Optimal Investment in Hedge Funds under Loss Aversion

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Abstract

We study optimal investment problems in hedge funds for a loss averse manager under the framework of cumulative prospect theory. We have solved the problems explicitly for general utility satisfying the Inada conditions and piece-wise exponential utility. Through a sensitivity analysis, we find that the manager will reduce the risk of the hedge fund when his/her loss aversion, risk aversion or ownership in the fund, or the management fee ratio increases. However, the increase of incentive fee ratio will drive the manager to seek more risk to achieve higher prospect utility.

Key Words: cumulative prospect theory; exponential utility; hedge funds; optimal investment; risk management; sensitivity analysis.

1 Introduction

In the business of hedge funds, fund managers usually charge investors a combination of two fees, known as “2 & 20”, namely, 2% management fees on the funds and 20% incentive fees on the excess of funds over a benchmark. This benchmark is related to hurdle rate, which is the minimum return that
fund managers need to beat in order to receive incentive fees. The existence of hurdle rate naturally sets up a benchmark (reference point) for hedge fund managers to separate gains from losses, and hence inspires us to apply cumulative prospect theory (CPT) to model the preference of hedge fund managers.

Kahneman and Tversky (1979) propose prospect theory to overcome some drawbacks of expected utility theory (EUT), e.g., violation of independence axiom in Allais paradox, and explain “irrational” behaviors of investors, e.g., risk seeking for gains of small probability. The first-order stochastic dominance fails under the original prospect theory, which motivates the development of CPT in Tversky and Kahneman (1992). Other alternative theories to EUT include rand-dependent theory by Quiggin (1982) and SP/A theory by Lopes (1987).

In this paper, we study optimal investment problems for a hedge fund manager whose preference is characterized by CPT. Our main objective is to obtain optimal investment strategy that maximizes the prospect utility of the manager. In a standard CPT framework, there are three key components:

- **Reference point**
  Experimental evidence shows that people do not evaluate final outcomes directly, but rather compare them with a certain reference point, which separates all outcomes into gains and losses. If we denote the final outcome by $X$ and the reference point by $\theta$, then people evaluate the difference $X - \theta$, and treat $X - \theta$ as gains if being positive, and losses otherwise.

- **Prospect utility function**
  In EUT, people’s reference is represented by a concave utility function throughout the whole domain, which implies uniform risk aversion throughout the entire region. This characterization clearly violates human behavior towards risk, see Tversky and Kahneman (1992) for counterexamples and description of fourfold pattern of risk attitudes. In addition, people are more sensitive to losses than to gains, a behavior called loss aversion in the literature. To account for those features, a S-shaped prospect utility is proposed in CPT, which is concave in the region of gains but is convex in the region of losses, with the convex piece being steeper than the concave piece. The complete definition of prospect utility is given in Assumption 2.1.
• Probability weighting function

Probability weighting function (also called distortion function) serves as a change of probability measure, which overweights small probability events and underweights large probability events. Interested readers may refer to Barberis and Huang (2008) for details on probability weighting and its implications on asset pricing. In our analysis, we exclude probability weighting in the modeling, see Remark 2.1 for discussions.

Optimal investment problems for an individual investor under CPT have received considerable attention recently. In continuous time, Jin and Zhou (2008) provide complete analysis to this problem under a standard Black-Scholes model, see also Berkelaar et al. (2004) and He and Zhou (2011b). In single-period discrete time, the problems have been studied by Bernard and Ghossoub (2010), He and Zhou (2011a), Pirvu and Schulze (2012), and others. Carassus and Rásonyi (2015) set up a multi-period model and establish the existence of optimal investment strategies under CPT. However, there are not many papers studying optimal investment problems for hedge fund managers under CPT framework. Comparing optimal investment problems under CPT for individual investors and hedge fund managers, there is a significant difference: in the former problem, there is only one step comparison between the final outcome and the reference point; while in the latter problem, there are two comparisons, with the first one between the final fund value and the fund benchmark to decide whether incentives are awarded, and the second one between the manager’s wealth and the CPT reference point to distinguish gains from losses.

In all those papers mentioned above, piece-wise power utility is assumed to obtain the optimal investment strategy in explicit forms.\textsuperscript{1} A major drawback under the assumption of piece-wise power utility is that the CPT optimization problems can be easily \textit{ill-posed}, with either optimal investment being infinite or optimal prospect utility being infinite. For analysis on well-posedness of optimal investment problems under CPT and piece-wise power utility, please consult Jin and Zhou (2008) in continuous model, He and Zhou (2011a) in single-period discrete model and Carassus and Rásonyi (2015) in

\textsuperscript{1}In He and Zhou (2011a, Section 5.2), piece-wise linear utility is also considered, but optimal investment is found under strong assumptions. Pirvu and Schulze (2012, Sections 4.2 and 4.3) include piece-wise linear utility and piece-wise exponential utility, but fail to obtain optimal investment strategy under piece-wise exponential utility.
multi-period discrete model. Another criticism of power utility is that it cannot explain high risk averse behavior. This is intrinsically embedded with power utility, since its relative risk aversion is a constant $1 - p \in (0, 1)$ for power index $p \in (0, 1)$.\footnote{In Tversky and Kahneman (1992), $p$ is estimated to be 0.88.} Köbberling and Wakker (2005) show that piecewise power utility fails to capture the loss aversion behavior near 0 or at sufficiently large values; however, such problem does not exist if exponential utility is applied. To fill the blank in the literature and also address the drawbacks of power utility, we mainly work with piece-wise exponential utility, see Sections 4 and 5.

Optimal investment problems and risk sharing in hedge funds under EUT have been studied extensively in the literature. Carpenter (2000) considers a risk averse manager whose preference is modeled by a HARA utility and studies the impact of incentive fees on the optimal investment strategy, see also Agarwal et al. (2009) and Hodder and Jackwerth (2007). Guasoni and Obloj (2013) consider high-water marks (the highest value that a hedge fund has ever reached) in the modeling and find that the optimal portfolio is a constant under power utility. Bichuch and Sturm (2014) introduce convex incentive fee schemes and obtain the optimal portfolio in a general semi-martingale market model by duality theory.

Two related papers to our work are Kouwenberg and Ziemba (2007) and He and Kou (2014). The main objective of Kouwenberg and Ziemba (2007) is to investigate the relation between incentive fees and risk taking in hedge funds. Both analytical and empirical results are obtained in Kouwenberg and Ziemba (2007), suggesting that higher incentive fees will drive loss averse managers to take more risk, but such risk taking is largely reduced if managers own more than 30% of the hedge funds. He and Kou (2014) consider the problem from the profit side (other than risk) and compare the optimal prospect utility between traditional fee scheme and first-loss fee scheme. Their results show that it is possible to improve both managers’ and investors’ optimal prospect utility, and reduce the fund risk at the same time by replacing a traditional fee scheme with a first-loss scheme. This paper differs from those two papers in several directions. In both Kouwenberg and Ziemba (2007) and He and Kou (2014), only piece-wise power utility is considered; we consider general utility satisfying the Inada conditions (including power utility) and exponential utility. Kouwenberg and Ziemba (2007) only find the optimal terminal value of the hedge fund, but do not obtain the
optimal investment strategy nor the optimal prospect utility; Section 5 in this paper provides full details. In the sensitivity analysis part (Section 6), we not only focus on risk taking in the fund but also study the manager’s prospect utility. Furthermore, we take into account the impact of more factors, including loss aversion, risk aversion, and management fee ratio, which are not considered in those two papers.

The rest of this paper is organized as follows. In Section 2, we set up the model and formulate the main optimization problem. We then obtain the optimal terminal value of the hedge fund explicitly for general utility in Section 3 and exponential utility in Section 4, respectively. Under the assumptions of exponential utility and constant benchmark, we further provide the optimal investment strategy and the optimal prospect utility of the manager in explicit forms in Section 5. We conduct sensitivity analysis in Section 6 to study the effects of various factors on the fund risk and the optimal prospect utility. Final conclusions are summarized in Section 7.

