fluid turbulence. It is well established that the long-time behaviour of solutions to the 2D Navier-Stokes system, in a bounded domain, has a finite number of degrees of freedom. Several rigorous studies support this assertion, most notably the pioneering work of Foias and Prodi [16]. It is found that any solution of this system, for large time, is completely determined by the behaviour of its projection on the subspace spanned by the first m eigenfunctions of the linear Stokes operator, for sufficiently large m. More precisely, if the asymptotic behaviour of the first m modes of two solutions agree, then the entire solutions agree as $t \to \infty$. The minimum number of modes, m, with this property has been called *the number of determining modes.* This does not mean that each and every snapshot of this system, taken as transients have faded away, can be exactly written as a linear combination of the first m eigenfunctions of the Stokes operator. The best one can hope for is that the higher-order modes are enslaved, in an asymptotic manner, by the lower modes. Thus, one may seek in that case a global function that gives the high modes of every solution in terms of the lower modes. Nevertheless, the number m has been thought of as the minimum number of modes needed in order for the Galerkin approximation to describe 2D turbulence adequately.

In more recent developments, Foias and Titi [18] and Jones and Titi [26] proposed another way to characterize the degrees of freedom of the Navier-Stokes equations. These authors studied the behaviour of the averages of the velocity field on small subdomains and introduced the idea of *determining finite volume elements.* It has been shown, in a similar way as in the case of determining modes, that if the system domain is partitioned into N equal subdomains and if the asymptotic behaviour of the averages of any two solutions on those subdomains agree then the two solutions are equal, provided that N is large enough (i.e., small enough subdomains).

These results are important from the practical point of view. The existence of a finite number of determining modes or determining volume elements provides modellers guidance as to how many Fourier modes or grid points there should be in a numerical model. Unfortunately, those numbers are so huge for even moderate values of the Reynolds number that no modern computational device can even get close to them. Another setback is that they do not agree with each other nor do any of them agree with the Hausdorff dimension. In particular, the Hausdorff dimension is of the order of $Gr^{2/3}$, the number of determining modes is Gr, and the number of determining finite volume elements is Gr^2, where Gr is the Grashof number to be recalled in chapter 3.

1.5 Characteristics of atmospheric models

The earth's atmosphere is characteristically different from small fluid systems and its dynamics may be far from turbulent (i.e., only weakly chaotic). Being large makes its dynamics well approximated and most simply modelled with the 2D Navier-Stokes equations. This has dramatically different characteristics from the three-dimensional system because of the inverse energy cascade, making the concept of extensive chaos questionable. Being large also makes the atmosphere virtually inviscid. This statement will be made precise in chapter 2 where it will be shown that molecular viscosity alone provides a damping time on the order of billions of years!

Besides of its large-scale, two-dimensional and nearly inviscid nature, atmospheric dynamics is characterized by weak forcing. The thermal forcing is weak because its horizontal gradient is small, and the wind is driven by this gradient rather than by the heat itself. Although the forcing is weak, since viscosity is ineffective by any standard, energy loss due to other factors should be considered carefully. In this respect, surface friction due to the rough planetary surface would be caught by every observant mind (surface friction is molecular too, ultimately! But it is different in effect from interior viscosity). Observational evidence indicates that this friction dominates molecular viscosity. Thus, ignoring the loss of energy due to surface friction may be unrealistic or even disastrous. In this thesis, linear Ekman drag which represents this friction is studied in both chapter 2 and chapter 3.

Another small but important effect, not discussed in this thesis, is radiative cooling which has been overlooked in many thermally driven fluid systems. While the vast atmosphere can be hardly affected by molecular diffusion, outgoing radiation can substantially reduce the heat gradient. Physically, the role of radiative cooling in the thermal equation has certain similarities with the role of surface friction in the momentum equation. Parameterization of the effect of the former may be guided by what has been known as black-body radiation. Given the Stefan-Boltzmann law and the average temperature of the earth's atmosphere, radiative cooling is found to overwhelmingly dominate thermal diffusion for large-scale dynamics.

1.6 Objectives and organization of the thesis

This thesis examines the asymptotic behaviour of the 2D Navier-Stokes equations (with and without a large-scale dissipation mechanism), explores its dependence on various parameters including system size, and investigates the stability problem of simple stationary flows. These topics are organized in the remainder of the thesis as follows. In chapter 2, an asymptotic analysis of the 2D Navier-Stokes equations driven by a monoscale forcing is presented. The results feature some intrinsic properties of 2D turbulence and serve as a general motivation of the need for large-scale dissipation in large 2D fluid systems. In chapter 3, the question of dynamical complexities and the extensivity issue of the 2D Navier-Stokes equations on a square and rectangular domain and those of the modified system (by the addition of a linear Ekman drag) are addressed. Chapter 4 presents a stability analysis of some stationary laminar flows by the method of Liu [35, 36, 37], Iudovich [25], and Meshalkin and Sinai [47]. The stability problem is somewhat detached from the main flow of the thesis. Nevertheless, it is interesting and worth investigating in its own right. Moreover, the results in this chapter help clarify and consolidate the concept of the dissipation scale in the previous chapters as well. The main body of the thesis ends with some concluding remarks in the final chapter. Finally, the mathematical techniques on which the analyses in chapters 3 and 4 heavily rely are summarized in various appendices.

Chapter 2

The 2D Navier-Stokes system and 2D turbulence

This chapter deals with a viscous incompressible 2D fluid system driven by a monoscale forcing, and its organisation is as follows. Section 1 contains the governing equations (the 2D Navier-Stokes equations) and their mathematical setting. Section 2 reviews some classical ideas and arguments in two-dimensional turbulence. After those preliminaries, results from an asymptotic analysis are reported in section 3. Section 4 extends the results of section 3 in an attempt to explore a possible scaling law for the energy spectrum. In section 5 a dissipation mechanism which is responsible for absorbing the energy of the large scales (the linear Ekman drag is considered) is added to the 2D Navier-Stokes equations, and in section 6 the asymptotic behaviour of the modified system is analysed and compared with that of the original.

2.1 The Navier-Stokes system and its functional setting

As is well-known the motion of a viscous incompressible fluid is governed by the Navier-Stokes equations:

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u &= -\nabla p + \nu \Delta u + f \qquad (2.1) \\ \nabla \cdot u &= 0, \quad \text{in} \quad \Omega \times (0, \infty). \end{aligned}$$

In (2.1), $u = u(x,t)$ is the velocity vector, $p = p(x,t)$ is the pressure, $\nu > 0$ is the kinematic viscosity, and $f(x,t)$ is a body force. For most parts of the thesis f is assumed to be monoscale and time-independent but in some instances in the present chapter the second requirement is not crucial. The differential operators are standard. In 2D problems Ω is a subset of $\mathbf{R}^2$ with some regularity requirements. It should be clear from the outset that a 2D fluid is a highly idealized system. In such a fluid the third dimension is assumed to be either infinite or bounded by two frictionless parallel boundaries. Furthermore, the velocity field is supposed to have no component nor any variation in the ignored vertical dimension. To define an initial boundary value problem, (2.1) must be supplemented with initial and boundary conditions on the state variables. Most often studied are the rigid nonslip boundary and the doubly-periodic domain. Restriction to the latter case where $\Omega = L \times L/\alpha$ ($\alpha \in (0,1]$) is to be observed throughout this thesis. The present chapter requires no restriction on α, while in subsequent chapters the value of α needs to be specified in several instances. In all cases, the solution is assumed to satisfy

$$\begin{aligned} u(0,x_2,t) &= u(L,x_2,t), \quad 0 < x_2 < L/\alpha, \\ u(x_1,0,t) &= u(x_1,L/\alpha,t), \quad 0 < x_1 < L. \end{aligned} \tag{2.2}$$

The same periodicity is assumed for p and f as well. In addition, the space averages of all state variables and that of the forcing are assumed to vanish at all time, so that the Laplace operator can be properly inverted.

Existence and uniqueness of the solution for the 2D Navier-Stokes equations is well known. Also, there exists a global attractor (for time-independent f) which attracts all bounded sets in the phase space. The more mathematically inclined reader is referred to [60, 61, 62] for the proofs as well as details of the solution space. For the purposes of this thesis it is sufficient to give a brief description of the system phase space, so that some necessary notation can be introduced. Let

$W = \{u | \mathbf{R}^2 \to \mathbf{R}^2,$ vector-valued trigonometric polynomials with periods $L, L/\alpha,$

$$\nabla \cdot u = 0, \text{ and} \int_\Omega u dx = 0\}$$

with

$$\begin{aligned} H &= \text{the closure of } W \text{ in } (L_2(\Omega))^2, \\ V &= \text{the closure of } W \text{ in } (H^1(\Omega))^2, \end{aligned}$$

where $L_2(\Omega)$ is the space of square-integrable functions on Ω and $H^l(\Omega)$ denotes the usual L_2-Sobolev spaces. H and V are Hilbert spaces with scalar products and norms given respectively by

$$\begin{aligned} (u, v) &= \int_\Omega u \cdot v dx, \\ \| u \| &= (u, u)^{1/2}, \\ ((u, v)) &= \int_\Omega \nabla u \cdot \nabla v dx = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx, \\ \| u \|_1 &= ((u, u))^{1/2}, \end{aligned}$$

where ∇u is the 4-vector $(\partial_{x_1} u_1, \partial_{x_2} u_1, \partial_{x_1} u_2, \partial_{x_2} u_2)^T$. The squares of the H-norm and the V-norm are usually referred to as the total energy and enstrophy, respectively, in the literature. The enstrophy takes a simple form due to the non-divergent property of the velocity field:

$$\| u \|_1^2 = (u, Au) = \| \nabla \times u \|^2,$$

where A denotes the Stokes operator $Au = -P\Delta u$ $(= -\Delta u$ in this case), and P denotes the orthogonal projection in $(L_2(\Omega))^2$ onto H. The domain of A will be denoted by $D(A)$. Since A is symmetric the spectral theory can be utilized to define the powers A^η for $\eta \in \mathbf{R}$. The space $D(A^\eta)$ is endowed with the scalar product

$$(u, v)_{D(A^\eta)} = (A^\eta u, A^\eta v),$$

and the norm

$$\| u \|_{D(A^\eta)} = \| u \|_{2\eta} = \| A^\eta u \|,$$

which makes it a Hilbert space. The spaces most encountered in this thesis are special cases. When $\eta = 1$ one recovers $D(A)$, while for $\eta = 1/2$ and $\eta = 0$ one recovers V and H respectively. Finally, let B denote the bilinear operator

$$B(u,v) \quad = \quad P((u \cdot \nabla)v).$$

With all the notation in place, the abstract form of the initial boundary-value problem (2.1)-(2.2) is given, in H, by

$$\begin{aligned} \frac{du}{dt} + B(u,u) + \nu A u &= f, \\ u(t=0) &= u_0, \end{aligned} \tag{2.3}$$

where f $(= Pf)$ is now assumed to be a vector in H. The pressure gradient has been removed by the projection P. Physically, ∇p does not affect conservation properties and can be removed by considering the vorticity equation. From this point on this system will be referred to as the NS system (or equations).

A geometric constraint and a couple of well-known identities of the velocity field are of fundamental importance for this chapter. The constraint which is of a geometrical nature is the relation between the energy and enstrophy. It can be derived if one writes the velocity vector u in terms of the eigenfunctions of A which form an orthonormal basis of H,

$$u \quad = \quad \sum_k (u_k e_k + u'_k e'_k). \tag{2.4}$$

The normalized eigenfunctions in (2.4) have the form

$$\begin{aligned} e_k &= \frac{\sqrt{2\alpha}}{|k|L} k' \cos \frac{2\pi}{L} kx, \\ e'_k &= \frac{\sqrt{2\alpha}}{|k|L} k' \sin \frac{2\pi}{L} kx, \end{aligned}$$

where $k = (k_1, \alpha k_2)^T$, k_1 and k_2 are integers satisfying either $k_1 > 0$ or $k_1 = 0$ and $k_2 > 0$, and $k' = (\alpha k_2, -k_1)^T$. For convenience this set of wavevectors will be denoted by K. Now by definition, the norm in $D(A^{\eta/2})$ is

$$\| u \|_\eta^2 = \sum_k \lambda_k^\eta (u_k^2 + {u'_k}^2),$$

where $\lambda_k = 4\pi^2|k|^2/|\Omega|$ is the eigenvalue of A corresponding to e_k and e'_k. It is easy to see that

$$\| u \|_\eta^2 \geq \lambda_1^\eta \| u \|^2, \quad \forall \eta \geq 0, \tag{2.5}$$

where $\lambda_1 = 4\pi^2\alpha^2/L^2$ is the first eigenvalue of A in H (the eigenvalues of A are assumed to be listed in a nondecreasing order according to the nondecreasing sequence of $|k|$; the first and smallest eigenvalue corresponds to $|k| = \alpha$). When $\eta = 1$, (2.5) reduces to what is known as the Poincaré inequality

$$\| u \|_1^2 \geq \lambda_1 \| u \|^2 . \tag{2.6}$$

Ineqs. (2.5) and (2.6), especially the latter, will be used repeatedly in this chapter and the next. They will both be referred to as the Poincaré inequality.

A couple of well-known identities which will be used repeatedly in the sequel are now quoted. They arise by virtue of the non-divergent and periodic properties of the velocity field:

$$(Au, B(v, v)) + (Av, B(v, u)) + (Av, B(u, v)) = 0. \tag{2.7}$$

$$(u, B(v, w)) = -(w, B(v, u)). \tag{2.8}$$

In particular,

$$(u, B(u, u)) = 0, \tag{2.9}$$

$$(v, B(u, v)) = 0, \tag{2.10}$$

$$(Au, B(u, u)) = 0, \tag{2.11}$$

Note that (2.9) is a special case of (2.10) and (2.10) is in turn a special case of (2.8). Also (2.11) is a special case of (2.7). These identities will be collectively referred to as the orthogonality properties of the nonlinear term.

2.2 Spectral distribution of energy and the inverse cascade

This section examines how an initial spectral peak of energy, like the case of a monoscale energy injection, redistributes itself to different scales. It is instructive to review the argument for an unforced-inviscid fluid. Such a fluid is governed by

$$\frac{du}{dt} + B(u,u) = 0. \tag{2.12}$$

This equation conserves energy and enstrophy due to (2.9) and (2.11). There is a consensus that an initial energy distribution cascades towards larger scales with the downscale cascade almost forbidden. The "proof" of the upscale cascade was due to Fjørtoft [15]. In his argument, Fjørtoft considered the change of energy for three different scales. Because of conservation of enstrophy, energy must flow from the intermediate scale to the smaller and larger scales, and vice versa. It was shown that, at the expense of the energy of the intermediate scale, the larger scale acquires most of the energy. In the present setting the argument goes as follows. Let $|l| < |k| < |m|$ be the three wavenumbers (corresponding to three scales $|m|^{-1} < |k|^{-1} < |l|^{-1}$) that are involved in the energy transfer. Furthermore, let $\Delta E(\cdot)$ denote the change of energy in each scale. It is easy to see from the two conservation laws that

$$\begin{aligned} \Delta E(|l|) + \Delta E(|k|) + \Delta E(|m|) &= 0, \\ \lambda_l \Delta E(|l|) + \lambda_k \Delta E(|k|) + \lambda_m \Delta E(|m|) &= 0. \end{aligned}$$

Hence,

$$\begin{aligned} \Delta E(|l|) &= -\frac{\lambda_m - \lambda_k}{\lambda_m - \lambda_l} \Delta E(|k|), \\ \Delta E(|m|) &= -\frac{\lambda_k - \lambda_l}{\lambda_m - \lambda_l} \Delta E(|k|). \end{aligned}$$

It is argued in [15] that if one takes $|m| = 2|k| = 4|l|$, for example, then ($\lambda_m = 4\lambda_k = 16\lambda_l$)

$$\frac{\Delta E(|l|)}{\Delta E(|m|)} = 4.$$

Therefore, changes in the kinetic energy on a certain scale are distributed in the ratio 4:1 on the components with the double and half scale, respectively, if no other components are involved in the energy transformation.

It is interesting to note that the nature of the nonlinear interaction does not enter Fjørtoft's argument in an essential manner, just the two conservation laws. This is perhaps the reason why a subtle point in the nonlinear interaction was overlooked by [15]. It turns out that the three scales involved in the energy transformation must satisfy the triad requirement (see Appendix B). In fact, the wavevectors l, k, m have to obey $m \pm l = k$ in addition to $|l| < |k| < |m|$. This imposes restrictions on the three interacting scales and may lead to interpretations other than that in [15]. In fact, there are two possibilities:

$$|l| \sim |k| \sim |m|,$$

$$|l| \ll |k| \sim |m|.$$

The first possibility contains two cases since it is quite satisfactory for $|k|$ to be either closer to $|l|$ or closer to $|m|$. In the first case

$$\frac{\Delta E(|l|)}{\Delta E(|m|)} \quad > \quad 1, \tag{2.13}$$

so the conclusion in [15] is correct except for the fact that the scales of double and a half of the intermediate scale in the example is too liberal of a choice as they can not satisfy the triad requirement[1]. In the second case the energy cascade actually reverses direction as the above inequality reverses direction (this error of [15] was first pointed out by Merilees and Warn [46]). The second possibility is an extreme limit of

[1]Although the geometry considered in [15] is a sphere, interactions in the form of triads are to be observed [57]. For an interacting triad of similar scales where one wishes to have the smallest and largest scales as far away from the intermediate scale as possible, the scale ratio of 1:2:3 is a better approximation than 1:2:4. Consequently, the ratio of the upscale cascade energy to the downscale cascade energy is approximately 5:3.

the second case. It depicts the interaction between two small scales via a large scale. For this type of triad the energy transfer obeys

$$\begin{aligned} \frac{\Delta E(|m|)}{\Delta E(|k|)} &\sim -1 \\ \frac{\Delta E(|l|)}{\Delta E(|m|)} &\ll 1. \end{aligned} \tag{2.14}$$

Hence, it implies that the change of the energy for mode l is only a small fraction of those for modes k and m. As a result, most of the energy involved in the transformation is engaged in the exchange between the two small-scale modes, and the role of the large-scale mode is only to mediate this exchange.

There remains the question of which type of interacting triad is most likely to be activated for a given spectral distribution of energy. This highly nontrivial problem was explored in part by [46] in which the statistics of triads which transfer energy upscale and downscale was examined. However, the question remains unanswered to date.

2.3 Asymptotic behaviour: Dynamical constraint

In this section a satisfactory answer to the above question will be sought and the problem of energy distribution and energy transfer will be considered in the context of the NS system. It will be shown that energy tends to flow back and forth across the injection scale with certain rules. Careful examination of these rules reveals that although most of the system energy may reside in the larger scales, the realization of an upscale or a downscale cascade depends on the current state of the system.

By multiplying (2.3) by u and Au and integrating the resulting equations over the domain, one obtains the evolution equations for the kinetic energy and enstrophy respectively as follows

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \| u \|^2 + \nu \| u \|_1^2 &= (f, u), \\ \frac{1}{2}\frac{d}{dt} \| u \|_1^2 + \nu \| u \|_2^2 &= (f, Au), \end{aligned}$$

where the two identities (2.9) and (2.11) have been used. Now assume that f is a monoscale forcing, i.e., f belongs to an eigenspace of A of eigenvalue λ_s, say. To avoid repeated specification, such a forcing will be denoted by $f(|s|)$. Multiplying the former equation by λ_s and subtracting it from the latter yields [note that the forcing terms cancel since $(f, Au) = (Af, u) = \lambda_s(f, u)$]

$$\frac{1}{2}\frac{d}{dt}\left\{\| u \|_1^2 - \lambda_s \| u \|^2\right\} + \nu\left\{\| u \|_2^2 - \lambda_s \| u \|_1^2\right\} \quad = \quad 0. \tag{2.15}$$

As will be seen the energy at the injection scale spreads out with a definite rule, but the dynamics of the NS system renders no clear notion of energy or enstrophy cascades. Alternatively, an energetic downscale flow (upscale flow) phase can be defined according to (2.15). The system is said to be in a downscale flow (upscale flow) phase if the first term in (2.15), i.e. the time derivative term, is positive (negative). More discussion on this topic will follow shortly. Meanwhile, the main concern is how energy is distributed on the two sides of the injection scale.