2 The Setup

2.1 The Financial Model

We consider optimal investment problems for a hedge fund manager with initial endowment $M_0 > 0$. The hedge fund attracts a total amount $I_0$ of capital from investors. The manager contributes $F_0$ out of his own wealth to the hedge fund, where $0 \leq F_0 \leq M_0$, and invests the remaining amount $e_0 (e_0 = M_0 - F_0)$ in the exogenous opportunities. Hence, the initial value of the hedge fund is $X_0 = I_0 + F_0$, and the proportion of the manager’s ownership in the hedge fund, called the manager’s managerial ownership ratio, is defined by $w = \frac{F_0}{X_0}$. Notice that $0 \leq w \leq \bar{w} := \frac{M_0}{M_0 + I_0}$. Given $X_0$ and $w$, we can easily calculate $F_0 = wX_0$ and $I_0 = (1 - w)X_0$.

The traditional fee scheme of hedge funds consists of two types of fees:

- Management fee

The hedge fund manager charges a fix proportion $\alpha \geq 0$ of the fund’s value from the investors in exchange of his/her professional management. The common choice of $\alpha$ ranges from 1% to 2% in the industry. We assume the management fee is incurred at time $T > 0$, e.g., next evaluation time. The amount of management fee is $\alpha(1 - w)X_T$, where $X_T$ denotes the fund value at time $T$.  

[5]
• Incentive fee (also called Performance fee)

The incentive fee is to award the managers for excellent performance of the hedge fund. If the hedge fund value $X_T$ at time $T$ is above the benchmark, denoted by $B_T$, then the manager receives a proportion $\beta$ of the excess value of the fund, $\beta(1 - w)(X_T - B_T)^+$. $\beta$ is set to 20% for many hedge funds.

The manager invests the hedge fund in a financial market consisting of one risk-free asset and one risky asset. The risk-free asset earns an interest $r > 0$. The price process $S$ of the risky asset is given by the following equation:

$$dS_t = S_t(\mu dt + \sigma dW_t), \ t \in [0,T], S_0 > 0,$$

where $\mu, \sigma > 0$ are the appreciation rate and the volatility of the risky asset, and $W$ is a standard Brownian Motion defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$. The financial market described above is a complete market, and hence admits a unique pricing kernel $\xi_t$, which is governed by

$$d\xi_t = -\xi_t(rd_t + \kappa dW_t), \ \xi_0 = 1, \ (1)$$

where $\kappa := \frac{\mu - r}{\sigma}$ is known as the market price of risk. Denote $\xi := \xi_T$.

An investment strategy $\{\pi_t\}_{t \in [0,T]}$, where $\pi_t$ denotes the amount invested in the risky asset at time $t$, is a progressively measurable process with respect to the given filtration and satisfies

$$E\left(\int_0^T \pi_t^2 dt\right) < +\infty.$$

Hereinafter, $E[\cdot]$ means taking expectation under probability $\mathbb{P}$. The manager chooses an investment strategy $\{\pi_t\}_{t \in [0,T]}$ for the hedge fund, and the dynamics of the fund’s value is then governed by

$$dX_t = (rX_t + (\mu - r)\pi_t)dt + \sigma\pi_tdW_t. \ (2)$$

We assume the fund is not allowed to go bankrupt during the whole period $[0,T]$, i.e., $X_t \geq 0$ for all $t \in [0,T]$. Let $\mathcal{A}(X_0)$ denote the set of all investment strategies that start with initial wealth $X_0$ and satisfy the non-bankruptcy condition.
The manager’s wealth at the evaluation time $T$ is

$$M_T = wX_T + \alpha(1 - w)X_T + \beta(1 - w)(X_T - B_T)^+ + e_T$$

$$= \begin{cases} 
  w_+ X_T - \beta(1 - w)B_T + e_T, & \text{when } X_T \geq B_T \\
  w_- X_T + e_T, & \text{when } X_T < B_T 
\end{cases}$$

where $w_+$ and $w_-$ are defined by

$$w_+ := w + (\alpha + \beta)(1 - w), \text{ and } w_- := w + \alpha(1 - w),$$

respectively, and $e_T$ denotes the value at time $T$ from exogenous investment. For instance, we may assume $e_T = e^{R_0 T} \cdot e_0$, with $R_0$ denoting the return of exogenous investment.

### 2.2 The Problem

The manager’s preference is characterized by CPT. In CPT, the manager’s final wealth $M_T$ is not evaluated directly, but rather is compared to a reference point $\theta_T$ at the evaluation time $T$. If $M_T \geq \theta_T$, $M_T - \theta_T$ is considered as gains; otherwise, $\theta_T - M_T$ is treated as losses. Unlike EUT, in which all investors are assumed to be uniformly risk averse (represented by a concave utility function), CPT assigns a S-shaped prospect utility to the difference between the final wealth and the reference point. The prospect utility is concave in the gains region, but convex in the losses region. The complete characterization of the S-shaped prospect utility is provided in the assumption below.

**Assumption 2.1.** The S-shaped prospect utility $u(\cdot)$ is given by

$$u(x) = u_+(x) \cdot 1_{x \geq 0} - u_-(x) \cdot 1_{x < 0},$$

where both $u_+(\cdot)$ and $u_-(\cdot)$ are strictly increasing and concave in $[0, \infty)$, satisfying $u_+(0) = u_-(0) = 0$, and $1_A$ is an indicator function on set $A$, i.e., $1_A(x) = 0$ if $x \in A$, and 0 otherwise.

Let $D(X_T) := M_T - \theta_T$. The manager seeks to solve the following optimal hedge fund management problem:

$$\max_{\pi \in \mathcal{A}(X_0)} E[u(D(X_T))]. \quad (P1)$$
Remark 2.1. In a standard CPT framework, the expectation in Problem (P1) is taken under a distorted probability distribution. As pointed out in He and Kou (2014), incorporating probability distortion (probability weighting) will cause time inconsistency of the optimal investment, require more complicated analytical tools (considering an equivalent optimization problem over quantiles), and increase simulation difficulties in numerical analysis. Furthermore, He and Kou (2014) also show that most results obtained from the model without probability distortion still hold when probability distortion is included. So in this paper, we do not take into account probability distortion in the modeling.

Since the market is complete, there exists a one-to-one mapping between \( \pi \in A(X_0) \) and \( X_T \in \mathcal{X} := \{ X \in \mathcal{F}_T : X \geq 0 \text{ a.s.} \text{ and } E[\xi X] = X_0 \} \). Hence Problem (P1) is equivalent to

\[
\max_{X \in \mathcal{X}} E[u(D(X))].
\]  

(P2)

Denote the optimal investment strategy to Problem (P1) by \( \pi^* \), and the associated optimal value process by \( \{ X^*_t \} \), where \( t \in [0, T] \). Let \( X^*_t \in \mathcal{X} \) be the optimal solution to Problem (P2). Due to market completeness, \( X^*_T \) coincides with \( X^* \), and \( \xi_t X^*_t = E[\xi_T X^*_T | \mathcal{F}_t] = E[\xi_T X^* | \mathcal{F}_t] \) is a Doob martingale. By martingale representation theorem, there exists a progressively measurable and square integrable process \( \psi \) such that

\[
\xi_t X^*_t = E[\xi_T X^* | \mathcal{F}_t] = X_0 + \int_0^t \psi_\sigma dW_s.
\]

Such process \( \psi \) is also unique in the almost surely sense. By Ito’s formula, we obtain

\[
\xi_t X^*_t = X_0 + \int_0^t \xi_s (\sigma \pi^*_s - \kappa) dW_s.
\]

Hence, the optimal solution to Problem (P1) is given by

\[
\pi^*_t = \frac{1}{\sigma} \left( \kappa + \frac{\psi_t}{\xi_t} \right), \quad t \in [0, T].
\]

The equivalence between Problems (P1) and (P2) is then established below.

**Theorem 2.1.** If \( \pi^* \in A(X_0) \) is the optimal solution to Problem (P1), then \( X^*_T \in \mathcal{X} \) is the optimal solution to Problem (P2). If \( X^* \) is the optimal solution to Problem (P2), then there exists a unique \( \pi^* \in A(X_0) \) with final wealth \( X^*_T = X^* \) which solves Problem (P1).
In our setting, a natural choice for the reference point is the manager’s final wealth when $X_T = B_T$, that is

$$\theta_T = (w + \alpha(1 - w))B_T + e_T = w_B + e_T.$$  \hfill (3)

We assume that the reference point $\theta_T$ is given by (3) in the sequel of this paper. Under such assumption, $D(X_T)$ can be written explicitly as

$$D(X_T) = \begin{cases} 
  w_+ \cdot (X_T - B_T), & \text{when } X_T \geq B_T \\
  w_- \cdot (X_T - B_T), & \text{when } X_T < B_T
\end{cases}.$$  \hfill (4)

Here in (4), we further see the benefits of introducing $w_+$ and $w_-$, which can be approximately understood as the manager’s proportion in the fund’s terminal value.

### 3 The Analysis for General Utility and General Benchmark

In this section, we solve Problem (P2) under general utility (as in Assumption 2.1) and general benchmark $B_T$ ($B_T \in F_T$ may be random). To this purpose, we consider a piece-wise optimization problem (for a fixed state $\omega \in \Omega$), given by

$$\max_{x \geq 0} \left[u(D(x)) - y\xi x\right],$$  \hfill (P3)

where $y > 0$ is the Lagrange multiplier from the budget constraint. Denote $x^*$ as the global nonnegative maximizer to Problem (P3).

The piece-wise feature of $D(x)$ in (4) inspires us to separate Problem (P3) into two sub-problems:

$$\max_{0 \leq x \leq B_T} \left[u(D(x)) - y\xi x\right], \quad \text{and} \quad \max_{x \geq B_T} \left[u(D(x)) - y\xi x\right].$$

When $0 \leq x \leq B_T$, $u(\cdot) = -u_-(\cdot)$, is a convex function; hence, the candidates for the maximizer are $x_1^* = 0$ or $B_T$. When $x \geq B_T$, $u(\cdot) = u_+(\cdot)$ is a concave function, and then the objective function admits a unique maximizer

$$x_2^* = B_T + \frac{1}{w_+} \cdot I_+ \left(\frac{y\xi}{w_+}\right),$$  \hfill (5)
where \( I_+(\cdot) := (u'_+)^{-1}(\cdot) \).

Define function \( f(\xi) \) by
\[
f(\xi) := [u(D(x^*_2)) - y\xi x^*_2] - [u(D(x^*_1)) - y\xi x^*_1].
\]

If we take \( x^*_1 = B_T \), then
\[
f(\xi) = u_+ \left( I_+ \left( \frac{y\xi}{w_+} \right) \right) - \frac{y\xi}{w_+} \cdot I_+ \left( \frac{y\xi}{w_+} \right) + u_-(w_- B_T) - y\xi B_T.
\] (6)
The result obtained above shows that \( f(\xi) > 0 \) when \( \xi \leq \frac{u_-(w_- B_T)}{yB_T} \). The first derivatives of \( f \) is given by
\[
f'(\xi) = \frac{y}{w_+} \cdot I_+ \left( \frac{y\xi}{w_+} \right) - y B_T < 0.
\]

If \( u_+ \) satisfies the Inada conditions, i.e., \( u'_+(0) = \infty \) and \( u'_+(\infty) = 0 \), then \( \lim_{\xi \to \infty} f(\xi) = -\infty \). Thus, there exists a unique \( \xi^* \in \left( \frac{u_-(w_- B_T)}{yB_T}, \infty \right) \) such that \( f(\xi^*) = 0 \), and the global maximizer to Problem (P3) is \( x^* = x^*_2 \cdot 1_{\xi \leq \xi^*} \), where \( x^*_2 \) is given by (5).

**Theorem 3.1.** Assume the utility function is given by Assumption 2.1 and \( u_+(\cdot) \) satisfies the Inada conditions. The optimal solution \( X^* \) to Problem (P2) is given by
\[
X^* = \left( B_T + \frac{1}{w_+} \cdot I_+ \left( \frac{y\xi}{w_+} \right) \right) \cdot 1_{\xi \leq \xi^*},
\] (7)
where \( \xi^* \) is the unique solution to \( f(\xi) = 0 \) with \( f(\xi) \) defined by (6) and \( y \) is solved through \( E[\xi X^*] = X_0 \).

Theorems 2.1 and 3.1 guarantee the existence and uniqueness of the optimal investment strategy \( \pi^* \) to Problem (P1), and \( X^*_T = X^* \) as in (7). Since
the benchmark $B_T$ for rewarding incentive fees can be random, we cannot derive $\pi^*$ in explicit forms. However, as noticed in Kouwenberg and Ziemba (2007), the manager’s terminal wealth from incentive fees is a call option, with payoff

$$\beta(1-w)(X_T^* - B_T)^+ = \frac{\beta(1-w)}{w_+} \cdot I_+ \left( \frac{y\xi}{w_+} \right) \cdot 1_{\xi \leq \xi^*},$$

which can be easily priced using Black-Scholes formula.

**Remark 3.1.** To show the existence and uniqueness of $\xi^*$ and $y$ in Theorem 3.1, we make a substitution $\zeta := y\xi$ in (6), and write

$$f_\zeta(\zeta) := u_+ \left( I_+ \left( \frac{\zeta}{w_+} \right) \right) - \frac{\zeta}{w_+} \cdot I_+ \left( \frac{\zeta}{w_+} \right) + u_-(w_- B_T) - \zeta B_T.$$ 

It is straightforward to show that there exists a unique positive $\zeta^*$ such that $f_\zeta(\zeta^*) = 0$. We then obtain a candidate solution to Problem (P2) as

$$X^*(y) = \left( B_T + \frac{1}{w_+} \cdot I_+ \left( \frac{y\xi}{w_+} \right) \right) \cdot 1_{\xi \leq \zeta^*}. $$

We can show that $E[\xi X^*(y)]$ is a continuous and strictly decreasing function in $y$. Furthermore, if $u_+(\cdot)$ satisfies the Inada conditions,

$$\lim_{y \to 0} E[\xi X^*(y)] = +\infty, \text{ and } \lim_{y \to +\infty} E[\xi X^*(y)] = 0.$$

Hence, there exists a unique positive $y$ such that $E[\xi X^*(y)] = X_0$, and $\xi^* = \frac{\zeta^*}{y}$ is the unique solution to $f(\xi) = 0$.

**Remark 3.2.** If $u_+ = x^p$, with $0 < p < 1$, then the Inada conditions hold. Piece-wise power utility function was proposed in the original prospect theory (see Kahneman and Tversky (1979)) and cumulative prospect theory (see Tversky and Kahneman (1992)), and has been used extensively in the literature, e.g., Kouwenberg and Ziemba (2007) and He and Kou (2014) on optimal hedge fund investment problems. However, some common utility functions do not satisfy the Inada conditions, such as exponential utility, linear utility and quadratic utility. Consider $u_+(x) = 1 - e^{-\eta x}$ for some $\eta > 0$, then $u'_+(0) = \eta$ and $u'_+(\infty) = 0$. 

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4 The Analysis for Exponential Utility and General Benchmark

As pointed out in Remark 3.2, if exponential utility is considered, then the Inada conditions fail to hold, and so does Theorem 3.1. In this section, we study Problem (P2) under the assumption of exponential utility, but do not impose any assumptions on the benchmark $B_T$.

**Assumption 4.1.** The piece-wise exponential utility function is of the form
\[ u(x) = u_+(x) \cdot 1_{x \geq 0} - u_-(x) \cdot 1_{x < 0}, \]
with $u_+ (\cdot)$ and $u_-(\cdot)$ given by
\[ u_+(x) = 1 - e^{-\eta x}, \quad \text{and} \quad u_-(x) = k(1 - e^{-\eta x}), \quad x \in [0, \infty), \]
where $\eta > 0$ and $k > 1$.