One has, for $\phi \in D(A)$

$$\| (A - \lambda_s)\phi \|^2 \quad = \quad \| \phi \|_2^2 - 2\lambda_s \| \phi \|_1^2 + \lambda_s^2 \| \phi \|^2 \quad \geq \quad 0,$$

so

$$\| \phi \|_2^2 - \lambda_s \| \phi \|_1^2 \quad = \quad \lambda_s(\| \phi \|_1^2 - \lambda_s \| \phi \|^2) + \| (A - \lambda_s)\phi \|^2 . \tag{2.16}$$

It then follows from (2.15) that

$$\frac{1}{2}\frac{d\xi}{dt} + \nu\lambda_s\xi \quad = \quad -\nu \| (A - \lambda_s)u \|^2, \tag{2.17}$$

where $\xi \;=\; \| u \|_1^2 - \lambda_s \| u \|^2$. This equation indicates that any positive value of ξ will be dissipated away in a finite time, except for a special case which will be dealt with in a moment; while a non-positive value will evolve but remain non-positive for all time. To see this, consider the solution of (2.17), for all $t \geq t_0$, given below:

$$\xi(t) \quad = \quad \exp\{-2\nu\lambda_s t\}\left(\xi_0 \exp\{2\nu\lambda_s t_0\} - 2\nu\int_{t_0}^{t} \exp\{2\nu\lambda_s t'\} \| (A - \lambda_s)u(t') \|^2 \, dt'\right), \tag{2.18}$$

where $\xi_0 = \xi(t = t_0)$. It is easy to see that if $\xi_0 \leq 0$ then $\xi(t) \leq 0$ for all time. On the other hand, if $\xi_0 > 0$ then $\xi(t)$ becomes non-positive for all $t \geq T$ where T is the solution of

$$\xi_0 \exp\{2\nu\lambda_s t_0\} \quad = \quad 2\nu \int_{t_0}^{t} \exp\{2\nu\lambda_s t'\} \parallel (A - \lambda_s)u(t') \parallel^2 dt'. \tag{2.19}$$

The above equation always has a finite solution unless $\parallel (A - \lambda_s)u(t') \parallel$ decreases exponentially in time more rapidly than $\exp\{-2\nu\lambda_s t\}$. This may occur when u asymptotically approaches, in the $D(A)$-norm, the eigenspace of A with eigenvalue λ_s. For convenience this eigenspace will be denoted by $H(\lambda_s)$. Since $H(\lambda_s)$ is part of the stable manifold of $\bar{u} = (\nu\lambda_s)^{-1} f$ (this can be seen from the fact, by appendix B, that $B(u, u) = 0$ for all $u \in H(\lambda_s)$, so $H(\lambda_s)$ is invariant and every trajectory on it exponentially converges to $\bar{u}$ at the rate of $\nu\lambda_s$), any trajectories which asymptotically approach $H(\lambda_s)$ also converge to $\bar{u}$. Therefore, the case in which a finite solution T is questionable turns out to be restricted to trajectories on the stable manifold of $\bar{u}$ and the proof of this goes as follows. Let

$$u \quad = \quad U + v,$$

where U is the projection of u onto $H(\lambda_s)$ and v is the component of u outside $H(\lambda_s)$ which satisfies

$$\parallel v \parallel_2 \quad \to \quad 0 \quad \text{as} \quad t \to \infty.$$

This requirement implies, by the Poincaré inequality, that the energy and enstrophy norms of v behave in the same manner as well. Taking the scalar product of the NS equation with a basis function of $H(\lambda_s)$, say e_k, yields

$$\begin{aligned} \frac{d}{dt}u_k + \nu\lambda_s u_k \quad &= \quad -(e_k, B(U + v, U + v)) + f_k \\ &= \quad -(e_k, B(U, U) + B(v, U) + B(U, v) + B(v, v)) + f_k \\ &= \quad -(e_k, B(v, U) + B(U, v) + B(v, v)) + f_k, \end{aligned} \tag{2.20}$$

where f_k denotes the e_k-component of f and one of the trilinear forms drops because $B(U,U)$ identically vanishes. The solution of this equation can be written formally for all $t \geq t_0$ as

$$\begin{aligned} u_k(t) &= u_k(t_0)\exp\{-\nu\lambda_s(t-t_0)\} + \frac{f_k}{\nu\lambda_s}(1-\exp\{-\nu\lambda_s(t-t_0)\}) \\ &\quad - \int_{t_0}^{t} \exp\{-\nu\lambda_s(t-t')\}(e_k, B(v,U)+B(U,v)+B(v,v))dt'. \end{aligned} \tag{2.21}$$

The first term on the right-hand side of the above equation vanishes in the limit $t \to \infty$, while the second term approaches $f_k/\nu\lambda_s$ which is just the e_k-component of $\bar{u}$. The integral term also goes to zero. This fact is due to a couple of estimates below. One has

$$|(e_k, B(v,U)+B(U,v)+B(v,v))| \leq \sup|e_k|(\| v \|\| U \|_1 + \| U \|\| v \|_1 + \| v \|\| v \|_1)$$

and by assumption $\| v \|, \ \| v \|_1 \ \to 0$ as $t \to \infty$. So for all $\epsilon > 0$ there exists a sufficiently large t_0 such that

$$\begin{aligned} &\int_{t_0}^{t} \exp\{-\nu\lambda_s(t-t')\}(e_k, B(v,U)+B(U,v)+B(v,v))dt' \\ &\qquad \leq \epsilon \int_{t_0}^{t} \exp\{-\nu\lambda_s(t-t')\}dt' < \frac{\epsilon}{\nu\lambda_s}\,. \end{aligned} \tag{2.22}$$

Therefore, it can be concluded that the integral term vanishes in the limit $t \to \infty$. Finally, since $e_k \in H(\lambda_s)$ can be chosen at will the result implies that $u \to \bar{u}$. Thus,

$$u \longrightarrow \bar{u} \ \text{ if } \ \| (A-\lambda_s)u \| \longrightarrow 0, \tag{2.23}$$

as $t \to \infty$.

The argument in the last paragraph indicates that there can be no trajectories which asymptotically approach $H(\lambda_s)$ except for those on the stable manifold of $\bar{u}$. This means that for a general trajectory not on the stable manifold of $\bar{u}$, ξ acquires a non-positive value in a finite time. But before drawing a conclusion for ξ on the global attractor it is necessary to secure the non-positiveness of ξ on homoclinic trajectories of $\bar{u}$ which, if they exist, are part of the global attractor. Recall that a homoclinic trajectory (also known as a homoclinic orbit) of a stationary solution is the one which emanates from the stationary solution and terminates on that same stationary solution. It is a special limit cycle[2] with infinite period. It can be seen from (2.18) that ξ does not acquire any positive value on any homoclinic orbit of $\bar{u}$ during its infinitely long journey if such an orbit exists. It would rather start from $\bar{u}$ with $\xi = 0$, evolve with some negative value for ξ, and then ξ would increase to terminate at $\bar{u}$ with $\xi = 0$ again. Therefore, it can be concluded that in the permanent regime (on the global attractor) of the NS system, the energy and enstrophy satisfy

$$\lambda_s \parallel u \parallel^2 \;\; \geq \;\; \parallel u \parallel_1^2 . \tag{2.24}$$

It is noted, for the sake of completeness, that the argument leading to (2.24) also implies that $\bar{u}$ is the only point on the global attractor where the equality occurs. Ineq. (2.24), together with the Poincaré inequality, gives

$$\lambda_s \parallel u \parallel^2 \;\; \geq \;\; \parallel u \parallel_1^2 \;\; \geq \;\; \lambda_1 \parallel u \parallel^2 . \tag{2.25}$$

It is interesting to note that the dynamical constraint (2.24) is a generic property for the following class of linear dissipation mechanisms which may be collectively called the general viscosity. Consider the class of operators σA^η with $\eta \geq 0$ and σ a positive constant of appropriate dimension. η may be called the degree of viscosity. Note that one has hyperviscosity for $\eta > 1$, normal viscosity for $\eta = 1$, and hypoviscosity for $\eta < 1$. Also note that in the last case the hypoviscosity reduces to linear Ekman drag for $\eta = 0$. With νA replaced by σA^η (or by any linear combination of the elements of this class), (2.24) remains valid. This claim is substantiated by the

[2]Note that a regular limit cycle which would intersect $H(\lambda_s)$ does not exist because $H(\lambda_s)$ is part of the stable manifold of $\bar{u}$. More quantitatively, the existence of such a cycle would imply that $\int_0^{T_0} \exp\{2\nu\lambda_s t'\} \parallel (A - \lambda_s)u(t') \parallel^2 dt' \;=\; 0$, where T_0 would be the period of the cycle. But this would require the cycle to be entirely in $H(\lambda_s)$ which would further reduce the cycle to $\bar{u}$.

fact that for all $\phi \in D(A^{(1+\eta)/2})$, $\eta \geq 0$,

$$\sum_{|k|<|s|} (\lambda_s^\eta - \lambda_k^\eta)(\lambda_s - \lambda_k)E(|k|) + \sum_{|k|>|s|} (\lambda_k^\eta - \lambda_s^\eta)(\lambda_k - \lambda_s)E(|k|) = \epsilon(\eta, \lambda_s),$$

where $E(|k|)$ is the energy of ϕ associated with $H(\lambda_k)$ and $\epsilon(\eta, \lambda_s)$ is positive unless $\phi \in H(\lambda_s)$ or $\eta = 0$ in which cases $\epsilon(\eta, \lambda_s) = 0$. The steps below are straightforward:

$$\sum_k \lambda_k^\eta(\lambda_k - \lambda_s)E(|k|) = \sum_k \lambda_s^\eta(\lambda_k - \lambda_s)E(|k|) + \epsilon(\eta, \lambda_s),$$

so that

$$\| \phi \|_{1+\eta}^2 - \lambda_s \| \phi \|_\eta^2 = \lambda_s^\eta(\| \phi \|_1^2 - \lambda_s \| \phi \|^2) + \epsilon(\eta, \lambda_s). \tag{2.26}$$

It is easy to see, for positive η, that the replacement of νA by σA^η in the NS system only leads to replacing (2.16) by (2.26) in the argument leading to (2.24).

The case of Ekman drag is special since it leads to

$$\frac{1}{2}\frac{d}{dt}\left\{\| u \|_1^2 - \lambda_s \| u \|^2\right\} + \sigma\left\{\| u \|_1^2 - \lambda_s \| u \|^2\right\} = 0, \tag{2.27}$$

which renders the equality sign instead of the inequality sign in (2.24). There is no fluctuation in the value of ξ for this case. Instead, every value of ξ decreases or increases monotonically and asymptotically approaches zero. As it is an equality, the constraint in this case is stiffest. It also indicates that more energy is allowed to be distributed on the small scales than in the truly viscous cases (for all $\eta > 0$). This possibility is explored in the last section of the present chapter where Ekman drag is added to the traditional Navier-Stokes equations to model large-scale dissipation.

It should be admitted that the above result for hypoviscosity, including Ekman drag, is not rigorous from a mathematical point of view. There has been no attempt to demonstrate the existence of a solution for each case, let alone the existence of a global attractor. Nonetheless, the universal effect of the general viscosity in constraining the large-time behaviour of the solution is interesting and merits attention, especially for modellers in meteorology where an effective dissipation mechanism (or a combination of different mechanisms) has constantly been sought.

A couple of constraints on stationary solutions other than $\bar{u}$ of the NS system which, together with (2.24), serve as an indication of how little energy may reach the small scales, are stated below. For notational convenience the stationary solution $\bar{u} = (\nu A)^{-1} f$ will be called the primary stationary solution. A stationary solution other than $\bar{u}$ (if it exists) will be identified as a secondary stationary solution (the existence of such a solution for suitable conditions is demonstrated in [25]). Such a secondary solution $\bar{u}'$ satisfies

$$\| \bar{u}' \|_2^2 - \lambda_s \| \bar{u}' \|_1^2 = 0, \tag{2.28}$$

$$\lambda_s^2 \| \bar{u}' \|^2 - \| \bar{u}' \|_2^2 > 0. \tag{2.29}$$

The equality is obvious from (2.15). Meanwhile the strict inequality can be deduced from the fact that $\bar{u}$ is the only stationary solution which belongs to the eigenspace of A with eigenvalue λ_s. That means that $\| (A - \lambda_s)\bar{u}' \|^2$ is strictly positive. Expanding this expression and comparing with the equality yields

$$\| (A - \lambda_s)\bar{u}' \|^2 = -\lambda_s \| \bar{u}' \|_1^2 + \lambda_s^2 \| \bar{u}' \|^2 > 0.$$

Combining this with (2.28) yields (2.29) (note that this inequality is an equality for $\bar{u}$).

Ineq. (2.24) spells out an upper bound on the energy which the small scales (whose wavenumbers are greater than $|s|$) can acquire. But given an initial state satisfying (2.24), an alternation between the upscale flow and the downscale flow phases should be characteristic of the system evolution. To see that one may rewrite (2.15) as

$$\frac{1}{2}\frac{d}{dt}\left\{\lambda_s \| u \|^2 - \| u \|_1^2\right\} = \nu\left\{\| u \|_2^2 - \lambda_s \| u \|_1^2\right\}. \tag{2.30}$$

During the course of its evolution, a trajectory may undergo an upscale flow or downscale flow phase depending on whether the right-hand side of (2.30) is positive or negative. In each phase, there may be local cascades of opposite directions so that the right-hand side of (2.30) may stay positive (negative) longer, thus resulting in a more lasting phase. Whereas, if there is no or little local energy cascade in the direction opposite to the mainstream of the energy flow, the phase will be short-lived. In any case, the right-hand side of (2.30) is expected to oscillate around zero for a trajectory other than fixed points (i.e., stationary solutions of the NS equations). The average amplitude and frequency of this oscillation may be taken as two measures of dynamic complexity. Another possibility is that the system could asymptotically reach a solution with

$$\begin{aligned} \lambda_s \parallel u \parallel^2 - \parallel u \parallel_1^2 &= +\text{const}, \\ \parallel u \parallel_2^2 - \lambda_s \parallel u \parallel_1^2 &= 0. \end{aligned} \tag{2.31}$$

That solution could be a stable secondary stationary flow, a limit cycle, or just a turbulent solution. The first two cases are more likely to be realized for a weak forcing, while the last and more interesting case might be expected for a strong forcing. Inasmuch as it is a speculation, the idea of turbulence with the above constraints remains to be tested. A numerical approach might be plausible for this problem.

2.4 Dissipation ranges

In the classical theory of 2D turbulence there is a dissipation range, say $k^2 > \lambda_d$, which is separated from the forcing scale by an inertial range (or inertial subranges) and within which all dissipation is supposed to take place. The idea of inertial subranges was suggested by Kraichnan [27], Leith [30], and Batchelor [3] and was later advanced by Kraichnan [28] and widely accepted by researchers in fluid dynamics. This idea is found to be inapplicable to the present system, if not in serious jeopardy. In fact, the results from the last section indicate that both energy dissipation and enstrophy dissipation are confined to scales around the injection scale (assumed to be large).

This means that the energy spectrum adjusts so that molecular viscosity acts on the large scales. Thus, $\lambda_d > \lambda_s$ leads to a contradiction. This section will be devoted to making this statement more quantitative. A second aim is to explore the possible scaling forms of the energy spectrum.

To facilitate the analysis the following definition can be introduced for a trajectory on the global attractor. Let Λ_{ED} be defined by

$$\Lambda_{ED} \parallel u \parallel^2 \;=\; \parallel u \parallel_1^2, \tag{2.32}$$

It can be seen that Λ_{ED} defines the average wavenumber (length scale) about which molecular viscosity operates on the dissipation of energy and $\nu\Lambda_{ED}$ is the (instantaneous) energy dissipation rate. A comparison between (2.24) and (2.32) gives

$$\Lambda_{ED} \;\leq \lambda_s. \tag{2.33}$$

Hence, the energy dissipation rate is bounded above by $\nu\lambda_s$. A similar definition can be stated for the dissipation of enstrophy. It is, however, more convenience to give a definition with an average sense. Let Λ_{ZD} be defined by

$$\Lambda_{ZD}\frac{1}{t}\int_0^t \parallel u \parallel_1^2 d\tau \;=\; \frac{1}{t}\int_0^t \parallel u \parallel_2^2 d\tau, \tag{2.34}$$

where $t \geq T$ for some large T. The initial point of the above definition is immaterial in this analysis as long as it is not on the stable manifold of $\bar{u}$. It may be assumed to belong to a chaotic trajectory on the global attractor, so that the averages of $\parallel u \parallel_1^2$ and $\parallel u \parallel_2^2$ may be taken as those of the attractor for sufficiently large T (because a chaotic trajectory is believed to sample all regions of the attractor in the phase space). This subtle point is behind but does not enter the arguments to follow in any manner. One then has by taking the time average of (2.30) for large time

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t (\parallel u \parallel_2^2 - \lambda_s \parallel u \parallel_1^2) d\tau \;=\; 0. \tag{2.35}$$

It follows that

$$\Lambda_{ZD} \;\sim\; \lambda_s. \tag{2.36}$$

Therefore,

$$\Lambda_{ED} \leq \Lambda_{ZD} \sim \lambda_s. \tag{2.37}$$

An attempt to predict the scaling law of the energy spectrum shall be made based on the observation that both the dissipation of energy and the dissipation of enstrophy are confined to the large scales, as indicated by (2.37). Let $\epsilon(k)$ be defined so that the total energy is given by

$$E = \int_0^\infty \epsilon(k) dk,$$

Note that the discrete wavenumber has been changed to a continuous one and $|k|$ is replaced by k for convenience. Now suppose that $\epsilon(k) \propto k^{-\eta}$ over a range $[k_0, k_1]$ where $|s| < k_0 \ll k_1$, then the dissipation of energy and enstrophy over that range is given respectively by

$$\begin{aligned} ED &\propto \int_{k_0}^{k_1} k^{2-\eta} dk = \frac{1}{3-\eta}\left(k_1^{3-\eta} - k_0^{3-\eta}\right), \\ ZD &\propto \int_{k_0}^{k_1} k^{4-\eta} dk = \frac{1}{5-\eta}\left(k_1^{5-\eta} - k_0^{5-\eta}\right). \end{aligned} \tag{2.38}$$

It is required that both ED and ZD be dominated by the large scales. As a result, the satisfactory values of η are confined to $\eta > 5$.

It is interesting to note that a scaling law of the energy spectrum $k^{-\eta}$ with $\eta > 5$ is much steeper than the classical k^{-3} inertial subrange power law (see Kraichnan [27, 28]). Hence, the result of this section suggests that the latter scaling law is unrealizable in 2D turbulence.

Besides Kraichnan's theory which gives the predicted values for $\eta = 3$, as mentioned above, there exist a number of predictions for the numerical value of η. Saffman [55] proposes $\eta = 4$, while Moffatt [48] favours a slightly smaller value: $\eta = 11/3$. Sulem and Frisch [59], instead, propose an upper bound: $\eta \leq 11/3$. Most recently, Polyakov [52] considers an infinite number of possible spectra with $\eta > 3$. This prediction of Polyakov is seen to be compatible with the present results and has a far-reaching physical significance. It is possible that η is sensitive to the physical parameters. A reasonable speculation would be that the spectrum may not simply scale linearly with the forcing amplitude when all else remains equal, for example. Thus, the notion of a common value of η for quite different physical conditions is perhaps conceivable but highly speculative.

2.5 Large-scale dissipation: Ekman drag

For 2D fluid systems driven by a monoscale forcing of the scale $2\pi\lambda_s^{-1/2}$ the results in the previous section indicate that the dissipation rate is $\nu\lambda_s$ ($\leq \nu\lambda_s$) for the enstrophy (energy). Consider a fluid system the size and viscosity of which are $|\Omega| = O(10^{14}m^2)$ and $\nu = O(10^{-5}m^2/s)$ respectively (typical of surface air on a synoptic scale) driven by a forcing the scale of which is $O(10^6 m)$. The dissipation rate for this system is $O(10^{-17}/s)$ which translates to a dissipation time of billions of years. This rate is ineffective by any standard. Thus, such a system is virtually inviscid for practical purposes (a finite-dimensional and bounded attractor exists, though).

It is, incidentally, common practice for researchers and modellers in meteorology to raise ν up to many billion times. The value of ν, then known as eddy viscosity or turbulent viscosity, may range through several orders of magnitude depending on the domain scales. In global models, for example, the vertical scale is about $10^4 m$ and the corresponding (vertical) turbulent viscosity may be $10^2\nu$ or $10^3\nu$, whereas the horizontal scale is about $10^7 m$ and the corresponding (horizontal) turbulent viscosity may be $10^{10}\nu$ or $10^{11}\nu$ [20], where $\nu = 1.4\times10^{-5}m^2/s$ is a standard value of the surface air kinematic viscosity. It is noted that the latter turbulent viscosity is typical of very viscous substances like glucose rather than air or water. In mesoscale models as used in frontogenesis studies, the turbulent viscosities can be found to range from 10ν to $10^6\nu$ corresponding to the vertical and horizontal scales of $10^3 m$ and $10^5 m$ respectively [65, 66]. Thus, turbulent viscosity seems to be adjusted to grow as the square of the domain scale (as $|\Omega|$ in 2D problems).

The use of turbulent viscosity has been claimed to arise from the need to compensate for the loss of energy due to the smallest scales which are absent altogether in atmospheric numerical models. It represents the average dissipation by the small-scale fluctuations which are no longer resolvable by the models. More discussion of this subject can be found in [20, 45]. McComb [45] presents a derivation of turbulent viscosity based on closure approximations. Except in unbounded isotropic turbulence, turbulent viscosity depends on the length scales of the flow (the domain scale) and the smallest scale retained in a model (the model resolution). In the limit of small length scales the turbulent viscosity reduces to the molecular viscosity. Hence, it is intrinsically inappropriate to try to use it in analytical studies where molecular viscosity is in full force.

But according to the results of the previous sections, energy loss due to small-scale fluctuations should be small for the NS system since the dissipation of both energy and enstrophy is confined to the injection scales, which are presumably well resolved by most models. Hence, the practice of employing a turbulent viscosity in modelling two-dimensional Navier-Stokes flows does not seem to have a sound theoretical basis. What is behind the practice? A possible answer may be found in the next section. It will be demonstrated there that the practice might just be an attempt to incorporate other dissipation mechanisms together with the existing molecular viscosity.

The remainder of this section describes in detail a frictional mechanism that is proposed to model 2D fluid systems which are so large (driven by forces of large scale) that friction due to the bottom boundary (and possibly the top boundary) dominates or at least competes favourably with viscosity. This dissipation mechanism is not new to meteorologists who struggle with the modelling of the effects of the rough planetary surface on the wind field. While it is not clear how to model this dissipation channel most realistically, the following heuristic argument will serve as a starting point.

For a thin layer of viscous incompressible fluid on a planetary scale to be treated as a 2D system, i.e., governed by the 2D Navier-Stokes equations, what might have been overlooked in the dimension reduction? A clear omission would be friction caused by the planetary surface boundary on the main (horizontal) flow. A less obvious omission would be turbulent eddy activity in the thin (vertical) dimension as would be expected when the main flow is strong. Those tiny eddies, being continually created, extract energy from the main flow, return part of the energy as they move along and dissipate the rest in the form of heat. To account for these effects, it is reasonable to retain the full 3D Laplacian for the horizontal velocity in the 2D Navier-Stokes equations. In other words, the third dimension is dynamically but not energetically ignored in the problem. Hence, one obtains the following "hybrid" system by so doing

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u &= -\nabla p + \nu \Delta u + \nu \partial_{x_3}^2 u + f \\ \nabla \cdot u &= 0 \quad \text{in} \quad \Omega \times (0, \infty). \end{aligned} \tag{2.39}$$

All notation in (2.39) is the same as in the 2D Navier-Stokes system except for the extra term $\nu \partial_{x_3}^2 u$. This term can be approximated in the case of laminar flows using the model of the frictional boundary layer obtained by Ekman [13]. In Ekman's model the horizontal stress S and the wind shear are related by

$$S = \nu \partial_{x_3} u.$$

It follows that

$$\nu \partial_{x_3}^2 u = \partial_{x_3} S.$$

Now the stress is zero outside the boundary layer (or Ekman layer), so that $\partial_{x_3} S$ can be approximated by

$$\partial_{x_3} S = -\frac{1}{h} S_0,$$

where h is the thickness of the Ekman layer and S_0 is the surface stress.