The condition $k > 1$ ensures $u_-(x) > u_+(x)$ for all $x \geq 0$, and then the validness of loss aversion. Constant $k$ is called the degree of loss aversion.

It is easy to check that the piece-wise exponential utility specified in Assumption 4.1 satisfy all the conditions that are required for a prospect utility function in Assumption 2.1. Under Assumption 4.1, we obtain
\[ u_+(x) = \eta \cdot e^{-\eta x}, \quad \text{and} \quad I_+(x) = -\frac{1}{\eta} \ln \left( \frac{x}{\eta} \right), \]
where the domain of $I_+$ is $D_+ = (0, \eta]$.

By following the same methodology used in Section 3, we seek the optimal solution to Problem (P3) by dividing the feasible domain into two parts. Since $u(D(x)) = -u_-(D(x))$ is still a convex function when $x < B_T$, the candidates for the maximizer are $x_1^* = 0$ or $B_T$. Denote the value of the objective function under $x_1^*$ by $V_1(\xi)$, i.e.,
\[ V_1(\xi) = -k \left( 1 - e^{-\eta w_-(B_T - x_1^*)} \right) - y\xi x_1^*. \]

We next study Problem (P3) when $x \geq B_T$, which can be written as
\[ V_2(\xi) := \max_{x \geq B_T} \left[ 1 - e^{-\eta w_+(x - B_T)} - y\xi x \right]. \]

Let $x_2^*$ be the maximizer at which $V_2(\xi)$ is achieved.

If $\xi \leq \frac{w_+}{y} := \xi_1$, the first-order condition admits a unique solution
\[ x_2^* = B_T - \frac{1}{\eta w_+} \ln \left( \frac{\xi}{\xi_1} \right). \quad (8) \]
If $\xi > \xi_1$, the maximizer is $x_2^* = B_T$ since the objective function is a strictly decreasing function of $x$.

To find the optimal solution to Problem (P3), it remains to compare $V_1(\xi)$ with $V_2(\xi)$. By definition, $V_2(\xi) \geq V_1(\xi)$ when $x_1^* = B_T$. Hence, in what follows, we take $x_1^* = 0$, which immediately implies

$$V_1(\xi) = -k \left(1 - e^{-\eta w - B_T}\right).$$

- $\xi > \xi_1$

  In this scenario, $x_2^* = B_T$ and $V_2(\xi) = -y \xi B_T$. The unique solution $\xi_2$ to $V_1(\xi) = V_2(\xi)$ is obtained as

  $$\xi_2 = \frac{k \left(1 - e^{-\eta w - B_T}\right)}{y B_T}.$$  

  Then the optimal solution $x^*$ to Problem (P3) is given by

  $$x^* = \begin{cases} 0, & \text{if } \xi > \max\{\xi_1, \xi_2\} \\ B_T, & \text{if } \xi_1 < \xi \leq \xi_2 \end{cases}.$$  

  In addition, we calculate

  $$V_1(\xi) - V_2(\xi) = -k \left(1 - e^{-\eta w - B_T}\right) + y \xi B_T$$
  $$> -k \left(1 - e^{-\eta w - B_T}\right) + \eta w B_T$$
  $$> (w_+ - k w_-) \eta B_T.$$  

  Hence, if $1 < k \leq \frac{w_+}{w_-} = \frac{w+(\alpha+\beta)(1-w)}{w+\alpha(1-w)}$, we always have $x^* = 0$.

- $\xi \leq \xi_1$

  Define $\bar{f}(\xi) := V_2(\xi) - V_1(\xi)$, then

  $$\bar{f}(\xi) = 1 - \frac{\xi}{\xi_1} + \frac{\xi}{\xi_1} \ln \left(\frac{\xi}{\xi_1}\right) - y \xi B_T + k \left(1 - e^{-\eta w - B_T}\right).$$ (9)

  Apparently, $\bar{f}(\xi) > 0$ for all $\xi \leq \xi_2$. $\bar{f}(\xi)$ is a strictly decreasing function in $(0, \xi_1]$ since

  $$\bar{f}'(\xi) = \frac{1}{\xi_1} \ln \left(\frac{\xi}{\xi_1}\right) - y B_T < 0$$ for all $\xi \in (0, \xi_1]$.  

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In addition, we have
\[ \bar{f}(\xi_1) = -\eta w B_T + k \left( 1 - e^{-\eta w - B_T} \right) = y B_T (\xi_2 - \xi_1). \]

If \( \bar{f}(\xi_1) > 0 \) (or equivalently \( \xi_2 > \xi_1 \)), then \( V_2(\xi) > V_1(\xi) \) for all \( \xi \leq \xi_1 \), and thus \( x^* = x_2^* \). If \( \bar{f}(\xi_1) \leq 0 \) (or equivalently \( \xi_2 \leq \xi_1 \)), there exists a unique \( \bar{\xi}^* \in (\xi_2, \xi_1] \) such that \( \bar{f}(\bar{\xi}^*) = 0 \), and \( x^* = x_2^* \cdot 1_{\xi \leq \bar{\xi}^*} \).

The following theorem is an immediate consequence of the above analysis, and provides the optimal solution to Problem (P2).

**Theorem 4.1.** If the utility function is given by Assumption 4.1, the optimal solution \( X^* \) to Problem (P2) is obtained as
\[
X^* = \begin{cases} 
  x_2^* \cdot 1_{\xi \leq \xi_1} + B_T \cdot 1_{\xi_1 < \xi \leq \xi_2}, & \text{if } \xi_1 < \xi_2 \\
  x_2^* \cdot 1_{\xi \leq \bar{\xi}^*}, & \text{if } \xi_1 \geq \xi_2,
\end{cases}
\]
where \( x_2^* \) is given by (8).

Let \( \bar{\xi}^* := \xi_1 \) if \( \bar{f}(\xi_1) > 0 \) (equivalently, \( \xi_1 < \xi_2 \)) and define two sets by \( A_1 := \{ \omega : \xi_1 < \xi(\omega) \leq \xi_2(\omega) \} \), and \( A_2 := \{ \omega : \xi(\omega) \leq \bar{\xi}^*(\omega) \} \). Then the above optimal solution \( X^* \) to Problem (P2) can be rewritten as
\[
X^* = B_T \cdot 1_{A_1} + \left[ B_T - \frac{1}{\eta w_+} \ln \left( \frac{y \xi}{\eta w_+} \right) \right] \cdot 1_{A_2},
\]
where constant \( y \) is such that \( E[\xi X^*] = X_0 \).

**Remark 4.1.** The optimal terminal value of the hedge fund is a digital option in Section 3 under general utility (see Theorem 3.1), but is a combination of two digital options under exponential utility (see Theorem 4.1).

## 5 The Analysis for Exponential Utility and Constant Benchmark

In this section, we still work under exponential utility as in Assumption 4.1, but further assume that the benchmark \( B_T \) is a constant. Our main objective is to find the optimal solution to Problem (P1) explicitly. The non-randomness assumption on \( B_T \) is reasonable and also common in the
industries, since the hurdle rate is usually preset to be a constant $r_B$. Thus, a typical choice for such $B_T$ is $e^{r_B T} X_0$, for instance, He and Kou (2014) take $B_T = X_0$ (or equivalently $r_B = 0$).

The non-randomness of $B_T$ directly implies that $\xi_2$ is also a constant, and so is $\bar{f}(\xi_1)$. The result in Section 4 shows that

$$\bar{f}(\xi_1) > 0 \Leftrightarrow \xi_1 < \xi_2.$$ 

Therefore, we only have two disjoint cases to analyze, namely, (i) $\bar{f}(\xi_1) \leq 0$ and $\xi_1 \geq \xi_2$ and (ii) $\bar{f}(\xi_1) > 0$ and $\xi_1 < \xi_2$.

### 5.1 The Case of $\xi_1 \geq \xi_2$

In this case, there exists a unique $\bar{\xi}^* \in (\xi_2, \xi_1]$ such that $\bar{f}(\bar{\xi}^*) = 0$ (notice such $\bar{\xi}^*$ is a constant). By Theorem 4.1, the optimal terminal value of the hedge fund reads as

$$X^*_T = \left[ B_T - \frac{1}{\eta w_+} \ln \left( \frac{y \xi_T}{\eta w_+} \right) \right] \cdot 1_{\xi_T \leq \bar{\xi}^*}.$$

**Theorem 5.1.** If the utility $u$ is given by Assumption 4.1, $B_T$ is a constant, and $\xi_1 \geq \xi_2$, then the optimal value process of the hedge fund is

$$X^*_t = e^{-r(T-t)} \left\{ \left[ B_T + \frac{\kappa \sqrt{T-t}}{\eta w_+} d_{1,t}(\xi_1) \right] \cdot N(d_{1,t}(\bar{\xi}^*)) + \frac{\kappa \sqrt{T-t}}{\eta w_+} \cdot N'(d_{1,t}(\bar{\xi}^*)) \right\},$$

and the optimal investment strategy $\pi^*_t$ to Problem (P1) is

$$\pi^*_t = e^{-r(T-t)} \frac{\kappa}{\sigma} \left[ \left( \frac{B_T}{\kappa \sqrt{T-t}} + \frac{1}{\eta w_+} \ln \left( \frac{\xi_t}{\bar{\xi}^*} \right) \right) \cdot N'(d_{1,t}(\bar{\xi}^*)) + \frac{1}{\eta w_+} \cdot N(d_{1,t}(\bar{\xi}^*)) \right],$$

where $N(\cdot)$ and $N'(\cdot)$ denote the cumulative distribution function and the probability density function of a standard normal distribution, respectively,
and \( d_{1,t}(x) \) is defined by

\[
d_{1,t}(x) := \frac{\ln \left( \frac{x}{\xi_t} \right) + \left( r - \frac{1}{2} \kappa^2 \right) (T - t)}{\kappa \sqrt{T - t}}. \tag{13}
\]

The optimal prospect utility of the manager, \( U^* = E[u(D(X^*_T))] \), is given by

\[
U^* = N(d_{2,0}(\bar{\xi}^*)) - k \left( 1 - e^{-\eta w - B_T} \right) N(-d_{2,0}(\bar{\xi}^*)) - \frac{e^{-r T}}{\xi_1} N(d_{1,0}(\bar{\xi}^*)), \tag{14}
\]

where \( d_{2,t}(x) = d_{1,t}(x) + \kappa \sqrt{T - t} \).

Recall \( \xi_1 = \frac{\eta w}{y} \) and \( \bar{\xi}^* \) is the unique solution to \( \bar{f}(\xi) = 0 \), where \( \bar{f} \), defined by (9), depends on \( y \). So both \( \xi_1 = \xi_1(y) \) and \( \bar{\xi}^* = \bar{\xi}^*(y) \) are dependent on \( y \). It is trivial to verify that both mappings are injective (one-to-one) and decreasing. Due to the budget constraint, positive constant \( y \) is such that

\[
E[\xi_2 X^*_T] = X^*_0 = X_0, \text{ or equivalently,}
\]

\[
\left[ B_T + \frac{\kappa \sqrt{T}}{\eta w^+} d_{1,0}(\xi_1) \right] \cdot N(d_{1,0}(\bar{\xi}^*)) + \frac{\kappa \sqrt{T}}{\eta w^+} \cdot N'(d_{1,0}(\bar{\xi}^*)) = e^r T X_0. \tag{15}
\]

Hence, Theorem 5.1 is complete as long as the existence and uniqueness of such \( y \) is validated.

**Proposition 5.1.** If \( \xi_1 \geq \xi_2 \), there exist unique positive \( \bar{\xi}^* \) and \( y \) such that \( \bar{f}(\xi) = 0 \) for the chosen \( y \) and \( X^*_0 = X_0 \).

### 5.2 The Case of \( \xi_1 < \xi_2 \)

In this case, we find the optimal terminal value of the hedge fund \( X^*_T \) is given by

\[
X^*_T = B_T \cdot \mathbf{1}_{\xi_T \leq \xi_2} - \frac{1}{\eta w^+} \ln \left( \frac{y \xi_T}{\eta w^+} \right) \cdot \mathbf{1}_{\xi_T \leq \xi_1}.
\]

**Theorem 5.2.** If the utility \( u \) is given by Assumption 4.1, \( B_T \) is a constant, and \( \xi_1 < \xi_2 \), then the optimal value process of the hedge fund is

\[
X^*_t = e^{-r(T-t)} \left[ B_T \cdot N(d_{1,t}(\xi_2)) + \frac{\kappa \sqrt{T - t}}{\eta w^+} d_{1,t}(\xi_1) \cdot N(d_{1,t}(\xi_1)) + \frac{\kappa \sqrt{T - t}}{\eta w^+} \cdot N'(d_{1,t}(\xi_1)) \right], \tag{16}
\]
and the optimal investment strategy $\pi_t^*$ to Problem (P1) is
\[
\pi_t^* = e^{-r(T-t)} \frac{\kappa}{\sigma} \left[ \frac{B_T}{\kappa \sqrt{T-t}} \cdot N'(d_{1,t}(\xi_2)) + \frac{1}{\eta w_+} \cdot N(d_{1,t}(\xi_1)) \right].
\] (17)

The optimal prospect utility of the manager $U^*$ is given by
\[
U^* = N\left(d_{2,0}(\xi_1)\right) - k \left(1 - e^{-\eta w_- B_T}\right) N\left(-d_{2,0}(\xi_2)\right) - \frac{e^{-rT}}{\xi_1} N\left(-d_{1,0}(\xi_1)\right).
\] (18)

We observe that all the optimal values in Theorem 5.2 are semi-explicit, depending on $y$ (to be precise, both $\xi_1$ and $\xi_2$ depend on $y$). We select $y$ such that the budget constraint is satisfied. The following proposition claims the existence and uniqueness of such $y$.

**Proposition 5.2.** If $\xi_1 < \xi_2$, there exists a unique positive $y$ such that $X_0^* = X_0$, where $X_0^*$ is given by (16) with $t = 0$. That means the following equation admits a unique positive solution
\[
B_T \cdot N\left(d_{1,0}(\xi_2)\right) + \frac{\kappa \sqrt{T}}{\eta w_+} d_{1,0}(\xi_1) \cdot N\left(d_{1,0}(\xi_1)\right) + \frac{\kappa \sqrt{T}}{\eta w_+} \cdot N'\left(d_{1,0}(\xi_1)\right) = e^{rT} X_0.
\] (19)

### 6 Sensitivity Analysis

In this section, we conduct sensitivity analysis to study the impact of various factors on the risk of the hedge fund and the optimal prospect utility of the manager based on the results obtained in Section 5 (under the assumptions of piece-wise exponential utility and constant benchmark).

#### 6.1 Analysis on Risk

According to Theorem 5.1 and Theorem 5.2, the optimal terminal value $X_T^*$ of the hedge fund is either no less than $B_T$ or 0. So the size of losses for the manager is the same, $-w_+ B_T$, in both cases. Denote $p_t$ as the loss probability of the manager at the evaluation time $T$. In other words, $p_t = \mathbb{P}(X_T^* < B_T)$. The “break even” event, $\{X_T^* = B_T\}$, is not included in the event of losses. We obtain
\[
\begin{cases}
    p_t = N\left(-d_{2,0}(\xi_1)\right) & \text{when } \xi_1 \geq \xi_2 \quad \text{(Case 5.1)}; \\
    p_t = N\left(-d_{2,0}(\xi_2)\right) & \text{when } \xi_1 < \xi_2 \quad \text{(Case 5.2)}. \\
\end{cases}
\]
The risk of the hedge fund is fully characterized by the size of losses $-w \cdot B_T$ and the probability of losses $p_t$. We refer the threshold of the pricing kernel to $\xi^*$ when $\xi_1 \geq \xi_2$ and $\xi_2$ when $\xi_1 < \xi_2$; this threshold uniquely determines the loss probability $p_t$. In this subsection, we study how the risk of the hedge fund is affected by various factors.