The stress at the surface is regarded as a drag exerted by the surface which tends to reduce the wind and therefore is a form of friction. To calculate its magnitude, the stress must be related to the wind in some way. The usual formulation is given in terms of a drag law in which the stress is related to the wind at some standard height above the surface. The following form of the drag law is commonly adopted:

$$S_0 = c_d|u_d|u_d,$$

where c_d is a dimensionless coefficient called the drag coefficient, and u_d is the velocity at the level mentioned above which may be taken as u. With this form of the surface stress, (2.39) reads

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u &= -\nabla p + \nu \Delta u - \sigma u + f \qquad (2.40) \\ \nabla \cdot u &= 0 \quad \text{in} \quad \Omega \times (0, \infty), \end{aligned}$$

where σ is defined by

$$\sigma = \frac{c_d}{h}|u|. \qquad (2.41)$$

Although the nonlinear form of σ which arises from the nonlinear nature of the drag law may be appropriate, it is not a desirable form. Instead, a linear operator is easier to handle. One way to linearize this term is by substituting close estimates of the parameters involved. Measurements in the atmospheric boundary layer give $c_d = O(10^{-4})$ [20], while the average wind speed of several meters per second can be taken for $|u|$, and the depth of the Ekman layer h of a couple of hundred meters might be considered appropriate. These values produce a constant $\sigma = O(10^{-5}s^{-1})$ or a dissipation time of a couple of days.

The main goal of this section, which has already been achieved, is to “derive” the constant σ (this dissipation mechanism will be referred to as the Ekman drag). However, a constant σ is not the only product of the linearization process. Another possible form of σ is one of the whole class of hypoviscous operators. The reason why this form of σ is appealing is that the cubic term which represents the total dissipation due to σ cannot be properly estimated by the energy-norm alone. That means that the replacement of σ by a constant suffers from the same shortcoming. A standard estimation would involve the enstrophy-norm as by Ladyzhenskaya's inequality (3.2):

$$\int_\Omega |u|^3 dx \;\le\; \left(\int_\Omega |u|^4 dx\right)^{1/2} \left(\int_\Omega |u|^2 dx\right)^{1/2} \;\le\; c_4^{1/2} \parallel u \parallel^2 \parallel u \parallel_1 .$$

Now a linearization of σ entails a "quadratization" of the above cubic term. It is quite reasonable that an intermediate norm $\parallel u \parallel_\eta$ with $0 < \eta < 1$ can be used as a compromise for that purpose. Thus, a hypoviscous operator to model the large-scale dissipation instead of Ekman drag may not be a bad idea.

2.6 Energy spectrum and dissipation ranges revisited

By the same formulation as in the case of the traditional 2D Navier-Stokes system, Eqs. (2.40) supplemented by the periodic boundary condition can be written in abstract form in H as follows

$$\begin{aligned} \frac{du}{dt} + B(u,u) + (\nu A + \sigma)u &= f, \\ u(0) &= u_0, \end{aligned} \tag{2.42}$$

where, from now on, σ denotes a constant. For convenience the initial boundary value problem (2.42) will from here on be referred to as the modified Navier-Stokes (MNS) system so as to distinguish it from the traditional form which has been designated as the NS system.

Although most of the results derived for the NS equations in section 3 are valid for the MNS system, in particular the asymptotic dynamical constraint (2.24), there is, however, a possible discrepancy in the energy spectrum of the chaotic attractors of the two systems. Physically, Ekman drag may suppress the upscale flow of energy since it is scale-independent. As a result, the spectrum of the MNS system may be less steep than that of the NS system. This section will assess how Ekman drag may bring about that difference by examining the spectrum of a secondary stationary solution if such a solution exists (the primary stationary solution in this case is $\bar{u} = (\nu A + \sigma)^{-1} f$), and the average spectrum of a chaotic trajectory.

Let $\bar{u}'$ be a secondary stationary solution of the MNS system; then $\bar{u}'$ satisfies

$$\begin{aligned} \nu \parallel \bar{u}' \parallel_1^2 + \sigma \parallel \bar{u}' \parallel^2 &= (\bar{u}', f), \\ \nu \parallel \bar{u}' \parallel_2^2 + \sigma \parallel \bar{u}' \parallel_1^2 &= \lambda_s (\bar{u}', f). \end{aligned}$$

It follows that

$$\nu \left\{ \parallel \bar{u}' \parallel_2^2 - \lambda_s \parallel \bar{u}' \parallel_1^2 \right\} = \sigma \left\{ \lambda_s \parallel \bar{u}' \parallel^2 - \parallel \bar{u}' \parallel_1^2 \right\}.$$

Since $\bar{u}'$ is necessarily on the global attractor, ineq. (2.24) holds. Moreover, the strict inequality of (2.24) applies to $\bar{u}'$ because the equality would imply

$$\parallel \bar{u}' \parallel_2^2 - \lambda_s \parallel \bar{u}' \parallel_1^2 = \lambda_s \parallel \bar{u}' \parallel^2 - \parallel \bar{u}' \parallel_1^2 = 0,$$

which would lead to $\parallel (A - \lambda_s) \bar{u}' \parallel^2 = 0$. This in turn would imply $\bar{u}' = \bar{u}$. Thus $\bar{u}'$ satisfies

$$\parallel \bar{u}' \parallel_2^2 > \lambda_s \parallel \bar{u}' \parallel_1^2 . \tag{2.43}$$

Note that this strict inequality would be an equality for the NS system. This difference indicates that the spectrum of a secondary stationary solution for the MNS system is less steep than that for the NS system, in the sense that it has more enstrophy for a given energy.

A similar result can be derived for a chaotic trajectory in the time-average sense. Following the same steps which lead to eq. (2.15), one obtains

$$\frac{1}{2} \frac{d}{dt} \left\{ \lambda_s \parallel u \parallel^2 - \parallel u \parallel_1^2 \right\} = \nu \left\{ \parallel u \parallel_2^2 - \lambda_s \parallel u \parallel_1^2 \right\} - \sigma \left\{ \lambda_s \parallel u \parallel^2 - \parallel u \parallel_1^2 \right\}. \tag{2.44}$$

It follows that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \left(\nu\left\{\| u \|_2^2 - \lambda_s \| u \|_1^2\right\} - \sigma\left\{\lambda_s \| u \|^2 - \| u \|_1^2\right\}\right) d\tau \quad = \quad 0. \qquad (2.45)$$

Hence, the following balance is reached for large time:

$$\frac{\nu}{t}\int_0^t \left\{\| u \|_2^2 - \lambda_s \| u \|_1^2\right\} d\tau \quad \sim \quad \frac{\sigma}{t}\int_0^t \left\{\lambda_s \| u \|^2 - \| u \|_1^2\right\} d\tau. \qquad (2.46)$$

It is recalled that the NS version of this expression has a zero on the right-hand side. Now because $\lambda_s \| u \|^2 - \| u \|_1^2$ is non-negative on the global attractor, the right-hand side is expected to be positive for a typical chaotic trajectory on the attractor. Hence, the above balance expression indicates that the distribution of energy is shifted toward the small scales as compared with the NS spectrum, for the same energy. To make this statement more explicit let E, Z, and P denote the large-time averages of $\| u \|^2$, $\| u \|_1^2$, and $\| u \|_2^2$ respectively. Expression (2.46) can be recast in the following form

$$\frac{P}{\lambda_s Z} \quad \sim \quad 1 + \frac{\sigma}{\nu\lambda_s}\left(\frac{\lambda_s E}{Z} - 1\right). \qquad (2.47)$$

The ratio $P/\lambda_s Z$ is likely to be greater than unity because $\lambda_s E/Z$ is likely to be greater than unity. The question is whether it can be much greater than unity for a system with $\sigma \gg \nu\lambda_s$, where the large-scale dissipation is dominated by Ekman drag. This is the same thing as asking whether the second term on the right-hand side of the above equation can be much greater than unity when $\sigma \gg \nu\lambda_s$. There is no evidence in this thesis that would support a definite answer to that question. It is not known how the ratio $\lambda_s E/Z$ would be adjusted with respect to the ratio $\sigma/\nu\lambda_s$. Nonetheless, it is clear that a slight increase in the ratio $\lambda_s E/Z$ would entail an enormous increase in $P/\lambda_s Z$ when $\sigma \gg \nu\lambda_s$. Hence, an upscale energy cascade would entail a downscale enstrophy cascade. Thus, a dual cascade appears to be possible for the MNS system.

The following remark is made in passing for the sake of completeness. For a trajectory emanating from the primary stationary solution and evolving along the unstable manifold the integrated value of the derivative on the left-hand side of (2.44) is always non-negative, so that a stronger statement can be made:

$$\nu \int_0^t \left\{ \| u \|_2^2 - \lambda_s \| u \|_1^2 \right\} d\tau \;\; \geq \;\; \sigma \int_0^t \left\{ \lambda_s \| u \|^2 - \| u \|_1^2 \right\} d\tau,$$

for all $t > 0$.

Finally, an exploration of the scaling law of the energy spectrum of the MNS system shall conclude this chapter. With the same notation as for the NS equations employed in section 2 one finds

$$\begin{aligned} ED &\propto \int_{k_0}^{k_1} \left(\sigma k^{-\eta} + \nu k^{2-\eta} \right) dk = \frac{\sigma}{1-\eta} \left(k_1^{1-\eta} - k_0^{1-\eta} \right) + \frac{\nu}{3-\eta} \left(k_1^{3-\eta} - k_0^{3-\eta} \right), \\ ZD &\propto \int_{k_0}^{k_1} \left(\sigma k^{2-\eta} + \nu k^{4-\eta} \right) dk = \frac{\sigma}{3-\eta} \left(k_1^{3-\eta} - k_0^{3-\eta} \right) + \frac{\nu}{5-\eta} \left(k_1^{5-\eta} - k_0^{5-\eta} \right). \end{aligned} \tag{2.48}$$

The requirement that ED be dominated by the large scales implies that $\eta > 3$. But this requirement is not necessary for ZD. In fact, in the spirit of (2.47) the dissipation term due to viscosity in the equation of ZD should be dominated by the small scales. Hence a value of η in the range $3 < \eta < 5$ would be satisfactory.

Chapter 3

Attractor dimension: Extensivity

The need for estimating the number of degrees of freedom of the Navier-Stokes equations arises from the ambitious quest of modelling turbulence by a finite number of ordinary differential equations. It was somewhat disappointing when the first works on this problem were reported. The results from those works revealed an astronomical number for the attractor dimension of typical fluid systems. Since then the original goal has shifted toward more theoretical aspects. One of the aspects is: How does the number of degrees of freedom grow with the Reynolds number, the Grashof number, or other physical parameters? The answers to this question constitute a rich literature. This chapter seeks answers to the same question where the emphasis is on the domain size.

This chapter consists of three parallel sections. Section 1 presents both a novel and a previously known estimate of the attractor dimension of the NS system on a square domain. The estimates are then examined from the perspective of extensive chaos. In section 2, an estimate of the attractor dimension of the NS system on a rectangular domain is presented. The derivation follows that of Ziane [67] with some variations in accordance with the theme of the thesis. Section 3 repeats the same calculations and analysis as in section 1 for the MNS system.

3.1 Attractor dimension of the NS system ($\alpha = 1$)

In this section the extensivity of the NS system driven by a forcing of a fixed scale and a fixed amplitude (i.e., $\| f \|^2$ grows linearly with Ω while λ_s remains constant) will be demonstrated. This condition implies that the quantity $\| f \|_{-1}^2 /|\Omega|$ and the density of the enstrophy forcing, $\| f \|_1^2 /|\Omega|$, are fixed. The dependence of D_H on other parameters such as the dissipation length and the forcing length will also be discussed. In addition, a new length scale which dictates the regime of extensive chaos for sufficiently large systems will be introduced.

3.1.1 Basic inequalities

Several inequalities are needed for the dimension calculation. Most of these inequalities were derived from much more general assumptions and therefore have had wider application. In this thesis, however, the main concern lies with the spaces H, V, $D(A)$ so that a general formulation would entail unnecessary complication. The inequalities will, therefore, be restated in a "narrow sense" and the interested reader is directed to relevant references for more information.

The most important inequality used in this thesis is one of the celebrated Lieb-Thirring inequalities. This class of sophisticated inequalities is an improvement of the classical Sobolev-Gagliardo-Nirenberg inequalities and was first proven by Lieb and Thirring [33] (see also [23]), hence the name. It gives improved estimates for the trace of linear Schrödinger-type operators, and provides in theoretical physics a proof of the stability of matter. Apart from the initial quantum-mechanical applications, it is essential in the derivation of sharp estimates of the dimension of attractors of the semigroups generated by dissipative partial differential equations.

The idea of using Lieb-Thirring inequalities in the study of attractors of the Navier-Stokes equations was first suggested by D. Ruelle [54], and some of the conjectures by [54] were later proven by E. Lieb [34]. But it was R. Temam who first realized Ruelle's suggestion by applying the natural role of the Lieb-Thirring inequalities in the estimate of the trace of a linear differential operator which appears in the dimension calculation process [60]. The version employed by Temam can be stated as follows: Let $\{\phi_i\}_{i=1}^m$ be a family of functions in V that are orthonormal in H, then

$$\| \sum_{i=1}^{m} |\phi_i|^2 \|^2 \ \leq \ c_1 \sum_{i=1}^{m} \| \phi_i \|_1^2, \tag{3.1}$$

where c_1 is a dimensionless constant independent of $\{\phi_i\}_{i=1}^m$ and of the domain size $|\Omega|$. Note that c_1 and other constants to follow in this section may depend on α. This dependence is important for $\alpha \ll 1$ and will be taken care of in the next chapter. However, all constants involved are absolute constants for a square domain.

An immediate corollary of (3.1) can be obtained by replacing the set $\{\phi_i\}$ by a single function $u/ \| u \|$ in V. But because it precedes (3.1) and is of special importance in the literature, it deserves to be quoted as an inequality of its own (see [29] for the original derivation, also see [16] for a sharp value of the constant). Let u be in V, then

$$\| u^2 \| \ \leq \ c_4 \| u \| \| u \|_1, \tag{3.2}$$

where c_4 is a dimensionless constant independent of u and of the domain size $|\Omega|$. Note that the best value for the constant in (3.2) is at most equal to $c_1^{1/2}$; hence, it is denoted differently.

The following fact is well-known of the eigenvalues λ_j of A (see Temam [60] for example):

$$\lambda_j \sim cj\lambda_1 \quad \text{as} \quad j \to \infty,$$

where c is a constant. Hence one can further deduce that

$$\sum_{i=1}^{m} \| \phi_i \|_\eta^2 \ \geq \ \sum_{i=1}^{m} \lambda_i^\eta \ \geq \ (c_2\lambda_1)^\eta m^{1+\eta}, \tag{3.3}$$

where η is a non-negative number, $\{\phi_i\}_{i=1}^m$ is a family of functions in $D(A^{\eta/2})$ that are orthonormal in H, and c_2 is a dimensionless constant independent of $\{\phi_i\}_{i=1}^m$ and of the domain size $|\Omega|$. The proof of the first ineq. can be found in [60], while the second ineq. is quite elementary.

Finally, the following ineq. is proven by P. Constantin in [6] (see also [32]). Let $\{\phi_i\}_{i=1}^m$ be a family of functions in $D(A)$ (i.e., with $\| A\phi_i \|^2 < \infty$) that are orthonormal in V, then

$$\sum_{i=1}^{m} |\phi_i|^2 \quad \leq \quad c_3 \left(1 + \ln(\lambda_1^{-1} \sum_{i=1}^{m} \| A\phi_i \|^2)\right) \tag{3.4}$$

where c_3 is yet another dimensionless constant independent of $\{\phi_i\}_{i=1}^m$ and of the domain size $|\Omega|$.

3.1.2 Preliminary estimates

A couple of estimates on the energy and enstrophy dissipation are needed for the calculations of the attractor dimension. As recalled from the last chapter, the energy evolution is governed by

$$\frac{1}{2}\frac{d}{dt} \| u \|^2 \quad = \quad -\nu \| u \|_1^2 + (u, f).$$

This equation is majorized by Schwarz's and Young's inequalities [1] respectively as follows

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \| u \|^2 \quad &\leq \quad -\nu \| u \|_1^2 + \| u \|_1 \| f \|_{-1} \\ &\leq \quad -\frac{\nu}{2} \| u \|_1^2 + \frac{\| f \|_{-1}^2}{2\nu}. \end{aligned}$$

Hence in the limit of large time,

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \| u \|_1^2 \, d\tau \quad \leq \quad \frac{\| f \|_{-1}^2}{\nu^2}.$$

For a forcing of single scale ($f(|s|)$) the above bound reads

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \| u \|_1^2 \, d\tau \quad \leq \quad \frac{\| f \|^2}{\nu^2 \lambda_s}. \tag{3.5}$$

[1] $ab \leq \frac{\alpha}{p} a^p + \frac{1}{p' \alpha^{p'/p}} b^{p'}, \quad \forall a, b, \alpha > 0,\ 1 < p < \infty,\ \frac{1}{p} + \frac{1}{p'} = 1.$
Here $\alpha = \nu,\ p = p' = 2.$

Similarly, the enstrophy equation gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\parallel u \parallel_1^2 &\leq -\nu \parallel u \parallel_2^2 + \parallel u \parallel_2 \parallel f \parallel \\ &\leq -\frac{\nu}{2}\parallel u \parallel_2^2 + \frac{\parallel f \parallel^2}{2\nu}.\end{aligned}$$

Therefore,

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \parallel u \parallel_2^2 \, d\tau \leq \frac{\parallel f \parallel^2}{\nu^2}. \tag{3.6}$$

3.1.3 Attractor dimension

Some well-known results on the attractor dimension of the NS equations are now reviewed. As shown in [2, 60] the upper bound on the Hausdorff dimension, D_H, of the global attractor of (2.3), where the domain Ω has a smooth rigid and nonslip boundary, is proportional to the generalized Grashof number, Gr, which is defined by

$$Gr = \frac{\parallel f \parallel}{\nu^2 \lambda_1}, \tag{3.7}$$

where λ_1 is the smallest eigenvalue of A. An improved estimate and sharpest to date for the doubly-periodic case is obtained in [7] in which

$$D_H \leq c \, Gr^{2/3}(1 + \ln Gr)^{1/3}, \tag{3.8}$$

where c is a constant independent of G_r. A similar result was also reported for the Navier-Stokes equations on a rotating sphere [23]. These results, besides serving to prove the finite dimensionality of the attractor, show that D_H depends on Gr in a fixed functional form regardless of the relative magnitudes of the parameters. A close examination of (3.8) reveals that D_H scales almost linearly with $|\Omega|$. To see this one may write

$$\parallel f \parallel = f_{rms}|\Omega|^{1/2}$$

where f_{rms} is the root mean square of f. Substituting (3.7) into (3.8) and noting $\lambda_1 = 4\pi^2/|\Omega|$, one has

$$D_H \leq c|\Omega| \left(\frac{f_{rms}}{4\pi^2\nu^2}\right)^{2/3} \left(1 + \ln \frac{f_{rms}|\Omega|^{3/2}}{4\pi^2\nu^2}\right)^{1/3}. \tag{3.9}$$

Thus, D_H scales linearly with $|\Omega|$ except for a $\ln|\Omega|$ correction term. This result has been claimed to be relevant to the classical theory of turbulence [7]. However, it is not in complete accordance with the idea of extensive chaos. Moreover, the result does not indicate how the structure of the forcing (its variation in space) affects D_H. It is not known whether long and short-wavelength forces of the same amplitude (i.e., the same $\| f \|$) would give rise to the same D_H, or what dependence D_H might have on this factor. This, along with the extensivity issue, will be addressed in the present chapter.

The attractor dimension for the NS system is estimated by the method of the CFT theorem which is briefly reviewed in appendix D. The first variation equation for (2.3) is given by

$$\frac{dU}{dt} + B(u, U) + B(U, u) + \nu AU = 0, \tag{3.10}$$

where U is a variation from u about which the linearization is taken. Let $\{\phi_i, i = 1, ..., m\}$ be an orthonormal basis of an m-dimensional subspace of H, and P_m be an m-dimensional projection in H; then the trace of the operator in the theorem is given by

$$\begin{aligned}
\mathrm{Tr}F'(u(\tau)) \circ P_m(\tau) &= \sum_{i=1}^{m} (\phi_i, -B(\phi_i, u) - \nu A\phi_i) \\
&= \sum_{i=1}^{m} \Big((\phi_i, -B(\phi_i, u)) - \nu \| \phi_i \|_1^2 \Big) \\
&\leq \| u \|_1 \| \sum_{i=1}^{m} \phi_i^2 \| - \nu \sum_{i=1}^{m} \| \phi_i \|_1^2 \\
&\leq \| u \|_1 \, (c_1 \sum_{i=1}^{m} \| \phi_i \|_1^2)^{1/2} - \nu \sum_{i=1}^{m} \| \phi_i \|_1^2
\end{aligned}$$

$$\begin{aligned} &\leq \frac{c_1 \parallel u \parallel_1^2}{2\nu} - \frac{\nu}{2}\sum_{i=1}^{m} \parallel \phi_i \parallel_1^2 \\ &\leq \frac{c_1 \parallel u \parallel_1^2}{2\nu} - \frac{c_2}{2}\nu\lambda_1 m^2. \end{aligned} \tag{3.11}$$

In the first line of (3.11) one of the trilinear forms drops out by virtue of (2.10). The third step involves simple manipulations of the remaining trilinear form $(\phi_i, B(\phi_i, u))$. The fourth step is from the Lieb-Thirring inequality where the constant c_1 is bounded by $c_1 \leq 2.89$ [23]. The fifth step is by Young's inequality. Finally the last step is by (3.3). The number q_m, i.e., the time average of (3.11), can be readily evaluated thanks to (3.5). It is found that

$$q_m \leq -\frac{c_2}{2}\nu\lambda_1 m^2 + \frac{c_1 \parallel f \parallel^2}{2\nu^3\lambda_s} . \tag{3.12}$$

A lower bound on m above which q_m is negative can be obtained from the above ineq. Such a lower bound on m is an upper bound for D_H by the CFT theorem, and the result is

$$D_H \leq \left(\frac{c_1}{c_2}\right)^{1/2} \frac{\parallel f \parallel}{\nu^2\lambda_s^{1/2}\lambda_1^{1/2}}. \tag{3.13}$$

This result differs from that of [2, 60] by having the factor $\lambda_s^{1/2}\lambda_1^{1/2}$ (instead of λ_1) in the denominator of the bound on the attractor dimension. This difference might seem marginal from the mathematical point of view. However, its physical implication is quite significant as can be seen from the following argument. For $\alpha = 1$, $\parallel f \parallel$ grows as L for a fixed forcing amplitude $|f|$ and $\lambda_1^{1/2} = 2\pi/L$. Now if ν and λ_s are assumed fixed, i.e., the kinematic viscosity and the forcing scale are held constant, then (3.13) apparently grows linearly with $|\Omega|$. Thus, (3.13) suggests the extensivity of the NS system. Moreover, it explicitly gives the dependence of D_H on the forcing scale. More discussion on this matter will be seen in the next section.