In Assumption 4.1, we require $k > 1$ to capture the loss aversion behavior. Apparently, loss aversion does not affect the size of losses. The impact of loss aversion on the probability of losses is summarized below.

**Proposition 6.1.** Under the assumptions of piece-wise exponential utility and constant $B_T$, the Lagrange multiplier $y$ and the threshold of the pricing kernel ($\xi^*$ in Case 5.1 and $\xi_2$ in Case 5.2) are strictly increasing functions of loss aversion $k$. The loss probability of the manager is a strictly decreasing function of loss aversion $k$. Hence, a more loss averse manager will reduce the fund risk.

When economy is bad, i.e., $\xi_T > \xi^*$ when $\xi_1 \geq \xi_2$ or $\xi_T > \xi_2$ when $\xi_1 < \xi_2$, the optimal terminal value $X^*_T$ is 0; when economy is good, the optimal terminal value $X^*_T$ is a decreasing function of $y$. Hence, a more loss averse manager (equipped with large $k$) reduces the risk in the hedge fund through twofold activities: (1) reducing the loss probability $p_t$; (2) seeking less payoff in good economy. Our findings on the impact of loss aversion on risk taking here are consistent with the results obtained via piece-wise power utility, see, for instance, Proposition 4 in Berkelaar et al. (2004).

To further illustrate the results obtained in Proposition 6.1, we carry out numerical calculations. We set up a standard numerical model with parameters chosen as follows:

- **Financial Market Parameters**
  $r = 5\%, \mu = 10\%, \sigma = 25\%$;

- **Hedge Fund Related Parameters**
  $\alpha = 2\%, \beta = 20\%, T = 1, B_T = e^{rT}X_0$;

- **Manager Related Parameters**
  $X_0 = 1, w = 7.1\%, \eta = 5, k = 2.25$.

In Agarwal et al. (2009), the average of $w$ from 1994 to 2002 is estimated as 7.1%. Bliss and Panigirtzoglou (2004) use bootstrap tests to estimate the relative risk aversion.
Each time when we study the impact of a certain parameter (loss aversion \( k \) in this case) on the fund risk, we treat this parameter as a random variable and keep the other parameters as given by the standard numerical model. We calculate the optimal fund value \( X^*_T \) at time \( T \) as a function of pricing kernel \( \xi_T \) when \( k = 1.01, 1.5, 2.25, \) and \( 3 \); the results are drawn in Figure 1. The twofold impact of loss aversion on the fund risk is clearly shown in Figure 1: (1) greater \( k \) will shift the threshold point (the minimal \( \xi_T \) such that \( X^*_T = 0 \) on the graph) towards the right hand side, and hence lead to a smaller loss probability \( p_l \); (2) greater \( k \) will also drive the manager to seek less payoff in good economy.

![Figure 1: Impact of \( k \) on risk](image)

Different from optimal investment problems for an individual, optimal investment problems for a hedge fund manager involve three important parameters: management fee ratio \( \alpha \), incentive fee ratio \( \beta \) and managerial ownership ratio \( w \). Hereinafter we call them *contractual parameters*. Recall that the size of loss is \(-w \cdot B_T = -(w + \alpha(1 - w))B_T\), the following observations are immediate.

(RRA) as 5.11 (mean) using the FTSE 100 Index; with \( X_0 = 1, \eta = 5 \) is a good estimate (Recall RRA = \( \eta \cdot X_T \) under exponential utility). Loss aversion parameter \( k \) is estimated as 2.25 in Tversky and Kahneman (1992).
Proposition 6.2. Under the assumption of constant $B_T$, the size of losses for the manager is a strictly increasing function of the management fee ratio $\alpha$ and the managerial ownership ratio $w$, but is independent of incentive fee ratio $\beta$.

The impact of these contractual parameters on the loss probability $p_t$ is more complicated, and full analytical results are not available. We first carry out a numerical example to investigate how the managerial ownership ratio $w$ affects the risk of the hedge fund. We consider $w = 0\%, 10\%, 20\%$, and $30\%$ under the standard numerical model. We draw the optimal weight in the stock $\pi_t^*$ as a function of the optimal fund value $X_t^*$ at time $t = 0.2$ (left panel) and $t = 0.8$ (right panel) in Figure 2. In both graphs, we observe that the increase of $w$, i.e., the manager increases his/her ownership ratio in the hedge fund, will lead to the decrease of the optimal weight invested in the stock, and then the reduce of risk in the hedge fund. The discounted benchmark $B_t := e^{-r(T-t)}B_T = e^{rt}X_0$ is 1.01 when $t = 0.2$, and 1.04 when $t = 0.8$. We notice that the optimal weight in the stock increases dramatically when $X_t^*$ is below the discounted benchmark (corresponding to low probability of having gains at time $T$), but is almost immune to the change of $X_t^*$ when $X_t^*$ is greater than 2.5 (corresponding to high probability of having gains at time $T$). This finding confirms the risk attitudes found in Tversky and Kahneman (1992): risk seeking for gains of low probability and risk aversion for gains of high probability. Despite the similarity in shape for both graphs in Figure 2, their magnitudes differ significantly when $X_T^*$ is small enough (e.g., around 0.5). This result shows that as $t$ approaches the evaluation time $T$, the manager is more eager to get out of the losing situation and will seek very risky investment strategies to achieve so. In our example, the risk of the hedge fund is largely reduced when the manager’s managerial ownership ratio $w$ is greater than 10%.

We next focus on the impact of the incentive fee ratio $\beta$ on the fund risk. In the standard numerical model with $\beta$ being a variable, the condition $\xi_1 < \xi_2$ is equivalent to $\beta < 7.66\%$. In the following numerical example, we separate into two cases, and pick: (1) $\beta = 8\%, 10\%$, and $20\%$ when $\xi_1 \geq \xi_2$; (2) $\beta = 1\%, 3\%, 5\%$, and $7\%$ when $\xi_1 < \xi_2$. We draw the graph for the optimal fund value $X_T^*$ at terminal time as a function of the pricing kernel $\xi_T$ in Figure 3 and Figure 4, respectively. In both figures, the increase of $\beta$ will shift the threshold of pricing kernel to the left, leading to a greater loss probability $p_t$. In addition, we observe in Figure 4 that the manager will seek
higher payoff in good economy with the increase of $\beta$ when $\xi_1 < \xi_2$. However, this finding does not hold in general when $\xi_1 \geq \xi_2$. The graphs in Figures 3 and 4 also show that impact of $\beta$ on the fund risk is more significant when $\beta$ is relatively small (in the case of $\xi_1 < \xi_2$).

The impact of $\alpha$ on the fund risk is examined under the standard numerical model at four different levels $\alpha = 0\%, 1\%, 2\%, \text{and } 3\%$. The graphs in Figure 5 show that the management fee ratio $\alpha$ has the exactly opposite effect on the loss probability $p_l$ as the incentive fee ratio, i.e., the increase of $\alpha$ will result in a smaller loss probability $p_l$. This is because at the evaluation time $T$ the management fee is guaranteed (gains of certainty) while the incentive fee is contingent (gains of uncertainty). We also observe that the impact of $\alpha$ on the fund risk is not significant comparing with those of $\beta$.

One possible explanation is that the increase of $\alpha$ will lead to the increase of $w_+$ and $w_-$ simultaneously (the increase of both gains and losses) while the increase of $\beta$ will only result in greater gains but same losses.

**Remark 6.1.** Under the assumptions of piece-wise exponential utility and constant $B_T$, if $\xi_1 \geq \xi_2$, the produce $y \cdot \xi^*$ is a strictly increasing function of all contractual parameters; if $\xi_1 < \xi_2$, the produce $y \cdot \xi_2$ is a strictly increasing function of $\alpha$ and $w$. 

Figure 2: Impact of $w$ on risk
In the classical expected utility theory, risk aversion plays an important role in related portfolio selection problems. Under the assumption of piecewise exponential utility, the absolute risk aversion is a constant, $\eta$, for both
The optimal prospect utility $U^*$ of the manager is given by (14) when $\xi_1 \geq \xi_2$ and (18) when $\xi_1 < \xi_2$. In this subsection, we are interested in the effect of loss aversion and contractual parameters on $U^*$.

We still conduct numerical analysis under the standard numerical model introduced in the previous subsection. First, we consider loss aversion $k$ as a variable taking values in $(1, 4)$, and draw the graph of $U^*$ correspondingly in Figure 7. We find that the optimal prospect utility $U^*$ is a strictly decreasing
function of loss aversion $k$. This result is due to the definition of prospect utility; the higher the loss aversion $k$ the bigger the punishment from the losses.

Figure 6: Impact of $\eta$ on risk

Figure 7: Impact of $k$ on the optimal prospect utility $U^*$

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Next we study the impact of the incentive fee ratio $\beta$ on $U^*$. In the numerical example, we consider $\beta \in (0\%, 30\%)$, and draw the graph of $U^*$ in Figure 8. The graph clearly shows that the increase of the incentive fee ratio $\beta$ will make the manager better off. This observation is expected, since the higher the $\beta$ the bigger the gains.

The management fee ratio $\alpha$ and the managerial ownership ratio $w$ pose the same effect on the optimal prospect utility $U^*$ as loss aversion $k$, i.e., the increase of $\alpha$ or $w$ is accompanied by the decrease of $U^*$.

![Figure 8: Impact of $\beta$ on the optimal prospect utility $U^*$](image)

**7 Conclusions**

We consider optimal investment problems in hedge funds for a loss averse manager under the framework of cumulative prospect theory. The financial market is assumed to follow the classical Black-Scholes model (a complete market model); the complete market assumption allows us to optimize over the set of all attainable terminal value of the hedge fund instead of all admissible investment strategies. We have solved the problem explicitly and

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4To shorten the pages of this page, I do not include the graphs here. Simulation results can be provided upon request.
have obtained the optimal terminal value of the hedge fund under two types of utility functions: (1) piece-wise prospect utility with the gain part satisfying the Inada conditions, and (2) piece-wise exponential utility. When the prospect utility is given by piece-wise exponential utility and the benchmark of the hedge fund $B_T$ is a constant, we further provide the optimal value process of the hedge fund and the corresponding optimal investment strategy, and obtain the optimal prospect utility of the manager, all in explicit forms.

A thorough sensitivity analysis is conducted to study the impact of loss aversion, contractual parameters and risk aversion on the risk of the hedge fund and the optimal prospect utility of the manager.

From the sensitivity analysis, we find that the fund risk will be reduced if the manager’s loss aversion $k$, managerial ownership ratio $w$ or risk aversion $\eta$, or the management fee ratio $\alpha$ increases. The risk of the hedge fund is largely reduced when the manager owns a significant proportion of the fund (10% in our numerical example). But the increase of the incentive fee ratio $\beta$ will drive the manager to take more risky investment strategies, and hence amplify the risk of the hedge fund. We also observe that the incentive fee ratio $\beta$ imposes a stronger effect on the fund risk than the management fee ratio $\alpha$. Those factors influence the optimal prospect utility $U^*$ of the manager in the same direction as on the risk of the hedge fund.

Appendix

We provide the proof for Theorem 5.1 below, but will leave the proof for Theorems 5.2 to readers, which can be done by following the same vein.

Proof for Theorem 5.1. From the SDE (1) of the pricing kernel $\xi_t$, we obtain

$$\xi_T = \xi_t \cdot \exp \left( - \left( r + \frac{1}{2} \kappa^2 \right) (T - t) - \kappa (W_T - W_t) \right),$$

and thus

$$\xi_T \leq x \iff Z_t := \frac{W_T - W_t}{\sqrt{T - t}} \geq - \frac{\ln \left( \frac{x}{\xi_t} \right) + \left( r + \frac{1}{2} \kappa^2 \right) (T - t)}{\kappa \sqrt{T - t}} = -d_{1,t}(x) - \kappa \sqrt{T - t} = -d_{2,t}(x),$$

and
where $d_{1,t}$ is given by (13), and $Z_t$ follows a standard normal distribution and is independent of $F_t$. According to the deduction of Theorem 2.1, we have

$$X_t^* = \frac{1}{\xi_t} E [\xi_T X_T^* | F_t]$$

$$= e^{-(r + \frac{1}{2}\kappa^2)(T-t)} \left[ B_T + \left( r + \frac{1}{2}\kappa^2 \right) \frac{T-t}{\eta w_+} + \frac{1}{\eta w_+} \ln \left( \frac{\xi_1}{\xi_t} \right) \right]$$

$$\cdot \int_{-d_{1,t}(\xi^*)-\kappa\sqrt{T-t}}^{\infty} e^{-\kappa \sqrt{T-t} z} dN(z)$$

$$+ e^{-(r + \frac{1}{2}\kappa^2)(T-t)} \frac{\kappa \sqrt{T-t}}{\eta w_+} \int_{-d_{1,t}(\xi^*)-\kappa\sqrt{T-t}}^{\infty} z e^{-\kappa \sqrt{T-t} z} dN(z)$$

$$= e^{-r(T-t)} \left[ B_T + \left( r + \frac{1}{2}\kappa^2 \right) \frac{T-t}{\eta w_+} + \frac{1}{\eta w_+} \ln \left( \frac{\xi_1}{\xi_t} \right) \right] \cdot N(d_{1,t}(\tilde{\xi}^*))$$

$$+ e^{-r(T-t)} \frac{\kappa \sqrt{T-t}}{\eta w_+} \left[ \int_{-d_{1,t}(\xi^*)}^{\infty} \frac{1}{\sqrt{2\pi}} \left( \tilde{z} - \kappa \sqrt{T-t} \right) e^{-\frac{1}{2} \tilde{z}^2} d\tilde{z} \right]$$

$$= e^{-r(T-t)} \left\{ \left[ B_T + \left( r + \frac{1}{2}\kappa^2 \right) \frac{T-t}{\eta w_+} + \frac{1}{\eta w_+} \ln \left( \frac{\xi_1}{\xi_t} \right) \right] \cdot N(d_{1,t}(\tilde{\xi}^*)) \right\}$$

$$+ \frac{\kappa \sqrt{T-t}}{\eta w_+} \cdot N'(d_{1,t}(\tilde{\xi}^*)),$$

which can be simplified as (11) by using the definition of $d_{1,t}(x)$.

The optimal value process $X_t^*$ derived above can be written as $X_t^* = g(t, \xi_t)$ for some function $g$. By applying Ito’s formula to $g(t, \xi_t)$ and comparing with SDE (2), we obtain

$$\pi_t^* = -\frac{\kappa g(t, \xi_t)}{\sigma \frac{\partial g}{\partial \xi_t}} \xi_t,$$

and eventually the expression in (12) after simplifications.

In this case, we have

$$D(X_T^*) = \begin{cases} \frac{1}{\eta} \ln \left( \frac{\xi_T}{\xi_T} \right), & \text{when } \xi_T \leq \tilde{\xi}^* \\ -w_- B_T, & \text{when } \xi_T > \tilde{\xi}^* \end{cases}$$
We then proceed to calculate the optimal utility of the manager as follows:

\[ U^* = E[u(D(X_T^*))] = E \left[ \left( 1 - \frac{\xi_T}{\xi_1} \right) \cdot 1_{\xi_T \leq \xi^*} - k \left( 1 - e^{-\eta w_{-B_T}} \right) \cdot 1_{\xi_T > \xi^*} \right] \]

\[ = P(\xi_T \leq \xi^*) - k \left( 1 - e^{-\eta w_{-B_T}} \right) P(\xi_T > \xi^* - \frac{1}{\xi_1} E \left[ \xi_T \cdot 1_{\xi_T \leq \xi^*} \right] \]

\[ = P(Z_0 \geq -d_{2.0}(\xi^*)) - k \left( 1 - e^{-\eta w_{-B_T}} \right) P(Z_0 < -d_{2.0}(\xi^*)) \]

Recall \( d_{2,t}(x) = d_{1,t}(x) + \kappa \sqrt{T-t} \), where \( x \in \mathbb{R}, t \in [0, T] \).