It is worthwhile to note that the restriction of the forcing to a monoscale force is not crucial for the extensivity of the NS system. Rather, it may be required that

$L^{-1} \parallel f \parallel_{-1}$ stay constant in a variable domain size for a more general driving force. This condition is much less restrictive and can be satisfied by a variety of body forces. Take, for example, a multiscale force, each and every Fourier component of which has a fixed amplitude and a fixed scale. In this case the condition is trivially satisfied. Another example is for a force, especially a force confined to a narrow range of wavenumbers, where the average forcing amplitude defined by

$$\overline{|f|} = \frac{\parallel f \parallel}{L}$$

also remains fixed. In this case a fixed average forcing scale can be defined by

$$\overline{L_s} = 2\pi \frac{\parallel f \parallel_{-1}}{\parallel f \parallel}.$$

These are a few examples of the driving forces that give rise to the extensivity of the NS system.

Another estimate which turns out to be optimal for a strong forcing and a small domain can be obtained by carrying out the above calculation in V, i.e., using the enstrophy norm instead of the energy norm. This procedure is possible thanks to a proposition of [3,21]. Let $\{\phi_i, i = 1, ..., m\}$ be an orthonormal basis (with respect to the V-norm) of an m-dimensional subspace of V. Let $P_m(\tau)$ be an m-dimensional projection in V. Then the trace $\text{Tr}F'(u(\tau)) \circ P_m(\tau)$ can be estimated, following [7], as follows

$$\begin{aligned}
& \text{Tr}F'(u(\tau)) \circ P_m(\tau) \\
= & \sum_{i=1}^{m} (A\phi_i, -B(\phi_i, u) - B(u, \phi_i) - \nu A\phi_i) \\
= & \sum_{i=1}^{m} \Big((A\phi_i, -B(\phi_i, u) - B(u, \phi_i)) - \nu \parallel \phi_i \parallel_2^2 \Big) \\
= & \sum_{i=1}^{m} (Au, B(\phi_i, \phi_i)) - \nu \sum_{i=1}^{m} \parallel \phi_i \parallel_2^2 \qquad (3.14) \\
\leq & \parallel u \parallel_2 \sup_{x\in\Omega} (\sum_{i=1}^{m} |\phi_i|^2)^{1/2} (\sum_{i=1}^{m} \parallel \phi_i \parallel_1^2)^{1/2} - \nu \sum_{i=1}^{m} \parallel \phi_i \parallel_2^2 \\
\leq & c_3^{1/2} \parallel u \parallel_2 (1 + \ln(\lambda_1^{-1} \sum_{i=1}^{m} \parallel \phi_i \parallel_2^2))^{1/2} (\sum_{i=1}^{m} \parallel \phi_i \parallel_1^2)^{1/2} - \nu \sum_{i=1}^{m} \parallel \phi_i \parallel_2^2 .
\end{aligned}$$

In (3.14) use has been made of (2.7) in the third step. In the fourth step, Hölder's ineq. for series was employed. The last step is possible by (3.4). To further majorize the quantity above, let us recall Hölder inequality in its simplest form

$$\| \phi \|^2 \leq |\Omega|^{1/2} \| \phi^2 \| .$$

Since $\{\phi_i\}$ is an orthonormal set in V, this ineq. and (3.1) yield

$$\sum_{i=1}^{m} \| \phi_i \|_1^2 \leq (|\Omega| c_1 \sum_{i=1}^{m} \| \phi_i \|_2^2)^{1/2} = \frac{1}{2\pi}(\lambda_1^{-1} \sum_{i=1}^{m} \| \phi_i \|_2^2)^{1/2}.$$

Upon substitution of this ineq., the trace of (3.14) can be bounded by

$$\begin{aligned} \mathrm{Tr} F'(u(\tau)) \circ P_m(\tau) \quad \leq \quad & \frac{c_1^{1/4} c_3^{1/2}}{(2\pi)^{1/2}} \| u \|_2 \left(1 + \ln(\lambda_1^{-1} \sum_{i=1}^{m} \| \phi_i \|_2^2)\right)^{1/2} (\lambda_1^{-1} \sum_{i=1}^{m} \| \phi_i \|_2^2)^{1/4} \\ & -\nu \sum_{i=1}^{m} \| \phi_i \|_2^2 . \end{aligned} \tag{3.15}$$

The time-average of (3.42) for large time is estimated through a couple of steps. First, let x_m and y_m denote the following

$$\begin{aligned} x_m &= \lambda_1^{-1} \sum_{i=1}^{m} \| \phi_i \|_2^2, \\ y_m &= \frac{1}{t} \int_0^t x_m d\tau, \end{aligned}$$

then

$$\begin{aligned} & \frac{1}{t} \int_0^t \mathrm{Tr} F'(u(\tau)) \circ P_m(\tau) d\tau \\ \leq \quad & \frac{c_1^{1/4} c_3^{1/2}}{(2\pi)^{1/2}} \left(\frac{1}{t} \int_0^t \| u \|_2^2 \, d\tau\right)^{1/2} \left(\frac{1}{t} \int_0^t (1 + \ln x_m) x_m^{1/2} d\tau\right)^{1/2} - \nu\lambda_1 \frac{1}{t} \int_0^t x_m d\tau \\ \leq \quad & \frac{c_1^{1/4} c_3^{1/2}}{(2\pi)^{1/2}} \left(\frac{1}{t} \int_0^t \| u \|_2^2 \, d\tau\right)^{1/2} (1 + \ln y_m)^{1/2} y_m^{1/4} - \nu\lambda_1 y_m, \end{aligned} \tag{3.16}$$

where the last step is possible by applying Jensen's inequality [2] to the concave function $(1 + \ln x) x^{1/2}$ with $x \geq m$. Second, it is true that for $y > 1$ and ξ, $\epsilon > 0$ the following

[2]Let f and g be finite, measurable functions a.e. on $E \subset \mathbf{R}^n$. Suppose that fg and g are integrable on E, $g \geq 0$, and $\int_E g > 0$. If ϕ is concave in an interval containing the range of f, then

$$\frac{\int_E \phi(f) g}{\int_E g} \leq \phi\left(\frac{\int_E fg}{\int_E g}\right).$$

Here $E = [0, t]$, $\phi(x) = 1 + \ln(x)$, $f(x) = x^{1/2}$, and $g(x) = 1$.

ineq. holds (see appendix A)

$$\xi y^{1/4}(1+\ln y)^{1/2} - 2\epsilon y \leq -\frac{\epsilon}{2}y + c\frac{\xi^{4/3}}{\epsilon^{1/3}}(1+\ln\frac{\xi}{\epsilon})^{2/3},$$

where $c < 3/4$. In the limit of large time (3.6) can be utilized, and with the help of this ineq. the number q_m is found to be bounded by

$$q_m \quad \leq \quad -\frac{c_2\nu\lambda_1}{4}m^2 + c'\frac{\| f \|^{4/3}}{\nu^{5/3}\lambda_1^{1/3}}\left(1+\ln\frac{\| f \|}{\nu^2\lambda_1}\right)^{2/3}, \tag{3.17}$$

where c' is an absolute constant of the same order as c, c_1, c_3 (i.e., unity). Again, the lower bound on m above which q_m is negative is an upper bound for D_H. Hence, it follows that

$$D_H \quad \leq \quad c''\left(\frac{\| f \|}{\nu^2\lambda_1}\right)^{2/3}\left(1+\ln\frac{\| f \|}{\nu^2\lambda_1}\right)^{1/3}, \tag{3.18}$$

where $c'' = 2\left(\frac{c'}{c_2}\right)^{1/2}$.

3.1.4 Characteristic length scales

There are some length scales that characterize the dynamics of a spatially extended system. Hohenberg and Shraiman [22] distinguish three lengths which are associated with dissipation, excitation, and correlation and suggest that it is the ratios of these lengths to one another and to the typical system length that will determine the dynamics of the system. The dissipation length l_D characterizes the length scale below which all modes are damped in a finite time. For fluid flow where molecular viscosity alone is responsible for dissipation, $l_D \sim (\nu\tau)^{1/2}$ where τ^{-1} is the local rate of shear [22]. This definition is applicable to both 2D and 3D flows. However, it is not the only definition of l_D in each case. In 2D flow, for example, Batchelor [3] and Kraichnan [27] define this length as $l_D = (\nu^3/\chi)^{1/6}$, where $\chi = \nu\langle\Delta u\rangle^2$ ($\langle\cdot\rangle$ denotes ensemble average) is the enstrophy flux. It should be noted that these definitions have their roots entirely in physical and dimensional considerations. The excitation length is the scale on which energy is injected into the system by external forcing.

It is noted that for the atmosphere, direct solar thermal forcing has a length scale comparable to the size of the earth itself, while the heat contrast between oceans and continents has a length scale comparable to that of the solar forcing reduced by a factor of two or three. The concept of correlation length has a relatively long history in fluid dynamics (dating back to early in the 1900s) in particular, and in dynamical systems theory in general. There has been more than one definition of this length and two of them that are most relevant to the present discussion will be briefly reviewed. The first and simpler one is the two-point correlation length defined in terms of a time correlation function

$$C(x, x') \equiv \langle (u(x,t) - \langle u \rangle_x)(u(x',t) - \langle u \rangle_x) \rangle_t,$$

where the angles with subscript x denote the spatial average, the angles with subscript t denote the time average, and $u(x,t)$ is some local variable. In cases where $C(x,x') \sim \exp\{-|x-x'|/\xi\}$ as $|x-x'| \to \infty$ one can then define the two-point correlation length ξ. This length scale has not been found for the earth weather system [5]. The second correlation length is called the dimension correlation length, the definition of which is due to Cross and Hohenberg [10]. According to extensive chaos, there exists a dimension density δ being the ratio of the fractal dimension (or equivalently Hausdorff dimension) to the system volume

$$\delta \equiv \frac{D_H}{|\Omega|}.$$

Since δ has the physical units of inverse volume, Cross and Hohenberg suggest defining a dimension correlation length ξ_δ:

$$\xi_\delta \equiv \delta^{-1/d}, \tag{3.19}$$

where d is the dimension of the system's physical space. It is highly probable that the dimension correlation length so defined might not be a new length scale. The justification for this claim lies in the heuristic argument in favor of extensive chaos in chapter 1. It is argued there that the attractor dimension is of the order of $(L/l_D)^d$, so that $\delta = l_D^{-d}$. Substituting this into (3.19) one obtains $\xi_\delta = l_D$. Thus, def. (3.19) might just be the familiar dissipation length.

There are basically four physical parameters in the present problem. They are the system linear length scale $L = 2\pi/\lambda_1^{1/2}$, the forcing length scale $L_s = 2\pi/\lambda_s^{1/2}$, the forcing amplitude $|f|$, and the molecular viscosity ν. The first two parameters make obvious and natural length scales. Meanwhile, the last two parameters can be grouped, on dimensional grounds, to form a third length scale. It is seen that the quantity $(\nu^2/|f|)^{1/3}$ has the physical units of length. Hence, it is tempting to designate this length as a dissipation length. It turns out, however, that by grouping ν, $|f|$, and L_s one can form a more physically plausible length scale which will be called the dissipation length L_D:

$$L_D = \left(\frac{\nu^2}{|f|L_s}\right)^{1/2}. \tag{3.20}$$

A closer look at this expression reveals that the ratio on the right-hand side is of the order of the square root of the ratio of ν^2 to the average enstrophy density (see (3.5) for this average). It also agrees with the definition of l_D in [22] because $|f|L_s/\nu$ is sort of an average rate of shear. Besides that physical relevance, this definition is shown to conform to the intended physical significance of L_D as the terminology suggests. This task is postponed until the next chapter. It will be shown there that L_D is the scale at which energy may not be ready for exchange. In other words, it is truly the scale below which viscous effects entirely determine the dynamics.

In terms of the above length scales the results of the last section can be written as

$$D_H \sim \left(\frac{L}{L_D}\right)^2, \tag{3.21}$$

$$D_H \sim \left(\frac{L^3}{L_D^2 L_s}\right)^{2/3} \left(1 + \ln\frac{L^3}{L_D^2 L_s}\right)^{1/3}, \tag{3.22}$$

where a constant of order unity has been dropped from each expression. A couple of remarks are in order. First, not only does expression (3.21) agree with the idea of extensive chaos but it also takes the form suggested by the heuristic argument in the introductory chapter. Second, it is clear that the logarithmic correction term in expression (3.22) surely diverges as $L \to \infty$; however, (3.22) may well be optimal for small domain size. In that case the system is slightly "super-extensive". When L is large enough, the system may behave extensively as (3.21) becomes optimal. The length scale which separates the two regimes is now called the extensivity length scale L_e. A comparison of (3.21) and (3.22) gives

$$\left(\frac{L_e}{L_D}\right)^2 \sim \left(\frac{L_e^3}{L_D^2 L_s}\right)^{2/3} \left(1 + \ln \frac{L_e^3}{L_D^2 L_s}\right)^{1/3}.$$

It follows that

$$L_e \sim \exp\{(L_s/L_D)^2/3\}(L_D^2 L_s)^{1/3}. \tag{3.23}$$

Finally, the dimension correlation length as defined above by [10] and advanced by [11, 12] is just the newly defined dissipation length L_D.

3.2 Attractor dimension of the NS system ($\alpha < 1$)

This investigation has, thus far, been focused on the dependence of D_H on the domain size with the assumption that the domain is a square. In the square case, as is well-known, all the constants involved in the estimates of the last section are absolute constants. They are not dependent on the domain size, nor do they depend on the set of functions under consideration. On the contrary, most of the constants depend on the shape of the domain when one of the dimensions is elongated. This dependence renders the estimates in the last section too generous although they remain valid. An example of such estimates is ineq. (3.3). If (3.3) were to be applied to the present case it would read

$$\sum_{i=1}^{m} \| \phi_i \|_1^2 \geq \sum_{i=1}^{m} \lambda_i \geq c(\alpha)\lambda_1 m^2$$

$$\geq \quad c(\alpha)\frac{4\pi^2\alpha^2}{L^2}m^2, \tag{3.24}$$

where α is the domain shape ratio. It is obvious that unless $c(\alpha) \propto 1/\alpha$, ineq. (3.24) would not be sharp as $\alpha \to 0$. Hence there is a need to reexamine some of the basic inequalities employed in the last section.

The generalization from a square domain to an elongated one has been considered for the NS system in the context of its attractor dimension by Ziane [67], Temam and Ziane [63], and Raugel and Sell [53]. A similar analysis, which shows that the NS system on an elongated domain may also behave extensively for various forms of the body force, will be carried out presently.

3.2.1 Basic inequalities

This section will sort out the dependence on α of the constants in (3.1) and (3.3). The following result which is known as the anisotropic Lieb-Thirring inequality is proven by Ziane [67]. Let $\{\phi_i\}_{i=1}^m$ be a family of functions in $V(\Omega_\alpha)$ which is orthonormal in $H(\Omega_\alpha)$, then there exists an absolute constant c_0 such that

$$\begin{aligned} \| \sum_{i=1}^m \phi_i^2 \|^2 &\leq c_0 \left(\sum_{i=1}^m \| \frac{\partial \phi_i}{\partial x_1} \|^2 + \frac{\alpha^2}{L^2} \| \phi_i \|^2\right)^{1/2} \left(\sum_{i=1}^m \| \frac{\partial \phi_i}{\partial x_2} \|^2 + \frac{1}{L^2} \| \phi_i \|^2\right)^{1/2}, \\ &\leq \frac{c_0}{2}\left(\sum_{i=1}^m \| \phi_i \|_1^2 + \frac{\alpha^2+1}{L^2}\sum_{i=1}^m \| \phi_i \|^2\right) \end{aligned} \tag{3.25}$$

The dependence of c_2 in (3.3) on α can be obtained by geometrical arguments. However, (3.25) offers a more convenient way. First, for an orthonormal set $\{\phi_i\}_{i=1}^m$ in $H(\Omega_\alpha)$ it is observed that

$$m = \sum_{i=1}^m \| \phi_i \|^2 \leq |\Omega_\alpha|^{1/2} \| \sum_{i=1}^m \phi_i^2 \|$$

and then by (3.25)

$$\frac{2\alpha}{c_0 L^2}m^2 \leq \sum_{i=1}^m \| \phi_i \|_1^2 + \frac{\alpha^2+1}{L^2}\sum_{i=1}^m \| \phi_i \|^2 . \tag{3.26}$$

This inequality gives an estimate of the sum of $\| \phi_i \|_1^2$ in terms of α instead of α^2, but this cannot be achieved without a cost (the second term on the right-hand side of (3.26)).

3.2.2 Preliminary estimates

In this chapter two types of driving force are considered. The first one, which is an across-channel forcing, is defined by

$$f_T = f_s e_s + f_s' e_s',$$

where $s = (0, \alpha s_2)^T$ with $s_2 > 1$. This type will be considered in the stability problem in the next chapter as well (where f_s is set to zero for simplicity). The second type which has a non-vanishing along-channel component is defined by

$$f_A = \sum_{|k|=|s|} (f_k e_k + f_k' e_k'),$$

where $s = (s_1, \alpha s_2)^T$ with $s_1 > 0$. When $s_2 = 0$, F_A is called an along-channel forcing; otherwise, it will be called a mixed-mode forcing. An upper bound on the time-averaged enstrophy for each forcing which is needed in the dimension calculation will be recalled:

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \| u \|_1^2 \, d\tau \leq \frac{\| f \|^2}{\nu^2 \lambda_s}. \tag{3.27}$$

An appropriate substitution is all that is required for each individual case. When $f = f_T$, $\lambda_s = 4\pi^2\alpha^2 s_2^2/L^2$, while $\lambda_s = 4\pi^2(s_1^2 + \alpha^2 s_2^2)/L^2$ corresponds to $f = f_A$.

3.2.3 Attractor dimension

The attractor dimension will be estimated by the same procedure as in the previous section, but with a new set of inequalities. Due to technical difficulties, a refined estimate of the attractor dimension in V as in the square domain case is not available and will not be pursued in this thesis. The familiar trace, as recalled from the third line of (3.11), is bounded by

$$\mathrm{Tr}F'(u(\tau)) \circ P_m(\tau) \quad \leq \quad \| u \|_1 \| \sum_{i=1}^{m} \phi_i^2 \| - \nu \sum_{i=1}^{m} \| \phi_i \|_1^2 . \tag{3.28}$$

By the anisotropic Lieb-Thirring ineq. (3.25), the Young ineq., and (3.26), the above expression is successively majorized as follows

$$\begin{aligned}
& \mathrm{Tr}F'(u(\tau)) \circ P_m(\tau) \\
\leq \quad & (\frac{c_0}{2})^{1/2} \| u \|_1 \left(\sum_{i=1}^{m} \| \phi_i \|_1^2 + \frac{\alpha^2+1}{L^2} \sum_{i=1}^{m} \| \phi_i \|^2 \right)^{1/2} - \nu \sum_{i=1}^{m} \| \phi_i \|_1^2 \\
\leq \quad & \frac{c_0 \| u \|_1^2}{8\nu(1-\epsilon)} + \nu(1-\epsilon) \left(\sum_{i=1}^{m} \| \phi_i \|_1^2 + \frac{\alpha^2+1}{L^2} \sum_{i=1}^{m} \| \phi_i \|^2 \right) - \nu \sum_{i=1}^{m} \| \phi_i \|_1^2 \\
\leq \quad & \frac{c_0 \| u \|_1^2}{8\nu(1-\epsilon)} - \epsilon\nu \left(\sum_{i=1}^{m} \| \phi_i \|_1^2 + \frac{\alpha^2+1}{L^2} \sum_{i=1}^{m} \| \phi_i \|^2 \right) + \nu \frac{\alpha^2+1}{L^2} \sum_{i=1}^{m} \| \phi_i \|^2 \\
\leq \quad & \frac{c_0 \| u \|_1^2}{8\nu(1-\epsilon)} - \frac{2\epsilon\nu\alpha}{c_0 L^2} m^2 + \nu \frac{\alpha^2+1}{L^2} m.
\end{aligned} \tag{3.29}$$

In the above, $\epsilon \in [0, 1)$ has been introduced through the Young inequality. Since the enstrophy of a trajectory on the global attractor is bounded according to (3.5), the quantity q_m is majorized by

$$q_m \leq \frac{c_0 \| f \|^2}{8\nu^3 \lambda_s (1-\epsilon)} - \frac{2\epsilon\nu\alpha}{c_0 L^2} m^2 + \nu \frac{\alpha^2+1}{L^2} m. \tag{3.30}$$

An upper bound on the Hausdorff dimension can be obtained by determining the lower bound on m above which q_m is negative. Such a lower bound is obtained by varying ϵ in the admissible range. It is found by the method of appendix A that

$$D_H \quad \leq \quad \frac{c_0}{2} \left(\frac{L \| f \|}{\sqrt{\alpha}\nu^2 \lambda_s^{1/2}} + \frac{\alpha^2+1}{\alpha} \right), \tag{3.31}$$

or

$$D_H \quad \leq \quad \frac{c_0}{2} \left(\frac{L^2 |f|}{\alpha\nu^2 \lambda_s^{1/2}} + \frac{\alpha^2+1}{\alpha} \right). \tag{3.32}$$

This expression as it stands applies to a general forcing of wavenumber $|s|$. The expression clearly grows as α^{-1} for small α if all else is fixed, and thus qualifies the system as a candidate for extensive chaos. To further explore the extensivity of the system, however, it is more illuminating to distinguish three forcing classes as previously mentioned: (i) an across-channel forcing, (ii) an along-channel forcing, (iii) a mixed-mode forcing. For (i), $\lambda_s = 4\pi^2\alpha^2 s_2^2/L^2$, so that D_H grows as α^{-1} for small α if αs_2 is held fixed, i.e., if the forcing scale is held fixed. For (ii), $\lambda_s = 4\pi^2 s_1^2/L^2$ is independent of α, so that D_H grows as α^{-1} for small α (this result can be easily generalized to a multiscale forcing which is any linear combination of pure x_1 modes, see [67]). Finally for a mixed-mode forcing, $\lambda_s = 4\pi^2(s_1^2 + \alpha^2 s_2^2)/L^2$ which is bounded from below by $4\pi^2 s_1^2/L^2$ (which is a constant when s_1 is held fixed) when $\alpha \to 0$; so D_H grows as α^{-1} for small α where a formal constraint on the forcing scale is not necessary. In conclusion, the NS system on a long channel driven by the body forces considered above is an extensive system.