**Proof for Proposition 5.1.** Introduce \( \zeta := y\xi \) and rewrite \( \bar{f} \) defined in (9) as a new function of \( \zeta \) as follows:

\[ \bar{f}_\zeta(\zeta) = 1 - \frac{\zeta}{\eta w_+} + \frac{\zeta}{\eta w_+} \ln \left( \frac{\zeta}{\eta w_+} \right) - B_T \zeta + k(1 - e^{-\eta w_{-B_T}}) \cdot \text{where } \zeta \leq \eta w_+. \]

It is trivial to show that there exists a unique solution \( \zeta^* \in (0, \eta w_+] \) to \( \bar{f}_\zeta(\zeta) = 0 \). For any given positive \( y \), define \( \xi^* = \frac{\zeta^*}{y} \). Then such \( \xi^* \) is the unique solution to \( \tilde{f}(\xi) = 0 \) for the chosen \( y \). The optimal value of the hedge fund at time 0 can be obtained as

\[ X^*_0(y) = e^{-rT} \left\{ B_T + \frac{\kappa \sqrt{T}}{\eta w_+} d_{1,0} \left( \frac{\eta w_+}{y} \right) \cdot N \left( d_{1,0} \left( \frac{\zeta^*}{y} \right) \right) \right\} \]

Straightforward calculus gives

\[ (X_0^*)'(y) = -\frac{e^{-rT}}{y \eta w_+ \kappa \sqrt{T}} \left[ \left( \eta w_+ B_T + \ln \left( \frac{\eta w_+}{\zeta^*} \right) \right) \cdot N' \left( d_{1,0} \left( \frac{\zeta^*}{y} \right) \right) + \kappa \sqrt{T} \cdot N \left( d_{1,0} \left( \frac{\zeta^*}{y} \right) \right) \right], \]
which is strictly negative due to $\zeta^* \leq \eta w_+$. In addition,
\[
\lim_{y \to 0} X^*_0(y) = +\infty, \text{ and } \lim_{y \to +\infty} X^*_0(y) = -\infty.
\]
Therefore, there exists a unique \( y \) such that \( X^*_0(y) = X_0 \).

\textbf{Proof for Proposition 5.2.} Define
\[
h(y) := B_T \cdot N \left( d_{1,0} \left( \frac{k(1 - e^{-\eta w - B_T y})}{y B_T} \right) \right) + \frac{\kappa \sqrt{T}}{\eta w_+} \cdot N' \left( d_{1,0} \left( \frac{\eta w_+}{y} \right) \right)
+ \frac{\kappa \sqrt{T}}{\eta w_+} \cdot d_{1,0} \left( \frac{\eta w_+}{y} \right) \cdot N \left( d_{1,0} \left( \frac{\eta w_+}{y} \right) \right), \quad y > 0,
\]
and then Equation (19) becomes \( h(y) = e^{rT} X_0 \). Furthermore,
\[
\lim_{y \to 0} h(y) = +\infty, \quad \lim_{y \to +\infty} h(y) = 0, \quad \text{and} \quad h'(y) < 0.
\]
Hence, Equation (19) has a unique positive solution. \hfill \Box

\textbf{Proof for Proposition 6.1.} We break the proof into two disjoint cases.

- $\xi_1 \geq \xi_2$

  To emphasize the dependence on \( k \), we rewrite the function \( \tilde{f}(\xi) \), defined in (9), as \( \tilde{f}(k, \xi) \). With the substitution \( \zeta := y \xi \), the function \( \tilde{f} \) becomes a function of \( \zeta \), denoted by \( \tilde{f}_\zeta(k, \zeta) \). Clearly, \( \tilde{f}_\zeta \) is a strictly increasing function of \( k \) and a strictly decreasing function of \( \zeta \). By definition, \( \xi^* \) is the only solution to \( \tilde{f}(k, \xi) = 0 \). Let \( \zeta^* = y \xi^* \), then \( \zeta^* \) is the only solution to \( \tilde{f}_\zeta(k, \zeta) = 0 \). If \( k \) increases, \( \zeta^* \) will increase as well, which can be realized by three scenarios: (i) \( y \) decreases and \( \tilde{f}_\zeta(k, \zeta) \) decreases; (ii) \( y \) increases and \( \xi^* \) decreases; (iii) \( y \) increases and \( \tilde{f}_\zeta(k, \zeta) \) increases. If Scenario (i) holds, the increase of \( k \) will yield a strictly larger \( X^*_T \), and then a contradiction to the budget constraint \( E[\xi_T X^*_T] = X_0 \). Similarly, the assumption of Scenario (ii) will lead to a strictly smaller \( X^*_T \). Hence, only the last scenario holds when \( k \) increases. Please refer to Berkelaar et al. (2004) for more details.

- $\xi_1 < \xi_2$

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We rewrite the function $h$ defined above in the **Proof for Proposition 5.2** as $h(k, y)$. Then we obtain

$$
\frac{\partial}{\partial k} h(k, y) > 0 \text{ and } \frac{\partial}{\partial y} h(k, y) < 0.
$$

Since the Lagrange multiplier $y$ is such that $h(k, y) = e^{T}X_{0}$, if $k$ increases, $y$ will increase. According to Theorem 5.2, the optimal terminal value of the hedge fund in this case is given by

$$
X^*_T = B_T \cdot 1_{\xi_T \leq \xi_2} + \frac{1}{\eta w_+} \ln \left( \frac{\eta w_+}{y \xi_T} \right) \cdot 1_{\xi_T \leq \eta w_+}.
$$

If we assume $\xi_2$ does not increase when $k$ increases, then the first term of $X^*_T$ above will not increase but the second term will decrease, together yielding a strictly smaller $X^*_T$; hence the budget constraint $E[\xi_T X^*_T] = X_0$ will be violated.

Since both $d_{2,0}(\cdot)$ and $N(\cdot)$ are strictly increasing functions of $k$, $p_t$ is a strictly decreasing function of $k$. \hfill \Box

**Proof for Remark 6.1.** We study the impact of one contractual parameter each time and keep the other two unchanged. With slight misuse of notations, we write $\bar{f}$ as $\bar{f}(\alpha, \zeta)$ when analyzing the impact of $\alpha$ on $p_t$. $\bar{f}(\beta, \zeta)$ and $\bar{f}(w, \zeta)$ are introduced similarly. Through straightforward calculus, we obtain

$$
\frac{\partial \bar{f}(\alpha, \zeta)}{\partial \alpha} = (1 - w) \left[ k \eta B_T e^{-\eta w_- B_T} + \frac{\zeta}{\eta w_+} \ln \left( \frac{\eta w_+}{\zeta} \right) \right];
$$

$$
\frac{\partial \bar{f}(\beta, \zeta)}{\partial \beta} = \frac{\zeta (1 - w)}{\eta w_+^2} \ln \left( \frac{\eta w_+}{\zeta} \right);$$

$$
\frac{\partial \bar{f}(w, \zeta)}{\partial w} = (1 - \alpha) \left[ k \eta B_T e^{-\eta w_- B_T} + \frac{\zeta}{\eta w_+^2} \ln \left( \frac{\eta w_+}{\zeta} \right) \right].
$$

Notice that $\bar{f}$ is defined for $\xi \leq \xi_1 = \frac{\eta w_+}{y}$, or equivalently, $\zeta = y \xi \leq \eta w_+$. Hence, all the partial derivatives are strictly positive, and then the increase of any contractual parameter will lead to the increase of $\zeta^* = y \cdot \xi^*$ (since $\frac{\partial F}{\partial \xi}$ is always strictly negative). However, the arguments used in the **Proof for Proposition 6.1** to rule out Scenarios (i) and (ii) are not longer applicable here, since the change of any contractual parameter naturally causes the change of $w_+$. \hfill \Box

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