Finally, the dimension estimate can be written in terms of the forcing length L_s, the dissipation length L_D, the channel width L, and the domain shape ratio α as follows (the second term is dropped and a factor of the order of unity is ignored)

$$D_H \sim \frac{L^2}{\alpha L_D^2}. \tag{3.33}$$

3.3 Attractor dimension of the MNS system

It is seen that the presence of Ekman drag alters the energy spectrum of the NS system to some extent. How does it affect the attractor dimension? In this section a couple of upper bounds are derived for the attractor dimension of the MNS equations. It will be demonstrated that one of the bounds exhibits extensive behaviour for sufficiently large systems.

3.3.1 Preliminary estimates

Bounds on the time averages of the enstrophy and of $\| u \|_2^2$ which are needed in the dimension calculations will be derived. As there is an extra term due to the Ekman drag one has more options for the estimates than in the previous cases. The bounds to be derived below seem to be sharp, especially for $\sigma \gg \nu\lambda_s$.

For the enstrophy, an upper bound for the energy on the attractor can be derived and then the dynamical constraint in the last chapter can be utilized to obtain an estimate. The following procedure (which is more direct) yields as sharp a result. Taking the scalar product of (2.42) with Au one finds after invoking (2.9)

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \parallel u \parallel_1^2 &= -\nu \parallel u \parallel_2^2 - \sigma \parallel u \parallel_1^2 + (Au, f) \\
&\leq -(\nu\lambda_1 + \sigma) \parallel u \parallel_1^2 - \nu\lambda_s(\lambda_s - \lambda_1)E(|s|) + \lambda_s E(|s|)^{1/2} \parallel f \parallel \\
&\leq -(1-\epsilon)(\nu\lambda_1 + \sigma) \parallel u \parallel_1^2 - \left(\epsilon(\nu\lambda_1 + \sigma) + \nu(\lambda_s - \lambda_1)\right)\lambda_s E(|s|) \\
&\quad + \lambda_s E(|s|)^{1/2} \parallel f \parallel \\
&\leq -(1-\epsilon)(\nu\lambda_1 + \sigma) \parallel u \parallel_1^2 + \frac{\lambda_s \parallel f \parallel^2}{4(\epsilon(\nu\lambda_1 + \sigma) + \nu(\lambda_s - \lambda_1))},
\end{aligned}$$

where $E(|s|)$ denotes the system's energy at the forcing scale. The majorizations above are due to the Poincaré, Schwarz, and Young inequalities. The introduction of ϵ (the admissible range of ϵ is $\epsilon \in [0, 1)$) is to optimize the bound on the enstrophy as will be seen in a moment. By the Gronwall lemma the enstrophy satisfies

$$\begin{aligned}
\parallel u \parallel_1^2 \leq\ & \parallel u(t=0) \parallel_1^2 \exp\{-2(1-\epsilon)(\nu\lambda_1 + \sigma)t\} \\
&+ \frac{\lambda_s \parallel f \parallel^2 \left(1 - \exp\{-2(1-\epsilon)(\nu\lambda_1 + \sigma)t\}\right)}{4(1-\epsilon)(\nu\lambda_1 + \sigma)(\epsilon(\nu\lambda_1 + \sigma) + \nu(\lambda_s - \lambda_1))}.
\end{aligned}$$

For large time $\parallel u \parallel_1^2$ is bounded above by

$$\parallel u \parallel_1^2 \leq \frac{\lambda_s \parallel f \parallel^2}{4(1-\epsilon)(\nu\lambda_1 + \sigma)(\epsilon(\nu\lambda_1 + \sigma) + \nu(\lambda_s - \lambda_1))}.$$

This bound can be minimized by varying ϵ in the admissible range. Some simple arithmetic steps give

$$\epsilon = \frac{\sigma + \nu(2\lambda_1 - \lambda_s)}{2(\sigma + \nu\lambda_1)},$$

for the minimum value of the above bound. Substituting this value of ϵ into the last ineq. yields

$$\| u \|_1^2 \leq \frac{\lambda_s \| f \|^2}{(\nu\lambda_s + \sigma)^2}. \tag{3.34}$$

This is a bound in V of the global attractor.

A bound on the time average of $\| u \|_2^2$ can be derived from the enstrophy evolution equation. The steps go as follows

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \| u \|_1^2 &= -\nu \| u \|_2^2 - \sigma \| u \|_1^2 + (Au, f), \\ &\leq -\nu(1-\epsilon) \| u \|_2^2 - \lambda_s(\epsilon\nu\lambda_s + \sigma)E(|s|) + \lambda_s \| f \| E(|s|)^{1/2} \\ &\leq -\nu(1-\epsilon) \| u \|_2^2 + \frac{\lambda_s \| f \|^2}{4(\epsilon\nu\lambda_s + \sigma)}. \end{aligned}$$

As usual, ϵ is in the range $\epsilon \in [0, 1)$. Integrating this differential inequality one find in the limit of large time:

$$\limsup_{t\to 0} \frac{1}{t}\int_0^t \| u \|_2^2 \, d\tau \leq \frac{\lambda_s \| f \|^2}{4\nu(1-\epsilon)(\epsilon\nu\lambda_s + \sigma)}.$$

This bound can be minimized by varying ϵ in the admissible range. It turns out that unless $\sigma < \nu\lambda_s$ (which is not considered here), $\epsilon = 0$ is the appropriate choice. Hence,

$$\limsup_{t\to 0} \frac{1}{t}\int_0^t \| u \|_2^2 \, d\tau \leq \frac{\lambda_s \| f \|^2}{4\nu\sigma}. \tag{3.35}$$

As the analysis in this chapter aims at systems where $\nu\lambda_s \ll \sigma$, i.e., systems the dynamics of which are dominated by Ekman drag, it is tempting to ignore the molecular viscosity term for the sake of simplicity. However, viscosity is responsible for rapid compression of volume elements in phase space and is therefore essential in the next section when it comes to the dimension calculation. Moreover, it is not known whether or not a finite-dimensional chaotic attractor exists for the case where Ekman drag alone is responsible for dissipation. In passing, it is worthwhile to pause a bit here for a little thought about boundedness versus finite dimensionality. It is quite easy to give examples of infinite-dimensional bounded dynamics, and it is equally easy to construct finite-dimensional unbounded systems. Hence boundedness and finite dimensionality do not usually go hand in hand and they are two distinct mathematical properties. The physical world in which we live might just happen to be both bounded and finite-dimensional.

3.3.2 Attractor dimension

In this section estimates of the attractor dimension for the MNS system in H and in V will be presented. The extensivity issue of the system will also be discussed. Only the square domain case will be considered, so that ineqs. of section 1 in which all constants are absolute apply.

As previously done for the NS system, upper bounds on the attractor dimension for the MNS system can be estimated using either the H-norm or the V-norm. The immediate calculation below uses the former. The first variation equation for (2.42) is given by

$$\frac{dU}{dt} + B(u, U) + B(U, u) + (\nu A + \sigma)U = 0, \tag{3.36}$$

where U is a variation from u about which the linearization is taken. With the familiar notation adopted for the present case, the trace of the operator in the CFT theorem is given by

$$\begin{aligned} \mathrm{Tr} F'(u(\tau)) \circ P_m(\tau) &= \sum_{i=1}^{m} (\phi_i, -B(\phi_i, u) - \nu A \phi_i - \sigma \phi_i), \\ &= \sum_{i=1}^{m} [(\phi_i, -B(\phi_i, u)) - \nu \parallel \phi_i \parallel_1^2 - \sigma \parallel \phi_i \parallel^2], \qquad (3.37) \\ &\leq \parallel u \parallel_1 \parallel \sum_{i=1}^{m} \phi_i^2 \parallel - \nu \sum_{i=1}^{m} \parallel \phi_i \parallel_1^2 - \sigma \sum_{i=1}^{m} \parallel \phi_i \parallel^2, \\ &\leq \parallel u \parallel_1 (c_1 \sum_{i=1}^{m} \parallel \phi_i \parallel_1^2)^{1/2} - \nu \sum_{i=1}^{m} \parallel \phi_i \parallel_1^2 - \sigma \sum_{i=1}^{m} \parallel \phi_i \parallel^2 . \end{aligned}$$

In the first line of (3.37) one of the trilinear forms drops out by virtue of (2.10). The third step involves simple manipulations of the remaining trilinear form $(\phi_i, B(\phi_i, u))$.

The last step is from the Lieb and Thirring inequality where the constant c_1 is bounded by $c_1 \leq 2.89$ [23]. Thanks to Schwarz's inequality, the time average of the above trace is majorized by

$$\frac{1}{t}\int_0^t \mathrm{Tr}F'(u(\tau)) \circ P_m(\tau) d\tau \leq \left(\frac{c_1}{t}\int_0^t \| u \|_1^2 \, d\tau\right)^{1/2} \left(\frac{1}{t}\int_0^t \sum_{i=1}^m \| \phi_i \|_1^2 \, d\tau\right)^{1/2} - \frac{\nu}{t}\int_0^t \sum_{i=1}^m \| \phi_i \|_1^2 \, d\tau - \frac{\sigma}{t}\int_0^t \sum_{i=1}^m \| \phi_i \|^2 \, d\tau. \quad (3.38)$$

In the limit of large time (3.34) can be utilized, and the number q_m defined in the CFT theorem is found to be bounded by

$$q_m \leq \frac{(c_1\lambda_s)^{1/2} \| f \|}{\nu\lambda_s + \sigma} \left(\sum_{i=1}^m \| \phi_i \|_1^2\right)^{1/2} - \nu \sum_{i=1}^m \| \phi_i \|_1^2 - \sigma \sum_{i=1}^m \| \phi_i \|^2 . \quad (3.39)$$

A lower bound on m above which q_m is negative can be obtained by a variety of arithmetic procedures. Such a lower bound on m is an upper bound for D_H as by the CFT theorem. The following calculations seem to yield a sharp estimate. It is seen that q_m is non-positive when

$$\left(\sum_{i=1}^m \| \phi_i \|_1^2\right)^{1/2} \geq \frac{1}{2\nu}\left(\frac{(c_1\lambda_s)^{1/2} \| f \|}{\nu\lambda_s + \sigma} + \left(\frac{c_1\lambda_s \| f \|^2}{(\nu\lambda_s + \sigma)^2} - 4\nu\sigma \sum_{i=1}^m \| \phi_i \|^2\right)^{1/2}\right).$$

Since $\sum_{i=1}^m \| \phi_i \|^2 = m$ and $(\sum_{i=1}^m \| \phi_i \|_1^2)^{1/2} \geq (c_2\lambda_1)^{1/2} m$ by (3.3), the following condition guarantees the non-positiveness of q_m

$$m \geq \frac{1}{2\nu(c_2\lambda_1)^{1/2}}\left(\frac{(c_1\lambda_s)^{1/2} \| f \|}{\nu\lambda_s + \sigma} + \left(\frac{c_1\lambda_s \| f \|^2}{(\nu\lambda_s + \sigma)^2} - 4\nu\sigma m\right)^{1/2}\right).$$

For the NS system the above steps would yield an explicit expression for m. The present implicit expression for m is due to the Ekman drag. This ineq. can be solved for m in a couple of elementary steps. The minimum value of m which satisfies the solution is an upper bound of D_H. The result is

$$D_H \leq \left(\frac{c_1\lambda_s}{c_2\lambda_1}\right)^{1/2} \frac{\| f \|}{\nu(\nu\lambda_s + \sigma)} - \frac{\sigma}{c_2\nu\lambda_1}. \quad (3.40)$$

This estimate grows linearly with the domain area when all else is fixed. Hence, Ekman drag may not destroy the extensivity of the NS system.

Similar to the derivation in section 1 leading to (3.18), another estimate on the attractor dimension which is sharper than (3.40) for a small system can be obtained by carrying out the above calculation in V. All notation in this part remains the same except that the projection $P_m(\tau)$ is in V and $\{\phi_i, i = 1, ..., m\}$ is an orthonormal set with respect to the V-norm. The trace of (D-3) is estimated through several steps as follows

$$\begin{aligned}
& \mathrm{Tr} F'(u(\tau)) \circ P_m(\tau) \\
= & \sum_{i=1}^{m} (A\phi_i, -B(\phi_i, u) - B(u, \phi_i) - \nu A\phi_i - \sigma\phi_i) \\
= & \sum_{i=1}^{m} \Big((A\phi_i, -B(\phi_i, u) - B(u, \phi_i)) - \nu \parallel \phi_i \parallel_2^2 - \sigma \parallel \phi_i \parallel_1^2 \Big) \\
= & \sum_{i=1}^{m} (Au, B(\phi_i, \phi_i)) - \nu \sum_{i=1}^{m} \parallel \phi_i \parallel_2^2 - \sigma \sum_{i=1}^{m} \parallel \phi_i \parallel_1^2 \qquad (3.41) \\
\leq & \parallel u \parallel_2 \sup_{x\in\Omega} (\sum_{i=1}^{m} \phi_i^2)^{1/2} (\sum_{i=1}^{m} \parallel \phi_i \parallel_1^2)^{1/2} - \nu \sum_{i=1}^{m} \parallel \phi_i \parallel_2^2 - \sigma \sum_{i=1}^{m} \parallel \phi_i \parallel_1^2 \\
\leq & c_3^{1/2} \parallel u \parallel_2 (1 + \ln(\lambda_1^{-1} \sum_{i=1}^{m} \parallel \phi_i \parallel_2^2))^{1/2} (\sum_{i=1}^{m} \parallel \phi_i \parallel_1^2)^{1/2} \\
& \quad - \nu \sum_{i=1}^{m} \parallel \phi_i \parallel_2^2 - \sigma \sum_{i=1}^{m} \parallel \phi_i \parallel_1^2
\end{aligned}$$

In (3.41) use has been made of (2.7) in the third step. In the fourth step, Hölder's ineq. for series was employed. The last step is possible by (3.4). Further majorization of the quantity above is via the procedure employed earlier in this chapter for the NS system. In the intermediate steps one has

$$\begin{aligned}
\mathrm{Tr} F'(u(\tau)) \circ P_m(\tau) \leq & \frac{c_1^{1/4} c_3^{1/2}}{(2\pi)^{1/2}} \parallel u \parallel_2 \left(1 + \ln(\lambda_1^{-1} \sum_{i=1}^{m} \parallel \phi_i \parallel_2^2) \right)^{1/2} (\lambda_1^{-1} \sum_{i=1}^{m} \parallel \phi_i \parallel_2^2)^{1/4} \\
& - \nu \sum_{i=1}^{m} \parallel \phi_i \parallel_2^2 - \sigma \sum_{i=1}^{m} \parallel \phi_i \parallel_1^2 \qquad (3.42)
\end{aligned}$$

and

$$q_m \;\leq\; -\frac{c_2\nu\lambda_1}{4}m^2 - \sigma m + c'\left(\frac{\lambda_s^{1/2}\parallel f\parallel}{(\nu\sigma)^{1/2}(\nu\lambda_1)^{1/4}}\right)^{4/3}\left(1+\ln\frac{\lambda_s^{1/2}\parallel f\parallel}{\nu\lambda_1(\nu\sigma)^{1/2}}\right)^{2/3}, \tag{3.43}$$

where c' is an absolute constant of the same order as c_1, c_2, c_3 (i.e., unity). The lower bound on m above which q_m is negative is an upper bound for D_H as by the CFT theorem. It is found that

$$D_H \;\leq\; \frac{2}{c_2\nu\lambda_1}\left(\left(c_2c'\left(\frac{\lambda_1\lambda_s\parallel f\parallel^2}{\sigma}\right)^{2/3}\left(1+\ln\frac{\lambda_s^{1/2}\parallel f\parallel}{\nu\lambda_1(\nu\sigma)^{1/2}}\right)^{2/3}+\sigma^2\right)^{1/2}-\sigma\right). \tag{3.44}$$

Following the definition of the dissipation length for the NS system, the dissipation length for the MNS system L_D will be defined in exactly the same manner. However, as the average enstrophy takes a different form for the present system, L_D is given by

$$L_D \;=\; \left(\frac{L_s\nu(\nu\lambda_s+\sigma)}{|f|}\right)^{1/2} \;\sim\; \left(\frac{L_s\nu\sigma}{|f|}\right)^{1/2}, \tag{3.45}$$

after making the assumption $\sigma \gg \nu\lambda_s$. This allows the results in this section to be recast in the following forms

$$D_H \;\sim\; L^2\left(\frac{1}{L_D^2}-\frac{D_r}{L_s^2}\right), \tag{3.46}$$

$$D_H \;\sim\; L^2\left(\left(\left(\frac{D_r}{L_D^4L_s^2}\right)^{2/3}\left(1+\ln\frac{L^3D_r^{1/2}}{L_D^2L_s}\right)^{2/3}+\frac{D_r^2}{L_s^4}\right)^{1/2}-\frac{D_r}{L_s^2}\right). \tag{3.47}$$

In the above, $D_r = \sigma/\nu\lambda_s$ is the ratio of the dissipation rate due to the Ekman drag to that at wavenumber $|s|$ due to viscosity. In light of the analysis in chapter 2 (recall that the energy dissipation rate and the enstrophy dissipation rate due to viscosity were shown to be less than and equal to $\nu\lambda_s$, respectively), $\nu\lambda_s$ is an upper bound on the the rate of energy dissipation due to viscosity. Therefore, D_r is actually a lower bound on the relative strengths of the two dissipation channels.

A couple of remarks shall conclude this chapter. (i) Except for the term due to Ekman drag, the bounds on the attractor dimension for the MNS system are essentially the same as those for the NS system. The former reduce to the latter in the limit of $\sigma \to 0$ as expected. (ii) Similar to the NS system there is a length scale (called the extensivity scale L_e, which is not written out explicitly in this case) which separates the regime of extensive and super-extensive chaos. (iii) A peculiar property of the MNS system is that the enstrophy grows with the forcing wavenumber. Therefore, a small-scale forcing is more capable of exciting the high-order harmonics and maintaining a broader spectrum than a large-scale forcing. As a result, the attractor dimension grows with the forcing wavenumber accordingly. This is in sharp contrast to the NS system where the attractor dimension grows, instead, with the forcing scale. (iv) In the super-extensive regime, the dependence of D_H on the dissipation ratio D_r must be interpreted with caution. Since L_D scales as $D_r^{1/2}$ (for large D_r), D_H should decrease when Ekman drag becomes stronger.

Chapter 4

NS on an elongated domain: Stability analysis

In this chapter, the effect of the domain shape on the linear and nonlinear (global) stability problem is analysed. Section 1 presents a linear analysis of laminar across-channel flow and explores some properties of the unstable eigenmodes when the flow becomes unstable. The linear stability of this type of flow is shown to have a peculiar property due to the rectangular geometry of the domain. It will be demonstrated that there exist laminar monoscale across-channel flows which are linearly stable for arbitrary values of the physical parameters (arbitrarily large Reynolds number, for example). But that constitutes only half of the present chapter. The other half, section 2, aims to exploit a property of the nonlinear interactions for an elongated channel in the stability problem. It will be shown that any laminar monoscale across-channel flow, once nonlinearly stable, remains nonlinearly stable as $\alpha \to 0$. In addition, the physical significance of the dissipation length L_D previously defined naturally turns up in the nonlinear analysis as well.

4.1 Linear stability

The primary stationary laminar flow to be considered is

$$\bar{u} = \bar{u}_s e'_s = (\nu A)^{-1} f_0 e'_s, \tag{4.1}$$

where $f_0 > 0$ and $s = (0, \alpha s_2)^T$. It will be shown that when $\alpha s_2 \leq 1$, $\bar{u}$ remains linearly stable for arbitrary values of $\bar{u}_s$.

The linearized eq. around $\bar{u}$ of the NS system reads

$$\frac{dv}{dt} = -B(\bar{u}, v) - B(v, \bar{u}) - \nu A v. \tag{4.2}$$

Let the linear operator in the above eq. be denoted by $L(\bar{u})$, viz.,

$$L(\bar{u})v = -B(\bar{u}, v) - B(v, \bar{u}) - \nu A v. \tag{4.3}$$

As in a standard linear stability problem the focus is on the following eigenvalue problem:

$$L(\bar{u})W = \delta W, \tag{4.4}$$

where δ has a positive real part ($\mathrm{Re}(\delta) > 0$).

Let an eigenvector be written in the form

$$W = w + w',$$

where w and w' are the even and odd parts of W, i.e.,

$$w = \sum_{k \in K} w_k e_k, \tag{4.5}$$

$$w' = \sum_{k \in K} w'_k e'_k. \tag{4.6}$$

Note that w_k and w'_k are allowed to take complex values. Equivalently, W is assumed to be in the complex space H^* instead of H. Upon substitution of $W = w + w'$ into (4.4), and with reference to lemma B (see Appendix B), it is seen that the odd and even parts can be separated out. As a matter of fact, they satisfy two identical eigenvalue problems:

$$L(\bar{u})w = \delta w, \tag{4.7}$$

$$L(\bar{u})w' = \delta w'. \tag{4.8}$$

Hence, it is sufficient to consider either one, say the even part. Now substituting (4.5) into (4.7), and using lemma B with $l = s$ one obtains the recurrence relation

$$(\nu\lambda_k + \delta)w_k - \frac{\sqrt{2\alpha}\pi\bar{u}_s k' \cdot s}{L^2|k||s|}\left[\frac{|k-s|^2 - |s|^2}{|k-s|}w_{k-s} - \frac{|k+s|^2 - |s|^2}{|k+s|}w_{k+s}\right] = 0.$$

Equivalently (since $k' \cdot s = -k_1\alpha s_2,\ |s| = \alpha s_2$)

$$(\nu\lambda_k + \delta)w_k + \frac{\sqrt{2\alpha}\pi\bar{u}_s k_1}{L^2|k|}\left[\frac{|k-s|^2 - |s|^2}{|k-s|}w_{k-s} - \frac{|k+s|^2 - |s|^2}{|k+s|}w_{k+s}\right] = 0. \tag{4.9}$$

As $s = (0, \alpha s_2)^T$ the above equation gives a three-term recurrence relation among $w_{(k_1,k_2-\alpha s_2)^T}$, $w_{(k_1,k_2)^T}$, $w_{(k_1,k_2+\alpha s_2)^T}$ for each fixed k_1 (recall that $k_1 \geq 0$). It is easy to see that the case in which $k_1 = 0$ only yields the trivial solution to (4.9) for $\mathrm{Re}(\delta) > 0$. Therefore $k_1 \neq 0$ will be assumed for the rest of this section. Let

$$b_n = \frac{|k+ns|^2 - |s|^2}{|k+ns|}w_{k+ns}, \tag{4.10}$$

$$c_n = \frac{L^2|k+ns|^2(\nu\lambda_{k+ns} + \delta)}{\sqrt{2\alpha}\pi\bar{u}_s k_1(|k+ns|^2 - |s|^2)},$$

where n is an integer. Then (4.9) can be rewritten as

$$c_n b_n + b_{n-1} - b_{n+1} = 0. \tag{4.11}$$

Some properties of the unstable eigenmodes can be deduced from (4.11). In order for w to be in H^*, b_n has to satisfy

$$b_n \to 0 \quad \text{if } |n| \to \infty.$$

Since $\mathrm{Re}(c_n) \to \infty$ when $|n| \to \infty$ the above requirement, in connection with (4.11), yields

$$|k+ns|^\gamma b_n \to 0, \quad \forall\gamma \in \mathbf{R} \text{ if } |n| \to \infty. \tag{4.12}$$

Without loss of generality k_2 may be assumed to lie in the range $0 \leq k_2 < s_2$, so that there are at most two c_n's, namely c_0 and c_{-1}, that may have negative real parts for $k_1 < \alpha s_2$. When $k_1 > \alpha s_2$, $\mathrm{Re}(c_n) > 0\ \forall n$. With this observation a couple of important properties of b_n can be drawn.

First of all, the Fourier series of an unstable eigenvector is not terminating, i.e.,

$$b_n \neq 0, \quad \forall n.$$

Indeed, suppose otherwise that there is an $n' \geq 1$ such that $b_{n'} = 0$. It is easy to see that $b_n \neq 0\ \forall n > n'$. Let

$$p_n = \frac{b_n}{b_{n-1}}, \quad \forall n > n' + 1,$$

then it follows from (4.11) that

$$\begin{aligned} c_{n'+1} &= p_{n'+2}, \\ p_{n+1} &= c_n + \frac{1}{p_n}, \quad \forall n > n' + 1. \end{aligned}$$

Hence,

$$\begin{aligned} p_{n+1} &= c_n + \cfrac{1}{c_{n-1} + \frac{1}{p_{n-1}}} \\ &= c_n + \cfrac{1}{c_{n-1} + \cfrac{1}{c_{n-2} + \cfrac{}{\ddots + \cfrac{1}{c_{n'+2} + \frac{1}{c_{n'+1}}}}}}. \end{aligned}$$

Since $\mathrm{Re}(c_n) > 0$ for $n > n' \geq 1$, so

$$\mathrm{Re}(p_{n+1}) > \mathrm{Re}(c_n) \to \infty, \quad \text{when} \quad n \to \infty, \tag{4.13}$$

which is impossible since $b_n \to 0$ as $n \to \infty$ and $p_n = b_n/b_{n-1}$. Thus, $b_n \neq 0$ for all $n \geq 0$. Now suppose there is an $n' \leq 0$ such that $b_{n'} = 0$. A similar argument leads to

$$\mathrm{Re}(p_n^{-1}) \to -\infty, \quad \text{when} \quad n \to -\infty, \tag{4.14}$$

which is also not possible. Thus it has been shown, together with the above, that $b_n \neq 0\ \forall n$. It may also be concluded that when $|k| = |s|$, eq. (4.11) does not possess a nontrivial solution because $b_0 = 0$ (note that in this case c_0 is not well-defined, however the foregoing arguments apply by redefining $c_0 = 0$).

Second, there exists no nontrivial solution to (4.11) for $k_1 > \alpha s_2$. In that case $\mathrm{Re}(c_n) > 0\ \forall n$, so that according to (4.11)

$$\mathrm{Re}(p_n) < 0, \quad \forall n,$$

otherwise, $\mathrm{Re}(p_n) \to \infty$ as $n \to \infty$. But then,

$$p_n = \frac{1}{-c_n + p_{n+1}}$$

implies the impossibility of (4.14).

In conclusion the following have been proven. (I) The stationary flow $\bar{u} = \bar{u}_s e'_s$ of the NS system where $s = (0, \alpha s_2)^T$ with $\alpha s_2 \leq 1$ is linearly stable for arbitrary values of $\bar{u}_s$. (II) Unstable eigenvectors of $L(\bar{u})$ can only exist in the following forms

$$w(k) = \sum_n w_{k+ns} e_{k+ns}, \tag{4.15}$$

where n is an integer, k satisfies $|k| < |s|$ and $k_1 \neq 0$. Moreover,

$$w_{k+ns} \neq 0, \quad \forall n, \tag{4.16}$$

and for all real γ

$$|k + ns|^\gamma w_{k+ns} \to 0 \quad \text{if} \quad |n| \to \infty. \tag{4.17}$$

It is worth mentioning that the above results still apply if the molecular viscosity is replaced by any viscous operator of the general viscosity. In that case (I) requires no modification for its proof. Likewise, the proof of (II) only needs a little clarification for the case of Ekman drag. For this case c_n remains finite instead of approaching infinity as $n \to \infty$. But that fact poses no threats to the proof of (II).

Some properties of unstable eigenmodes of $\bar{u}$ are pointed out by (II) but the question of the existence of such modes is totally left open. At the moment, only "a few" unstable modes are known to exist, as by the following lemma (see Liu [35] for a proof).

Lemma : For any large enough $\bar{u}_s$, and for each k satisfying:

$$|s| > |k|, \quad |k - s| > |s|, \tag{4.18}$$

there exists a unique $\delta(k) > 0$ corresponding to a unique (up to a constant factor) eigenvector $w(k)$ that satisfies (4.7).

A remark based on the results of chapter 2 and the present section will conclude this section. It is seen that an infinite number of Fourier modes are excited along unstable eigenmodes of $\bar{u}$. The amplitudes of the high order Fourier modes, although finite, are "exponentially" small (beyond all orders). This agrees with the dynamical constraint derived earlier in chapter 2 and a spectral scaling law ($|k|^{-\eta}$ with $\eta > 5$) obtained afterward. In fact, one can show that $\lambda_s \parallel w \parallel^2 \;>\; \parallel w \parallel_1^2$ for an unstable eigenmode w, making the constraint even stronger in the vicinity of $\bar{u}$ (on the unstable manifold emanating from $\bar{u}$).

4.2 Nonlinear stability

It is seen from the last section that a laminar across-channel flow (pure x_2-mode) $\bar{u} = \bar{u}_s e'_s$, where $s = (0, \alpha s_2)^T$, does not become unstable by distributing its energy to any other pure x_2-modes, but rather to mixed modes (and possibly to pure x_1-modes). It is also noted that any unstable eigenvector emanating from the laminar flow has to contain at least one (and at most two) mode k the scale of which is larger than that of the laminar flow itself. Thus, in order for a stationary flow of a pure x_2-mode, s, to be linearly unstable it is necessary that there be at least a mode k which is not a pure x_2-mode such that $|k| < |s|$. Otherwise, such a flow remains linearly stable for any value of $\bar{u}_s$. There is, however, no guarantee that $\bar{u}$ remains nonlinearly stable for arbitrary values of $\bar{u}_s$. A couple of conditions for its global stability will be derived in this section.

Unlike the nonlinear stability problem for finite-dimensional systems in which all norms are equivalent, a nonlinear stability analysis for an infinite-dimensional system requires a specified norm. A stationary solution of an infinite-dimensional system that is shown to be nonlinearly stable with respect to a given norm may not necessarily be stable with respect to another norm. The present problem is an exceptional case in which the stability of $\bar{u}$ with respect to the energy norm implies the stability of $\bar{u}$ with respect to the enstrophy norm, and vice versa. This result is actually an immediate corollary of (2.24) and the proof goes as follows. Let the solution of the NS equation be written in the form

$$u = \bar{u} + v.$$

The stability of $\bar{u}$ with respect to the energy (enstrophy) norm is established if $\| v \|$ ($\| v \|_1$) can be shown to go to zero as $t \to \infty$. Due to the Poincaré inequality, $\| v \|_1 \to 0$ implies $\| v \| \to 0$; hence, the stability of $\bar{u}$ with respect to the enstrophy norm automatically implies that of $\bar{u}$ with respect to the energy norm. The reverse direction is established by the fact that

$$\lambda_s \| \bar{u} + v \|^2 - \| \bar{u} + v \|_1^2 \;=\; \lambda_s \| v \|^2 - \| v \|_1^2 .$$

Now the left-hand side of the above is, by (2.24), non-negative as $t \to \infty$. That renders $\lambda_s \| v \|^2 \geq \| v \|_1^2$ as $t \to \infty$. Thus, the following statements are equivalent: (i) $\bar{u}$ is nonlinearly stable with respect to the energy norm. (ii) $\bar{u}$ is nonlinearly stable with respect to the enstrophy norm.

Let v be defined as above. The governing equation for v is

$$\frac{dv}{dt} + B(\bar{u}, v) + B(v, \bar{u}) + B(v, v) + \nu A v = 0. \tag{4.19}$$

The energy evolves according to

$$\frac{1}{2}\frac{d}{dt} \| v \|^2 + (v, B(\bar{u}, v)) + (v, B(v, \bar{u})) + (v, B(v, v)) + \nu \| v \|_1^2 \;=\; 0. \tag{4.20}$$

Two of the trilinear forms drops out due to the orthogonality property of $B(\cdot,\cdot)$. So the above equation reduces to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\parallel v\parallel^2 &= (\bar{u}, B(v,v)) - \nu \parallel v \parallel_1^2 \\ &= \bar{u}_s(e'_s, B(v,v)) - \nu \parallel v \parallel_1^2 . \end{aligned} \tag{4.21}$$

The global stability of $\bar{u}$ is established if the right-hand side of (4.21) is negative. Commonly, the trilinear form and the dissipation term can be estimated via the inequalities reviewed earlier in the last chapter. But that practice would yield too generous estimates, especially when α is small. The present calculation exploits the fact that $(e'_s, B(v,v))$ does not involve any term with $v_{(0,\alpha n)^T}$ or $v'_{(0,\alpha n)^T}$ in the Fourier series of v where n is a positive integer. It also takes advantage of the fact that couplings among nearby modes are prohibited when $|s|$ is large. The former property enables one not to rely on the eigenvalues of the across-channel modes $\lambda_{(0,\alpha n)^T} = 4\pi^2\alpha^2 n^2/L^2$ (which are small when α is small) in the majorization of (4.21), while the latter prevents nonlinear couplings among the largest scale modes which have the greatest ability to create instability. These are made possible by explicitly expressing $B(v,v)$ in (4.21) via lemma B of appendix B:

$$\begin{aligned} B(v,v) = & \sum_{k,l\in K} \frac{\sqrt{2}\alpha\pi k'\cdot l(|k|^2-|l|^2)}{L^2|k||l|}\left(\frac{e'_{k+l}}{|k+l|}+\frac{e'_{k-l}}{|k-l|}\right)v_k v_l \\ & - \sum_{k,l\in K} \frac{\sqrt{2}\alpha\pi k'\cdot l(|k|^2-|l|^2)}{L^2|k||l|}\left(\frac{e'_{k+l}}{|k+l|}-\frac{e'_{k-l}}{|k-l|}\right)v'_k v'_l \\ & - \sum_{k,l\in K} \frac{\sqrt{2}\alpha\pi k'\cdot l(|k|^2-|l|^2)}{L^2|k||l|}\left(\frac{e_{k+l}}{|k+l|}-\frac{e_{k-l}}{|k-l|}\right)v_k v'_l . \end{aligned}$$

Since there exist no triads in K satisfying $k+l=s$ and because of the orthogonality of the basis functions the trilinear form $(e'_s, B(v,v))$ reduces to

$$(e'_s, B(v,v)) = \sum_{k-l=s} \frac{\sqrt{2}\alpha\pi k'\cdot l(|k|^2-|l|^2)}{L^2|k||l||s|}(v_k v_l + v'_k v'_l).$$

To evaluate the coefficients of $v_k v_l$ and $v'_k v'_l$ it is observed that

$$k'\cdot l \;=\; k_1\alpha s_2 \;=\; l_1 \alpha s_2$$

$$\begin{aligned} |k|^2 - |l|^2 &= |l+s|^2 - |l|^2 = \alpha^2 s_2^2 + 2\alpha^2 s_2 l_2 \\ |k||l||s| &= \alpha s_2 (l_1^2 + (\alpha s_2 + \alpha l_2)^2)^{1/2} (l_1^2 + \alpha^2 l_2^2)^{1/2}. \end{aligned}$$

Therefore,

$$\frac{k' \cdot l(|k|^2 - |l|^2)}{|k||l||s|} = \frac{l_1 \alpha s_2 (\alpha s_2 + 2\alpha l_2)}{(l_1^2 + (\alpha s_2 + \alpha l_2)^2)^{1/2} (l_1^2 + \alpha^2 l_2^2)^{1/2}}.$$

The trilinear form then reads upon substitution of the above identity

$$(e_s', B(v,v)) = \sum_{l_1=1}^{\infty} \sum_{l_2=-\infty}^{\infty} \frac{\sqrt{2\alpha}\pi l_1 \alpha s_2 (\alpha s_2 + 2\alpha l_2)(v_l v_{l+s} + v_l' v_{l+s}')}{L^2 (l_1^2 + (\alpha s_2 + \alpha l_2)^2)^{1/2} (l_1^2 + \alpha^2 l_2^2)^{1/2}}. \tag{4.22}$$

Note that each v_l (v_l'), for $l \neq (0, \alpha l_2)^T$, appears exactly twice in the above sum. Meanwhile, the dissipation term can be majorized as follows

$$\| v \|_1^2 = \sum_k \lambda_k (v_k^2 + v_k'^2) \geq \sum_k (\lambda_k \lambda_{k+s})^{1/2} (|v_k v_{k+s}| + |v_k' v_{k+s}'|). \tag{4.23}$$

Now by substituting (4.22) and (4.23) into the energy equation (4.21) yields (note the change of the dummy indices l_1 and l_2)

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \| v \|^2 \leq \sum_{k_1=1}^{\infty} \sum_{k_2=-\infty}^{\infty} \Bigg(& \frac{\bar{u}_s \sqrt{2\alpha}\pi k_1 \alpha s_2 |\alpha s_2 + 2\alpha k_2|}{L^2 (k_1^2 + (\alpha s_2 + \alpha k_2)^2)^{1/2} (k_1^2 + \alpha^2 k_2^2)^{1/2}} \\ & - \nu \frac{4\pi^2}{L^2} (k_1^2 + (\alpha s_2 + \alpha k_2)^2)^{1/2} (k_1^2 + \alpha^2 k_2^2)^{1/2} \Bigg) (|v_k v_{k+s}| + |v_k' v_{k+s}'|). \end{aligned} \tag{4.24}$$

The nonlinear stability of $\bar{u}$ requires that the right-hand side of (4.24) be non-positive. That can be satisfied when

$$\frac{\bar{u}_s \sqrt{2\alpha} k_1 \alpha s_2 |\alpha s_2 + 2\alpha k_2|}{4\pi\nu (k_1^2 + (\alpha s_2 + \alpha k_2)^2)(k_1^2 + \alpha^2 k_2^2)} \leq 1, \tag{4.25}$$

for all positive integers k_1 and all integers k_2. Since the left-hand side of (4.25) is greatest when $k_1 = 1$ for each k_2, the stability condition reduces to

$$\frac{\bar{u}_s \sqrt{2\alpha} \alpha s_2 |\alpha s_2 + 2\alpha k_2|}{4\pi\nu (1 + (\alpha s_2 + \alpha k_2)^2)(1 + \alpha^2 k_2^2)} \leq 1,$$

The problem of determining the nonlinear stability condition for $\bar{u}$ now reduces to determining the greatest value of the left-hand side of (4.25) for all integers k_2 and setting it ≤ 1. For that purpose, let us determine the maximum of the function

$$g(x) = \frac{|c+2x|}{(1+x^2)(1+(c+x)^2)},$$

where c is a positive parameter and x is a continuous real variable. It can be seen that $g(x)$ is symmetric about the line $x = -c/2$; therefore, the translation $x \to x - c/2$ helps reduce the problem to finding the maximum of

$$h(x) = \frac{2x}{(1+(x-c/2)^2)(1+(x+c/2)^2)}, \quad x > 0.$$

Differentiating $h(x)$ with respect to x, setting the derivative to zero, and solving the resultant equation for x in terms of c one obtains the only real solution below

$$x = \left(\frac{c^2 - 4 + 2(c^4 + 4c^2 + 16)^{1/2}}{12}\right)^{1/2},$$

which corresponds to a maximum value of $h(x)$. Therefore, $g(x)$ peaks at

$$x = -\frac{c}{2} + \left(\frac{c^2 - 4 + 2(c^4 + 4c^2 + 16)^{1/2}}{12}\right)^{1/2},$$

with the maximum value

$$\bar{g}(c) = \frac{\left((c^2 + \frac{2}{3}((c^4 + 4c^2 + 16)^{1/2} - c^2 - 2)\right)^{1/2}}{1 + \left(c^2 + \frac{1}{3}((c^4 + 4c^2 + 16)^{1/2} - c^2 - 2)\right) + \frac{1}{36}((c^4 + 4c^2 + 16)^{1/2} - c^2 - 2)^2}.$$

With this result $\bar{u}$ is nonlinearly stable when

$$\frac{\bar{u}_s \sqrt{2\alpha}\alpha s_2}{4\pi\nu}\bar{g}(\alpha s_2) \leq 1. \tag{4.26}$$

This is a sufficient condition for nonlinear stability. Note that the result is not restricted to cases with $\alpha s_2 \leq 1$ but applies, in general, to a laminar monoscale across-channel flow.

There is a feature of $\bar{g}(\alpha s_2)$ in (4.26) which leads to an interesting difference between the conditions for stability when the forcing scale is about the width of the channel or larger and when the forcing scale is much smaller than the channel width. When αs_2 is small, $\bar{g}(\alpha s_2)$ is of the order of unity, so that (4.26) reduces to

$$\frac{\bar{u}_s\sqrt{2\alpha}\alpha s_2}{4\pi\nu} \leq 1. \tag{4.27}$$

However, for large αs_2, $\bar{g}(\alpha s_2)$ becomes

$$\bar{g}(\alpha s_2) \sim \frac{\alpha s_2}{1+\alpha^2 s_2^2} \sim \frac{1}{\alpha s_2}.$$

As a result, the nonlinear stability of $\bar{u}$ prevails when

$$\frac{\bar{u}_s\sqrt{2\alpha}}{4\pi\nu} \leq 1. \tag{4.28}$$

The above analysis concerns only the stability of $\bar{u}$ with repect to the energy norm. A similar approach is now applied to derive a sufficient condition for the stability of $\bar{u}$ with respect to the enstrophy norm. Because of the equivalence of the two normed stability conditions a comparison between them can be made and an optimal condition for both normed stabilities can be deduced.

The governing equation for the enstrophy is obtain by taking the scalar product of (4.19) with Av

$$\frac{1}{2}\frac{d}{dt} \parallel v \parallel_1^2 + (Av, B(\bar{u}, v)) + (Av, B(v, \bar{u})) + (Av, B(v, v)) + \nu \parallel v \parallel_2^2 = 0.$$

Due to the orthogonality property of $B(\cdot,\cdot)$ the above equation reduces to

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \parallel v \parallel_1^2 &= (A\bar{u}, B(v,v)) - \nu \parallel v \parallel_2^2 \\ &= \lambda_s \bar{u}_s (e_s', B(v,v)) - \nu \parallel v \parallel_2^2 . \end{aligned} \tag{4.29}$$

The global stability of $\bar{u}$ is established if the right-hand side of (4.29) is negative. The trilinear form has already been estimated above. Meanwhile, the dissipation term can be majorized as follows

$$\parallel v \parallel_2^2 = \sum_k \lambda_k^2 (v_k^2 + v_k'^2) \geq \sum_k (\lambda_k \lambda_{k+s})(|v_k v_{k+s}| + |v_k' v_{k+s}'|). \tag{4.30}$$

Now by substituting (4.22) and (4.30) into the enstrophy equation (4.29) yields

$$\frac{1}{2}\frac{d}{dt}\parallel v \parallel_1^2 \leq \sum_{k_1=1}^{\infty}\sum_{k_2=-\infty}^{\infty}\left(\frac{\lambda_s \bar{u}_s\sqrt{2\alpha}\pi k_1 \alpha s_2|\alpha s_2+2\alpha k_2|}{L^2(k_1^2+(\alpha s_2+\alpha k_2)^2)^{1/2}(k_1^2+\alpha^2 k_2^2)^{1/2}} -\nu\frac{16\pi^4}{L^4}(k_1^2+(\alpha s_2+\alpha k_2)^2)(k_1^2+\alpha^2 k_2^2)\right)(|v_k v_{k+s}|+|v_k' v_{k+s}'|). \tag{4.31}$$

The nonlinear stability of $\bar{u}$ requires that the right-hand side of (4.31) be non-positive. That can be satisfied when

$$\frac{\alpha^2 s_2^2 \bar{u}_s\sqrt{2\alpha}k_1\alpha s_2|\alpha s_2+2\alpha k_2|}{4\pi\nu(k_1^2+(\alpha s_2+\alpha k_2)^2)^{3/2}(k_1^2+\alpha^2k_2^2)^{3/2}} \leq 1, \tag{4.32}$$

for all positive integers k_1 and all integers k_2. Since the left-hand side of (4.32) is greatest when $k_1 = 1$ for each k_2, the stability condition reduces to

$$\frac{\bar{u}_s\sqrt{2\alpha}\alpha^3 s_2^3|\alpha s_2+2\alpha k_2|}{4\pi\nu(1+(\alpha s_2+\alpha k_2)^2)^{3/2}(1+\alpha^2k_2^2)^{3/2}} \leq 1.$$

Similar to the previous calculation for the energy norm, the present problem reduces to determining the maximum of the function

$$g_1(x) = \frac{|c+2x|}{(1+x^2)^{3/2}(1+(c+x)^2)^{3/2}},$$

or equivalently the maximum of

$$h_1(x) = \frac{2x}{(1+(x-c/2)^2)^{3/2}(1+(x+c/2)^2)^{3/2}}, \quad x > 0,$$

where c is a positive parameter and x is a continuous real variable. It turns out that $g_1(x)$ peaks at

$$x = -\frac{c}{2}+\left(\frac{2c^2-8+(9c^4-12c^2+144)^{1/2}}{20}\right)^{1/2},$$

with the maximum value

$$\bar{g}_1(c) = \frac{10\left(c^2-4+\frac{1}{2}(9c^4-12c^2+144)^{1/2}\right)^{1/2}}{\left(2+7c^2+(9c^4-12c^2+144)^{1/2}+\frac{1}{40}(3c^2+8-(9c^4-12c^2+144)^{1/2})^2\right)^{3/2}}.$$

With this result $\bar{u}$ is nonlinearly stable when

$$\frac{\bar{u}_s\sqrt{2\alpha}\alpha^3 s_2^3}{4\pi\nu}\bar{g}_1(\alpha s_2) \quad \leq \quad 1. \tag{4.33}$$

It is not hard to see that $\bar{g}(\alpha s_2)$ and $\bar{g}_1(\alpha s_2)$ are of the same order when $\alpha s_2 \sim 1$. In that case either (4.26) or (4.33) may be approximately taken as a sufficient condition for the stability of $\bar{u}$. When $\alpha s_2 \ll 1$, $\bar{g}_1(\alpha s_2)$ is of the order of unity, so that (4.33) reduces to

$$\frac{\bar{u}_s\sqrt{2\alpha}\alpha^3 s_2^3}{4\pi\nu} \quad \leq \quad 1. \tag{4.34}$$

For large αs_2, $\bar{g}(\alpha s_2)$ becomes

$$\bar{g}(\alpha s_2) \quad \sim \quad \frac{1}{\alpha^2 s_2^2}.$$

As a result, the nonlinear stability of $\bar{u}$ prevails when

$$\frac{\bar{u}_s\sqrt{2\alpha}\alpha s_2}{4\pi\nu} \quad \leq \quad 1. \tag{4.35}$$

A comparison between (4.27) and (4.34) indicates that the latter is optimal, while a comparison between (4.28) and its counterpart (4.35) favours the former. These two optimal conditions are more naturally expressible in terms of the characteristic length scales defined in chapter 3. Since $\bar{u}_s = f_0/\nu\lambda_s$, a little algebra leads to the restatement of both (4.34) and (4.28) as follows

$$\frac{L^4}{16\pi^3 L_s^2 L_D^2} \quad \leq \quad 1, \quad \text{for} \quad \frac{L_s}{L} \gg 1, \tag{4.36}$$

$$\frac{L L_s}{16\pi^3 L_D^2} \quad \leq \quad 1, \quad \text{for} \quad \frac{L_s}{L} \ll 1. \tag{4.37}$$

These expressions are self-explanatory. It is sufficient to mention two properties. (i) Given a fixed kinematic viscosity, forcing amplitude and scale, the conditions are α-independent. This is true for any forcing scale, not just the two extreme cases pointed out above. The upshot is that the stationary flow $\bar{u}$, once nonlinearly stable, remains nonlinearly stable for arbitrarily small values of α. This result is seen to be consistent with the physical set-up of the problem: The stability of an across-channel flow of a fixed scale should not depend on the length of the channel. (ii) The dependence of the stability condition on the forcing scale is due to two effects. On the one hand, the amplitude of $\bar{u}$ is seen to grow as L_s^2. This factor should and does enter the stability condition directly. On the other hand, L_s dictates the nature of the nonlinear coupling which is responsible for instability. A mode k of large scale and a mode l are coupled only if the scale of the latter is approximately equal to L_s, so that a small L_s (large $|s|$) forbids coupling among the largest scales. Thus scale-separation coupling diminishes the ability of the nonlinear term to overcome the effects of molecular viscosity and cause instability. This effect, when factored in, results in the functions $\bar{g}(\alpha s_2)$ and $\bar{g}_1(\alpha s_2)$ which lead to the variable dependences of the stability condition on the forcing scale. More precisely, a forcing of a certain amplitude is most powerful in terms of its ability to create a nonlinearly unstable laminar flow (and hence chaos) when its scale is comparable to the channel width.

The promise to demonstrate the physical significance of L_D can be fulfilled by interpreting the conditions for nonlinear stability. Consider a domain of length scale L of the same order of magnitude as L_D. The nonlinear stability conditions dictate that no laminar flows the scales of which are below L may become unstable. It means that at L_D viscosity completely outplays nonlinearity. A somewhat liberal interpretation of this result is that L_D is the length scale such that the energies of the harmonics below that scale are not so ready for exchanges among themselves.

Chapter 5

Concluding remarks

This thesis presents a theoretical study of the Navier-Stokes system (NS) and its modified version (MNS) on the 2-torus, in which a linear dissipation mechanism called Ekman drag is added. It aims to determine the asymptotic behaviour, stability, and number of degrees of freedom of these systems.

An analysis of the dynamics of the NS system driven by a monoscale forcing is reported in chapter 2. A constraint is derived for the dynamics of this system. It is found that most of the energy of the system is concentrated on scales larger than the forcing scale, while most of the enstrophy is distributed about the forcing scale. As a result, there may not exist an enstrophy cascading inertial range [3, 30, 27, 28] in which the energy spectrum would scale as $|k|^{-3}$ (possibly with a logarithmic correction). In fact, it is highly probable that a much steeper scaling law ($|k|^{-\eta}$ with $\eta > 5$) applies to scales smaller than the forcing scale, as demonstrated in chapter 2.

Besides Kraichnan's theory [27, 28], which gives the predicted values of $\eta = 3$ as mentioned above, there exist a number of predictions for the numerical value of η. Saffman [55] proposes that $\eta = 4$, while Moffatt [48] favours a slightly smaller value: $\eta = 11/3$. Sulem and Frisch [59], instead, propose an upper bound: $\eta \leq 11/3$. Most recently, Polyakov [52] considers an infinite number of possible spectra with $\eta > 3$. While most of the previous predictions are either conjectural or hypothetical in nature the present prediction is rigorously substantiated by a proven fact.

A dissipation mechanism consisting of the usual molecular viscosity and a hypo-viscosity is proposed for large 2D fluid systems driven by large-scale forcing. While the former channel of dissipation is familiar and standard in fluid dynamics, the latter does not enjoy the same kind of popularity. It is intended, as proposed, to account for the loss of kinetic energy due to boundary friction and friction-generated turbulent eddy activity in the ignored vertical dimension. The physical effect of this term is to provide a reasonable damping rate for large-scale motion observed in fluid systems such as the atmosphere or water tank experiments [49].

A spectral analysis of the 2D Navier-Stokes equations with the addition of linear Ekman drag, which is a special form of hypoviscosity, was carried out. The result is compared and contrasted with that of the original system. It has been shown that Ekman drag is capable of causing the energy spectrum to adjust toward the small scales. As a result, a spectral distribution of $|k|^{-\eta}$ with $3 < \eta < 5$ for scales smaller than the forcing scale is possible for the modified system.

Both linear and nonlinear stability analyses of a laminar monoscale stationary flow for the 2D Navier-Stokes equations are reported in chapter 4. A necessary condition for the existence of unstable eigenmodes and their properties are derived. The results indicate that in the vicinity of the unstable laminar flow a trajectory is in the phase of an upscale energy cascade. Moreover, the energy possessed by a scale associated with a wavenumber $|k|$ goes as $|k|^{-n}$ when $|k| \to \infty$ for $\forall n > 0$. Those results can be readily generalized to systems with a general viscosity as well. Two interesting properties which are peculiar to elongated domains have been pointed out in chapter 4. One is the fact that there are laminar monoscale across-channel stationary flows which remain linearly stable for arbitrary values of the physical parameters (arbitrarily large Reynolds number, for instance). Note that these stationary flows are not the flow of the largest scale considered by Meshalkin and Sinai [47], Iudovich [25], and Marchioro [44] (see appendix C) and are not known to remain globally stable. The other is the result from the nonlinear stability analysis. It has been shown that a laminar monoscale across-channel stationary flow, once nonlinearly stable, remains nonlinearly stable for an arbitrary length of the channel.

Extensivity of the 2D Navier-Stokes system on both square and elongated domains has been demonstrated in chapter 3. The attractor dimension in each case is found to grow linearly with the domain area provided that the forcing scale and amplitude and the kinematic viscosity are held fixed. Although the results are in the form of upper bounds rather than equalities, the system's extensivity can be concluded with definiteness thanks to the results of [35] in which a lower bound on the attractor dimension for a similar system is also shown to grow linearly with the system size. However, extensive behaviour can only be found for systems larger than a certain scale called the extensivity scale (defined in chapter 3). For a system smaller than this scale, an optimal estimate of the attractor dimension grows slightly faster than the system size, being indicative of super-extensive behaviour.

It is found that the addition of linear Ekman drag does not destroy the extensivity of the NS system. Instead, this dissipation channel enlarges the dissipation length and reduces the attractor dimension in such a way that the dimension may still grow linearly with the domain area for large systems.

Most of the analysis in this work revolves about the forcing scale (injection scale) which is unambiguously defined by the introduction of a monoscale forcing. The role of this parameter in the dynamics and dynamical complexities of the NS system (with/without large-scale dissipation) is explored. It is demonstrated that a monoscale across-channel stationary flow (resulted from a monoscale across-channel forcing) is most likely to be nonlinearly unstable when the scale is on the same order as the channel width. Away from this range (either too small or too large as compared to the channel width) a forcing may lose its strength and eventually become incapable of creating instability, let alone exciting chaos. The forcing scale also plays a role in the attractor dimension. This role is hidden in the dissipation length in the regime of extensive chaos. For the 2D Navier-Stokes system the dissipation length is inversely proportional to the forcing scale, so that the attractor dimension grows linearly with the forcing scale. The explanation of this property rests on the viscous effects. It is totally due to viscosity that the bound on the enstrophy of the attractor grows as the square of the forcing scale. Although this dependence is not guaranteed to be sharp it is physically reasonable. A large-scale forcing can avoid viscous effects and promote intense dynamics of the large scales. These "strong" modes subsequently excite the small-scale modes via nonlinearity. In contrast, a small-scale forcing of the same magnitude suffers heavy damping and produces a much weaker dynamics of the large scales. It is, therefore, less capable of exciting and maintaining as broad a spectrum as its large-scale counterpart, despite the obvious advantage that the spectral peak in this case is closer to the smallest scales. Thus a larger-scale forcing results in a smaller dissipation length and an attractor of higher dimension. This picture completely reverses for the MNS system in which Ekman drag dominates the viscosity. In this case, the bound on the enstrophy of the attractor is inversely proportional to the square of the forcing scale, i.e., forcings of smaller scale bring about attractors of greater enstrophies provided that their amplitudes are the same. This means that a smaller-scale forcing is more capable of exciting and maintaining a broader spectrum than a larger-scale one, and hence yields a higher dimensional attractor.

An apparent difference between the systems analyzed in this thesis and a more realistic model of large-scale atmospheric motion is the lack of curvature and the addition of artificial boundary conditions in the former. It turns out, however, that spherical geometry is equivalent to the doubly-periodic flat space with respect to the estimates involved [12,13]. It is therefore a rather straightforward reformulation of the present problem in the spherical case, and one would have attractor dimensions of the same order of magnitude for both geometries.

It should be admitted that the estimates are uncomfortably large even for simple flows governed by the MNS equations. But given a large system full of small-scale fluctuations such as the atmosphere, one could hardly think of a small number of modes that would describe each and every snapshot of its dynamics adequately. The huge number of degrees of freedom is due to the presence of small-scale dynamics at the dissipation length. In practice one may wish to ignore “twists and turns” the sizes of which are many orders of magnitude larger than the dissipation size. Unfortunately, the governing equations admit those as legitimate solutions. The question is how one could go about revising the governing equations in such a way that “twists and turns” up to predefined scales are filtered out while those of larger scales remain unaffected or nearly so. The flow governed by the revised system would then need fewer modes to simulate. This step may be compared to the step of going from molecular dynamics to the Navier-Stokes system for fluids.

APPENDIX A

Optimal problems

This appendix considers two algebraic problems: An optimal problem of a one parameter quadratic form and a supremum problem of a function that are used in chapter 3.

I. Consider,

$$f(x) = -\epsilon a x^2 + bx + \frac{c}{1-\epsilon}$$

where a, b, c are fixed positive numbers, $\epsilon \in [0, 1)$, and $x > 0$. The problem is to find the smallest possible x_0 by varying ϵ in the allowed range such that $f(x) \leq 0$ for $x \geq x_0$. This problem can be taken care of by elementary calculus. By setting $f(x) = 0$, x can be solved in terms of ϵ

$$x = \left(\sqrt{b^2 + \frac{4ac\epsilon}{1-\epsilon}} + b \right) / 2a\epsilon$$

Taking the derivative of x with respect to ϵ and setting $dx/d\epsilon = 0$, it is found after some algebra

$$\epsilon = \frac{\sqrt{ac} + b}{2\sqrt{ac} + b} \ .$$

x_0 is found from the above equations to be

$$x_0 = \frac{2\sqrt{ac} + b}{a}.$$

II. Consider,

$$f(y) = \xi y^{1/4}(1+\ln y)^{1/2} - 2\epsilon y,$$

for $y > 1,\ \xi,\ \epsilon > 0$. The problem is to find $\sup f(y)$. This problem is solved in [60], and the steps go as follows. Let

$$g(y) = \xi^2(1+\ln y) - \epsilon^2 y^{3/2}.$$

The maximum of $g(y)$ is found to be

$$\max g(y) = \xi^2\left(\frac{1}{3}+\frac{2}{3}\ln\frac{2\xi^2}{3\epsilon^2}\right).$$

Hence,

$$\xi^2(1+\ln y) \le \epsilon^2 y^{3/2} + \xi^2\left(\frac{1}{3}+\frac{2}{3}\ln\frac{2\xi^2}{3\epsilon^2}\right).$$

And the following steps are straightforward

$$\xi(1+\ln y)^{1/2} \le \epsilon y^{3/4} + \xi\left(\frac{1}{3}+\frac{2}{3}\ln\frac{2\xi^2}{3\epsilon^2}\right)^{1/2},$$

$$\begin{aligned}\xi y^{1/4}(1+\ln y)^{1/2} &\le \epsilon y + \xi y^{1/4}\left(\frac{1}{3}+\frac{2}{3}\ln\frac{2\xi^2}{3\epsilon^2}\right)^{1/2} \\ &\le \frac{3}{2}\epsilon y + \frac{3\xi^{4/3}}{4(2\epsilon)^{1/3}}\left(\frac{1}{3}+\frac{2}{3}\ln\frac{2\xi^2}{3\epsilon^2}\right)^{2/3}.\end{aligned}$$

It follows that

$$\xi y^{1/4}(1+\ln y)^{1/2} - 2\epsilon y \le -\frac{\epsilon}{2}y + c\frac{\xi^{4/3}}{\epsilon^{1/3}}\left(1+\ln\frac{\xi}{\epsilon}\right)^{2/3},$$

where $c < 3/4$.

APPENDIX B

Spectral form of the vorticity equations

This appendix presents a spectral analysis of the nonlinear interactions in the NS (and MNS) equations, the results of which are used in the stability problems in chapter 4. For that purpose the forcing and dissipation terms are ignored. The inviscid and unforced system is essentially the familiar vorticity equation. This equation and its Fourier transform were studied by Lorenz [39], in which nonlinear interactions in the form of triads of three different scales were pointed out and a system of only three ordinary differential equations was proposed for some qualitative studies of the atmospheric general circulations. It is not intended of this appendix to further elaborate on the behaviour of the Lorenz three-component system. The main objective is rather to demonstrate an alternative route leading to the spectral form of the vorticity equation. The new route to be taken is via basis functions in vector form rather than those in scalar form as in [39]. It is hoped that more insights into the nature of these interactions can be gained from the discussion and that the new spectral equations will offer advantages in numerical analysis.

The inviscid and unforced version of the NS (and MNS) system is written in H as follows (initial condition is omitted)

$$\frac{du}{dt} + B(u,u) = 0. \tag{B-1}$$

The spectral equations can be derived by expanding u as in (2.4). The nonlinear term can then be handled by the following lemma which is taken from Liu [35, 36] (an elongated domain with a shape ratio of α is considered in the present calculation instead of a square domain in [35, 36]).

Lemma B: For every k and l in the definition of e_k and e'_k

$$B(e_k, e_l) + B(e_l, e_k) \quad = \quad \frac{\sqrt{2\alpha}\pi k' \cdot l(|k|^2 - |l|^2)}{L^2|k||l|}\left(\frac{e'_{k+l}}{|k+l|} + \frac{e'_{k-l}}{|k-l|}\right), \tag{B-2}$$

$$B(e'_k, e'_l) + B(e'_l, e'_k) = -\frac{\sqrt{2\alpha}\pi k' \cdot l(|k|^2 - |l|^2)}{L^2|k||l|}\left(\frac{e'_{k+l}}{|k+l|} - \frac{e'_{k-l}}{|k-l|}\right), \quad \text{(B-3)}$$

$$B(e_k, e'_l) + B(e'_l, e_k) = -\frac{\sqrt{2\alpha}\pi k' \cdot l(|k|^2 - |l|^2)}{L^2|k||l|}\left(\frac{e_{k+l}}{|k+l|} - \frac{e_{k-l}}{|k-l|}\right). \quad \text{(B-4)}$$

Proof: It suffices to give a proof for one of the equations. The others are proven in exactly the same manner. It is found by direct calculation that

$$\begin{aligned}(e_k \cdot \nabla)e_l &= -\frac{4\pi\alpha(k' \cdot l)l'}{L^3|k||l|}\cos\frac{2\pi}{L}kx \sin\frac{2\pi}{L}lx, \\ (e_l \cdot \nabla)e_k &= -\frac{4\pi\alpha(l' \cdot k)k'}{L^3|k||l|}\cos\frac{2\pi}{L}lx \sin\frac{2\pi}{L}kx. \end{aligned} \quad \text{(B-5)}$$

Adding the equations of (B-5), noting that $k' \cdot l = -l' \cdot k$ one finds

$$\begin{aligned} &(e_k \cdot \nabla)e_l + (e_l \cdot \nabla)e_k \\ = &-\frac{4\pi\alpha(k' \cdot l)}{L^3|k||l|}\left(l'\cos\frac{2\pi}{L}kx\sin\frac{2\pi}{L}lx - k'\cos\frac{2\pi}{L}lx\sin\frac{2\pi}{L}kx\right) \\ = &-\frac{2\pi\alpha(k' \cdot l)}{L^3|k||l|}\left((l'-k')\sin\frac{2\pi}{L}(k+l)x - (l'+k')\sin\frac{2\pi}{L}(k-l)x\right). \end{aligned} \quad \text{(B-6)}$$

It is noted that the right hand side of (B-6) is identically zero for $l = k$. When $l \neq k$ it can be projected onto H to yield

$$\begin{aligned} &B(e_k, e_l) + B(e_l, e_k) \\ = &-\frac{2\pi\alpha k' \cdot l}{L^3|k||l|}\left(\frac{L(l'-k')\cdot(k'+l')}{\sqrt{2\alpha}|k+l|}e'_{k+l} - \frac{L(l'+k')\cdot(k'-l')}{\sqrt{2\alpha}|k-l|}e'_{k-l}\right) \\ = &\frac{\sqrt{2\alpha}\pi k' \cdot l(|k|^2 - |l|^2)}{L^2|k||l|}\left(\frac{e'_{k+l}}{|k+l|} + \frac{e'_{k-l}}{|k-l|}\right), \end{aligned} \quad \text{(B-7)}$$

where the projection P of the vector sum involved onto H has been done term by term via

$$P(m\sin\frac{2\pi}{L}lx) = \frac{Lm \cdot l'}{\sqrt{2\alpha}|l|}e'_l. \quad \text{(B-8)}$$

The first equation of the lemma is thus proven. □

With the help of Lemma B the Fourier transform of (B-1) reads, upon substitution of (2.4) in (B-1)

$$\begin{aligned}\frac{d}{dt}\sum_k (u_k e_k + u'_k e'_k) &= \frac{1}{2}\sum_{l\neq m} \frac{\sqrt{2\alpha}\pi m'\cdot l(|m|^2-|l|^2)}{L^2|m||l|}\left(\frac{u'_m u'_l}{|m+l|}e'_{m+l} - \frac{u'_m u'_l}{|m-l|}e'_{m-l}\right)\\ &-\frac{1}{2}\sum_{l\neq m} \frac{\sqrt{2\alpha}\pi m'\cdot l(|m|^2-|l|^2)}{L^2|m||l|}\left(\frac{u_m u_l}{|m+l|}e'_{m+l} + \frac{u_m u_l}{|m-l|}e'_{m-l}\right)\\ &+\sum_{l\neq m} \frac{\sqrt{2\alpha}\pi m'\cdot l(|m|^2-|l|^2)}{L^2|m||l|}u_m u'_l\left(\frac{e_{m+l}}{|m+l|} - \frac{e_{m-l}}{|m-l|}\right), \quad \text{(B-9)}\end{aligned}$$

where a factor of one half has been inserted into the first two sums to compensate for the repetition of terms by the summation. The governing equations for the Fourier coefficients of u can be obtained by equating the coefficients of like Fourier modes. It is found for each k that

$$\begin{aligned}\frac{d}{dt}u_k &= \sum_{m\pm l=k} \frac{\pm\sqrt{2\alpha}\pi m'\cdot l(|m|^2-|l|^2)}{L^2|m||l||k|}u_m u'_l,\\ \frac{d}{dt}u'_k &= \sum_{m\pm l=k} \frac{\pm\sqrt{\alpha}\pi m'\cdot l(|m|^2-|l|^2)}{\sqrt{2}L^2|m||l||k|}(u'_m u'_l \mp u_m u_l). \quad \text{(B-10)}\end{aligned}$$

Naturally, the spectral equations inherit all properties of (B-1). For example, the energy and enstrophy are conserved, i.e.,

$$\begin{aligned}\frac{d}{dt}\sum_k (u_k^2 + {u'_k}^2) &= 0,\\ \frac{d}{dt}\sum_k \lambda_k (u_k^2 + {u'_k}^2) &= 0.\end{aligned}$$

That means that the sum of all triple products in each of the energy equation and enstrophy equation is identically zero. This property is inherited from the orthogonality properties of the nonlinear term: $(u, B(u,u)) = (Au, B(u,u)) = 0$. Also, by virtue of the orthogonality properties which clearly do not depend on the spectral composition of u, it is possible to omit reference to all but a finite number of Fourier modes by performing any finite truncation to the Fourier series of u without losing the orthogonality properties in the resulting finite system of spectral equations. However, it is clear that a maximum simplification would have to retain at least three dependent variables associated with a wave vector triad $m \pm l = k$. Lorenz [39] called such a system the minimum hydrodynamic equations. Obviously, the minimum hydrodynamic equations must involve three different scales or else the three nonlinear equations would reduce to two linear equations with the third variable becoming a parameter (see B-10). Furthermore, for similar reasons the three modes must be either all odd or two even and one odd.

It is interesting to note, for $\alpha = 1$, that the three largest scales, i.e., the first twelve modes associated with the six wavevectors $(0,1)^T, (1,0)^T, (1,1)^T, (1,-1)^T, (0,2)^T, (2,0)^T$ do not make nontrivial triads among themselves. As a result, the truncation of the first twelve terms of the Fourier series of u only yields trivial dynamics for any forcing term (see Smale [58]).

The important point to note at this stage is that the collective nature of the nonlinearity is inescapable. In principle each Fourier mode of the velocity field is coupled to every other mode and therefore the nonlinear term is very difficult to handle. It is not known, for example, how two triads (with one or two common scales) would interact, let alone a number of them. In this respect one is often at a loss as to which set of triads corresponds to a realistic model of the flow. As a matter of fact, numerical experiments [19] with the Lorenz convection model [38] (by adding more modes to the three-component system) indicate that some augmented models do not behave in a chaotic manner, and the flow simulations become periodic or even steady.

APPENDIX C

Examples of stationary and globally attracting solutions

This appendix features a fixed point analysis of the NS system. The stationary solutions are solved for two examples of the body force, and a simple proof will be presented that for one of the examples the NS equations admit trivial dynamics. The fact that the attractors of the NS system reduce to a single stationary solution for a particular choice of the body force (with arbitrary large Reynolds number) was demonstrated by Iudovich [25] and Marchioro [44] and a simpler proof was provided later by Constantin et al. [7]. The method of proof presented here is similar to that of [7]. However, a more general forcing than that in [7] will be considered. Let us recall the NS equations which are described in detail in chapter 2

$$\begin{aligned} \frac{du}{dt} + B(u,u) + \nu A u &= f, \qquad \text{(C-1)} \\ u(0) = u_0. \end{aligned}$$

In this appendix the volume force f is assumed to take either of the following forms

$$f_1 = \sum_{|k|=|s|} (f_k e_k + f'_k e'_k), \qquad \text{(C-2)}$$

$$f_2 = \sum_{k=ns} (f_k e_k + f'_k e'_k) \qquad \text{(C-3)}$$

where f_k, f'_k are constants and e_k, e'_k are the eigenvectors of A (see chapter 2), and n is a positive integer. Let $\bar{u}_1$ and $\bar{u}_2$ be defined by

$$\bar{u}_1 = (\nu A)^{-1} f_1 = (\nu \lambda_s)^{-1} \sum_{|k|=|s|} (f_k e_k + f'_k e'_k), \qquad \text{(C-4)}$$

$$\bar{u}_2 = (\nu A)^{-1} f_2 = \sum_{k=ns} (\nu \lambda_k)^{-1} (f_k e_k + f'_k e'_k) \qquad \text{(C-5)}$$

It is easy to see that $\bar{u}_1$ and $\bar{u}_2$ are stationary solutions of (C-1) with $f = f_1$ and $f = f_2$ respectively. The proof of this claim is supported by the following Lemmas which are essentially by-products from the process of proving Lemma B.

Lemma C1: The bilinear operator $B(u, v)$ satisfies, for all positive integer n

$$B(\varphi_k, \varphi_{nk}) = B(\varphi_{nk}, \varphi_k) = 0, \tag{C-6}$$

where φ_k is either e_k or e'_k.

Proof: There are six individual identities in (C-6). It suffices to demonstrate any one of them. Let us recall from (B-5)

$$(e_k \cdot \nabla)e_l = -\frac{4\pi\alpha k' \cdot l}{|k||l|L^3} l' \cos\frac{2\pi}{L}kx \sin\frac{2\pi}{L}lx. \tag{C-7}$$

Since $k' \cdot l = \alpha(k_2 l_1 - k_1 l_2)$, the right hand side of (C-7) vanishes when k and l are collinear, i.e. when $l = nk$. Thus, it is established that $B(e_k, e_{nk}) = B(e_{nk}, e_k) = 0$. Other identities in the lemma are proven similarly. □

Lemma C2: The bilinear operator $B(u, v)$ satisfies

$$B(e_k, e_l) + B(e_l, e_k) = B(e_k, e'_l) + B(e'_l, e_k) = B(e'_k, e'_l) + B(e'_l, e'_k) = 0, \tag{C-8}$$

whenever $|l| = |k|$.

Proof: Only the first identity will be demonstrated; others are proven in exactly the same manner. Let us recall eqs. (B-6) and (B-7) from the last appendix. It is easy to see that eq. (B-6) proves the lemma for $l = k$ while eq. (B-7) does the job for $l \neq k$. Thus, it has been shown that $B(e_k, e_l) + B(e_l, e_k) = 0$ for $|k| = |l|$. □

It is inferred from the two lemmas that $B(\bar{u}_1, \bar{u}_1) = B(\bar{u}_2, \bar{u}_2) = 0$. Hence,

$$B(\bar{u}_1, \bar{u}_1) + \nu A\bar{u}_1 = f_1, \tag{C-9}$$

$$B(\bar{u}_2, \bar{u}_2) + \nu A\bar{u}_2 = f_2. \tag{C-10}$$

Thus, $\bar{u}_i$ is a stationary solution of (C-1) corresponding to the force f_i $(i = 1, 2)$ as claimed.

The rest of this appendix shows that for $f = f_1(|s| = \alpha)$ the corresponding stationary solution is unique and globally attracting. Note that $f_1(|s|)$ is a linear combination of all degenerate eigenfunctions of A corresponding to the same eigenvalue

λ_s. In particular, $f_1(|s| = \alpha)$ is a linear combination of two degenerate eigenfunctions corresponding to the first eigenvalue λ_1 (when $\alpha = 1$ the first eigenvalue of A has four degenerate eigenfunctions). This fact is essential in the proof of trivial dynamics in this case as shall be seen.

Proof of uniqueness of $\bar{u}_1(|s| = \alpha)$.

Suppose that v is another fixed point of (C-1) corresponding to $f = f_1(|s| = \alpha)$, that is v satisfies

$$B(v, v) + \nu A v = f_1(|s| = \alpha). \tag{C-11}$$

Successively taking the scalar product in H of (C-11) with v and Av it can be shown after invoking (2.9), (2.11) and using the fact that $Af_1(|s| = \alpha) = \lambda_1 f_1(|s| = \alpha)$

$$\begin{aligned} \nu \parallel v \parallel_1^2 &= (v, f_1(|s| = \alpha)) \\ \nu \parallel Av \parallel^2 &= \lambda_1 (v, f_1(|s| = \alpha)) \end{aligned} \tag{C-12}$$

Multiplying the first equation of (C-12) by λ_1 and subtracting it from the other one yields

$$\nu(\parallel Av \parallel^2 - \lambda_1 \parallel v \parallel_1^2) = 0. \tag{C-13}$$

The left hand side of (C-13) is non-negative due to the Poincaré inequality. It means that (C-13) is true if and only if v belongs to the subspace spanned by the four degenerate eigenfunctions of A corresponding to λ_1. It is then inferred from lemma C2 that $B(v, v) = 0$ and that $v = \bar{u}_1(|s| = \alpha)$. □

Proof of global attraction of $\bar{u}_1(|s| = \alpha)$.

Similar to (C-12), it is derived from (C-1)

$$\frac{1}{2}\frac{d}{dt} \parallel u \parallel^2 + \nu \parallel u \parallel_1^2 = (u, f_1(|s| = \alpha)) \tag{C-14}$$

$$\frac{1}{2}\frac{d}{dt} \parallel u \parallel_1^2 + \nu \parallel Au \parallel^2 = \lambda_1 (u, f_1(|s| = \alpha)). \tag{C-15}$$

And by the same token

$$\frac{1}{2}\frac{d}{dt}(\| u \|_1^2 - \lambda_1 \| u \|^2) + \nu(\| Au \|^2 - \lambda_1 \| u \|_1^2) \quad = \quad 0. \tag{C-16}$$

Each of the two terms in the round brackets in (C-16) is non-negative by the Poincaré inequality. Hence, it can be deduced that

$$u \to H(\lambda_1) \quad \text{in the enstrophy norm as } t \to \infty. \tag{C-17}$$

Expression (C-17) implies that the global attractor is confined to the two-dimensional (four-dimensional for $\alpha = 1$) eigenspace corresponding to λ_1. To show that the attractor only consists of the single stationary solution it is convenient to expand a trajectory u on the attractor in a Fourier series

$$u \quad = \quad \sum_{|k|=\alpha} (u_k e_k + u_k' e_k'). \tag{C-18}$$

Successively taking the scalar product in H of (C-1) with the eigenvectors corresponding to λ_1, noting their orthogonality and using Lemma C2, one finds

$$\frac{du_k}{dt} + \nu\lambda_1 u_k \quad = \quad f_k, \tag{C-19}$$

$$\frac{du_k'}{dt} + \nu\lambda_1 u_k' \quad = \quad f_k', \tag{C-20}$$

for $|k| = \alpha$. These differential equations can be readily solved and it is apparent that u converges to $\bar{u}_1$ in H as $t \to \infty$. Thus, the global attractor reduces to a single stationary solution in this case. □

APPENDIX D

The CFT theorem

This appendix contains a brief review of the Constantin-Foias-Temam (CFT) theorem and its application in estimating the attractor dimension of dissipative dynamical systems. A complete treatment of this theorem and related subjects can be found in Temam [60].

Upper bounds on invariant sets or attractor dimensions for a dynamical system can be determined by considering the time evolution of finite-dimensional volume elements in the system phase space. If all m-dimensional volume elements with all possible orientations in the (infinite-dimensional) phase space contract to zero volume as $t \to \infty$, then the attractor contains no m-dimensional subsets and hence the Hausdorff dimension cannot exceed m. The goal is to determine the smallest possible m with this property. Such an m is an upper bound on the Hausdorff dimension of the attractor.

Consider the evolution equations in a Hilbert space H

$$\begin{aligned} \frac{d}{dt}u(t) &= F(u(t)), \\ u(t=0) &= u_0, \end{aligned} \tag{D-1}$$

which possess a global attractor, or more generally, an invariant set X. To monitor the evolution of volume elements in the phase space it is sufficient to consider the linearized version of (D-1) along phase space trajectories:

$$\frac{d}{dt}U(t) = F'(u(t))U(t). \tag{D-2}$$

In the above, F' is the Fréchet differential of F which is assumed to exist. For an infinitesimal m-dimensional volume situated in an m-dimensional subspace H_m of H, the volume $V(t)$ evolves along a trajectory $u(t)$ according to

$$V(t) = V_0 \exp\left\{ \int_0^t \mathrm{Tr} F'(u(\tau)) \circ P_m(\tau) d\tau \right\}.$$

In the above, P_m is the orthogonal projection in H into H_m. It is seen that when the integral in the above expression is negative $V(t)$ shrinks in time. It has been shown that if

$$q_m = \limsup_{t\to\infty} \sup_{u_0 \in X} \left(\frac{1}{t} \int_0^t \sup_{P_m} \mathrm{Tr} F'(u(\tau)) \circ P_m(\tau) d\tau \right) \tag{D-3}$$

is non-positive for all $m' \geq m$, then the Hausdorff dimension of X is less than or equal to m. Note that the supremum taken over P_m ensures that all possible orientations of the m-dimensional volume element are covered, while the supremum taken over the initial point $u_0 \in X$ takes care of all points of X.

The trace in (D-3) is evaluated via a basis of $P_m H$. If $\{\phi_i\}_{i=1}^m$ is a basis of $P_m H$ which are orthonormal in H then

$$\mathrm{Tr} F'(u(\tau)) \circ P_m = \sum_{i=1}^m (F'(u(\tau))\phi_i, \phi_i), \tag{D-4}$$

where $(\cdot, \cdot)$ denotes the inner product in H. The problem now reduces to the estimation of the sum above.

Equipped with this theorem one can, in principle, estimate the Hausdorff dimension of invariant sets of a dynamical system. The goal is to obtain an optimal estimate of the trace in (D-3) and then determine m. This process is highly technical and involves several steps in each of which there may be more than one way to estimate the terms in question. Because of that situation a knowledge of the magnitudes of the terms involved is necessary in order to have a possibly sharp estimate in each and every step. That makes one to decide, for example, which estimates to adopt for a given set of the physical parameters. Consequently, there may be more than one estimate of the Hausdorff dimension each of which is sharp in certain regions of the parameter space, and two such estimates do not necessarily reduce to each other as the parameters cross their optimal domains.

Bibliography

[1] I.S. Aranson, A.V. Gaponov-Grekhov, and M.I. Rabinovich, The development of chaos in dynamic structure ensembles, Sov. Phys. JETP, 62 (1985), No. 1, 52-59.

[2] A. V. Babin and M. I. Vishik, Attractors of partial differential equations and estimate of their dimensions, Russian Math. Surveys, 38:4 (1983), 151-213.

[3] G.K. Batchelor, Computation of the energy spectrum in homogeneous, two dimensional turbulence, Phys. Fluids, II 12 (1969), 233-239.

[4] H. Brézis and T. Gallouet, Nonlinear Schrödinger equations, Nonlinear Anal., TMA, 4 (1980), 677-681.

[5] C.E. Buell, The correlation between wind and height on an isobaric surface II: summer data, J. of Meteorology, 15 (1958), 502-512.

[6] P. Constantin, Collective L^∞ estimates for families of functions with orthonormal derivatives, Indiana Univ. Math. J., 36 (1987), 603-616.

[7] P. Constantin, C. Foias, and R. Temam, On the dimension of the attractors in two-dimensional turbulence, Physica D, 30 (1988), 284-296.

[8] P. Constantin and C. Foias, Global Lyapunov exponents, Kaplan-Yorke formulas and the dimension of the attractors for 2D Navier-Stokes equations, Comm. Pure Appl. Math., XXXVIII (1985), 1-27.

[9] P. Constantin, C. Foias, and R. Temam, Attractors representing turbulence flows, Memoirs of A.M.S., 53 (January, 1985), No. 314.

[10] M.C. Cross and P.C. Hohenberg, Pattern formation outside of equilibrium, Rev. Mod. Phys., 65 (1993), No. 3, 851-1112.

[11] D.A. Egolf, Characterization of extensively chaotic states and transitions, Ph.D. thesis, Department of Physics, Duke University, Durham, NC, 1994.

[12] D.A. Egolf and H.S. Greenside, Relation between fractal dimension and spatial correlation length for extensive chaos, Nature, 369 (1994), 129-131.

[13] V.W. Ekman, On the influence of the earth's rotation on ocean currents, Arch. Math. Astron. Phys., 2 (1905), No. 11.

[14] J.D. Farmer, E. Ott, and J.A. Yorke, The dimension of chaotic attractors, Physica D, 7 (1983), 153-180.

[15] R. Fjørtoft, On the changes in the spectral distribution of the kinetic energy for twodimensional, nondivergent flow, Tellus, 5 (1953), No. 3, 225-230.

[16] C. Foias and G. Prodi, Sur le comportement global des solutions non stationnaires des équations de Navier-Stokes en dimension 2, Rend. Sem. Mat. Univ. Padova, 39 (1967), 1-34.

[17] C. Foias, O.P. Manley, R. Temam, and Y.M. Treve, Asymptotic analysis of the Navier-Stokes equations, Physica D, 9 (1983), 157-188.

[18] C. Foias and E.S. Titi, Determining nodes, finite difference schemes and inertial manifolds, Nonlinearity, 4 (1991), 135-153.

[19] V. Franceschini and C. Tebaldi, A seven-mode truncation of the plane incompressible Navier-Stokes equations, J. Stat. Phys., 25 (1981), No. 3, 397-417.

[20] A. Gill, Atmosphere-Ocean Dynamics, (Academic Press, New York, 1982).

[21] H.S. Greenside, Spatiotemporal Chaos in Large Systems: The Scaling of Complexity with Size. Preprints of the Montreal Workshop of the CRM (Centre de Recherche en Mathematiques) under the title "Semi-analytic Methods for the Navier-Stokes Equations", Oct. 1995.

[22] P.C. Hohenberg and B.I. Shraiman, Chaotic behaviour of an extended system, Physica D, 37 (1989), 109-115.

[23] A.A. Il'in, Partly dissipative semigroups generated by the Navier-Stokes system on two-dimensional manifolds and their attractors, Russ. Acad. Sci. Sb. Math., 78 (1994), 47-76.

[24] A.A. Il'in, Lieb-Thirring inequalities on the N-sphere and in the plane, and some applications, Proc. London Math. Soc., 67 (1993), No. 3, 159-182.

[25] V.I. Iudovich, Example of the generation of a secondary stationary or periodic flow when there is loss of stability of the laminar flow of a viscous incompressible fluid, J. Appl. Math. Mech., 29 (1965), 527-544.

[26] D.A. Jones and E.S. Titi, Determining finite volume elements for the 2D Navier-Stokes equations, Physica D, 60 (1992), 165-174.

[27] R.H. Kraichnan, Inertial ranges in two-dimensional turbulence, Phys. Fluids, 10 (1967), 1417-1423.

[28] R.H. Kraichnan, Inertial-range transfer in two- and three-dimensional turbulence, J. Fluid Mech., 47 (1971), No. 3, 525-535.

[29] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Second English Edition, translated from the Russian by R.A. Silverman and J. Chu (Gordon and Breach, New York, 1969).

[30] C.E. Leith, Diffusion approximation for two-dimensional turbulence, Phys. Fluids, 11 (1968), 671-673.

[31] M. Lesieur, Turbulence in Fluids, 2nd edition (Kluwer Academic Publishers, Dordrecht, 1990).

[32] E. Lieb, An L^p bound for the Riesz and Bessel potentials of orthonormal functions, J. Funct. Anal., 51 (1983), No. 2, 159-165.

[33] E. Lieb and W. Thirring, Inequalities for the moments of the eigenvalues of the Schrödinger hamiltonian and their relation to Sobolev inequalities, Studies in Mathematical Physics (Princeton University Press, 1976).

[34] E. Lieb, On characteristic exponents in turbulence, Comm. Math. Phys., 92 (1984), 473-480.

[35] V.X. Liu, A sharp lower bound for the Hausdorff dimension of the global attractors of the 2D Navier-Stokes equations, Comm. Math. Phys., 158 (1993), 327-339.

[36] V.X. Liu, Instability for the Navier-Stokes equations on the 2-dimensional torus and a lower bound for the Hausdorff dimension of their global attractors, Comm. Math. Phys., 147 (1992), 217-230.

[37] V.X. Liu, An example of instability for the Navier-Stokes equations on the 2-dimensional torus, Comm. P.D.E., 17 (1992), No. 11 & 12, 1995-2012.

[38] E. Lorenz, Deterministic nonperiodic flow, J. Atmos. Sci., 20 (1963), 130-141.

[39] E. Lorenz, Maximum simplification of the dynamic equations, Tellus, 12 (1960), 243-254.

[40] B. Mandelbrot, Fractals: Form, Chance and Dimension (Freeman, San Francisco, 1977).

[41] B. Mandelbrot, The Fractal Geometry of Nature (Freeman, San Francisco, 1982).

[42] B. Mandelbrot, Fractals and turbulence: attractors and dispersion. Turbulence Seminar, Berkeley 1976/7. Lecture Notes in Mathematics 615, 83-93 (Springer, New York).

[43] P. Manneville, Lyapunov exponents for the Kuramoto-Savashinsky model, Macroscopic modeling of turbulent flows (O. Pironneau, ed.), Lecture Notes in Physics, vol. 230, (Springer-Verlag, New York, 1985), 319-326.

[44] C. Marchioro, An example of absence of turbulence for any Reynolds number, Comm. Math. Phys., 105 (1986), 99-106.

[45] W.D. McComb, The physics of fluid turbulence, (Oxford University Press, Oxford, 1990).

[46] P.E. Merilees and H. Warn, On energy and enstrophy exchanges in two-dimensional non-divergent flow, J. Fluid Mech., 69 (1975), No. 4, 625-630.

[47] L.D. Meshalkin and Ia.G. Sinai, Investigation of the stability of a stationary solution of a system of equations for the plane movement of an incompressible viscous fluid, J. Appl. Math. Mech., 25 (1961), 1700-1705.

[48] H.K. Moffatt, Advances in turbulence, eds. G. Comte-Bellot and J. Mathieu (Springer-Verlag, Berlin, 1986).

[49] S.W. Morris, Private communication.

[50] B. Nicolaenko, B. Schuerer, and R. Temam, Some global dynamical properties of a class of pattern formation equations, Commun. Partial Differential Equations, 14 (1989), 245-297.

[51] C. O'hern, D. Egolf, and H. Greenside, Lyapunov spectral analysis of a nonequilibrium Ising-like transition, Phys. Rev. E, 53 (1996), No. 4, 3374-3386.

[52] A.M. Polyakov, The theory of turbulence in two dimensions, preprint PUPT-1369 (1992).

[53] G. Raugel, Navier-Stokes equations on thin 3D domains. I: Global attractors and global regularity of solutions, J. Amer. Math. Soc., 6 (1993), 503-568.

[54] D. Ruelle, Large volume limit of the distribution of characteristic exponents in turbulence, Comm. Math. Phys., 87 (1982), 287-302.

[55] P.G. Saffman, On the spectrum and decay of random two-dimensional vorticity distributions of large Reynolds number, Stud. Appl. Math., 50 (1971), 377-383.

[56] R. Salmon, Lectures on geophysical fluid dynamics (Oxford University Press, Oxford, 1998).

[57] I. Silberman, Planetary waves in the atmosphere, J. Meteor., 11 (1954), 27-34.

[58] S. Smale, dynamics retrospective: Great problems, attempts that failed, For “Non-linear Science: The Next Decade”, Los Alamos, May 1990.

[59] P.L. Sulem and U. Frisch, Bounds on energy flux for finite energy turbulence, J. Fluid Mech., 72 (1975), 417-423.

[60] R. Temam, Infinite-dimensional dynamical systems in mechanics and physics, 2nd edition (Springer-Verlag, New York, 1997).

[61] R. Temam, Navier-Stokes equations, theory and numerical analysis, 2nd edition (North-Holland, Amsterdam, 1979).

[62] R. Temam, Navier-Stokes equations and nonlinear functional analysis, NSF/CBMS Regional Conference Series in Appl. Math. (SIAM, Philadelphia, 1983).

[63] R. Temam and M. Ziane, Navier-Stokes equations in three-dimensional thin domains with various boundary conditions, Adv. in Differential Equations, 1 (1996), 499-546

[64] H. Tennekes and J.L. Lumley, A first course in turbulence (The MIT Press, Cambridge, Massachusetts, 1974).

[65] R.T. William, Numerical simulations of steady-state fronts, J. Atmos. Sci., 31 (1974), 1286-1296.

[66] H. Zhao, Frontogenesis and mature fronts in the geostrophic theory, Ph.D. thesis, Department of Physics, University of Toronto, Toronto, ON., 1996.

[67] M. Ziane, Optimal bounds on the dimension of the attractor of the Navier-Stokes equations, Physica D, 105 (1997), 1-19.