Optimal Consumption, Investment and Insurance Problem in Infinite Time Horizon

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Abstract

We incorporate an insurable risk into the classical consumption and investment problem and intend to find optimal investment, consumption and insurance strategies for a risk-averse investor who wants to maximize her expected utility of consumption in infinite time horizon. We obtain explicit optimal strategies for HARA utility functions. Through an economic analysis, we examine the impact of market parameters and the investor’s risk aversion on optimal strategies and how insurance affects the value function.

Key Words: utility of consumption; deductible insurance; stochastic control; Hamilton-Jabobi-Bellman equation; economic analysis.

1 Introduction

The study of consumption and investment problems in continuous time began with Merton’s seminal paper [5], in which Merton found explicit solutions to this problem for the first time using classical stochastic control method.

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Karatzas et al. [3] revisited the same problem in a more rigorous mathematical analysis, which overcame several errors in Merton [5], for instance, bankruptcy and infinite value function. Karatzas et al. [4] applied martingale representation method to solve this problem in an incomplete market. Many initial contributions to consumption and investment problems can be found in Sethi’s monograph [10]. Notice that there is only source of uncertainty (risk) in the classical consumption/investment model, namely, the stochastic movements of stock prices. However, in real life, most investors also face other random risks apart from the one arisen from investment. For instance, a working employee faces death risk and she can insure against such risk by purchasing life insurance. Therefore, a natural extension to the classic investment and consumption framework is to incorporate insurable risks into the model and then seek optimal investment, consumption and insurance strategies.

Early research on optimal insurance problem did not take investment and/or consumption into consideration and most of them worked in static models. The pioneering work of optimal insurance problem originated from Arrow [1] in medical care. Under the assumptions that premium only depends on the expected value of insurance payout and a risk-averse insured wants to maximize her expected utility, optimal insurance is a deductible insurance in a two-period discrete time model. Arrow [2] extended the results to both state-dependent and state-independent utility functions and analyzed the effect of various factors on optimal insurance, such as benefit-premium ratio and actual probability measure. Mossin [7] obtained similar results as Arrow [1] and showed that full coverage insurance is never optimal if there is a positive loading in premium. Promislow and Young [9] reviewed optimal insurance problem (without investment/consumption) and presented a general framework to obtain optimal insurance under various premium principles, such as Wang’s premium principle, Swiss premium principle and so on.

For dynamic optimal insurance problem along with investment and consumption, please refer to Moore and Young [6] and the references therein. They successfully solved optimal investment, consumption and insurance problem in finite time horizon and random time horizon and found closed form solutions for certain utility functions. Perera [8] studied this problem in a more general Levy processes market and obtained explicit solutions in finite time horizon for exponential utility function.

In this paper, we consider optimal investment, consumption and insurance
problem for a risk-averse investor who wants to maximize her expected utility of consumption in infinite time horizon. The main contributions of this paper are to provide rigorous verification theorems associated with Hamilton-Jacobi-Bellman equations, find sufficient and necessary conditions for optimal insurance to be no insurance or deductible insurance in dynamic case, and verify the conjectured functions and strategies are indeed the value functions and optimal strategies.

The structure of this paper is organized as follows. In Section 2, we introduce the market model and formulate optimal investment, consumption and insurance problem. In Section 3, we derive the Hamilton-Jacobi-Bellman equation to our optimization problem and provide the corresponding verification theorems. In Section 4, we obtain explicit value functions and optimal strategies when utility function is of HARA class. In Section 5, we conduct an economic analysis to investigate how various factors affect optimal strategies and how insurance influences optimal investment and consumption. Our conclusions are listed in Section 6.

2 The Model

We assume there are two assets in the market for investment. The price process of riskless asset (Bond) $S_0$ is driven by

$$dS_{0,t} = rS_{0,t}dt, \text{ with } S_{0,0} = 1.$$ 

The price dynamics of risky asset (stock) $S_1$ is given by

$$dS_{1,t} = S_{1,t}((\mu dt + \sigma dB_t), \text{ with } S_{1,0} > 0,$$

where $\{B_t\}_{t \geq 0}$ is a one-dimensional Brownian Motion defined in a standard stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$. We assume $\mu > r > 0$ and $\sigma > 0$.

At time $t$, an investor chooses the proportion of wealth invested in the risky asset, denoted by $\pi(t)$, and consumption rate $c(t)$. Apart from the risk in trading assets, the investor also faces another insurable risk, which is assumed to be proportional to the investor’s wealth, denoted by $W$. We denote the loss proportion at time $t$ by $L(t)$, then the insurable risk will be $L(t)W(t)$. We assume $L$ is left continuous in time, $\forall t \geq 0$, $L(t) \in (0,1)$ and the distribution of $L(t)$ does not depend on $t$, or equivalently, $\forall t \neq s$, \ldots
$L(t)$ and $L(s)$ have the same distribution. We assume the occurrence of the insurable risk is governed by a Poisson process $N$ with constant intensity $\lambda$. In the market, there are insurance policies available to insure against the risk $LW$. The investor controls the payout of insurance policy at time $t$ by choosing $I(t)$. For example, if $\Delta N_t = 1$, then the investor suffers a loss of amount $L_t W_t$ but gets compensation $I_t(L_t W_t)$ meanwhile, so the net loss will be $L_t W_t - I_t(L_t W_t)$. Furthermore, we assume that to obtain insurance protection, the investor needs to pay premium continuously at the rate $P(t)$.

\[ P(t) = \lambda(1 + \theta)E[I_t(L_t W_t)], \]

where $\theta$ is a positive constant, known as premium loading in the insurance industry. Such extra loading comes from insurance companies’ administrative cost, tax, and profit e.t.c.

We assume that Brownian Motion $B$, Poisson process $N$ and loss proportion process $L$ are mutually independent. We take the augmented filtration generated by stochastic processes $L$, $B$ and $N$ as our filtration $\{\mathcal{F}_t\}_{t \geq 0}$.

**Remark 2.1** The above continuous premium setting is called expected value principle and has been commonly used in many research papers, for instance, [6] and [8]. Since the payout will only be triggered when $\Delta N = 1$, so in an ideal world, premium $P$ should satisfy $P(t)\Delta t = E[I_t(L_t W_t)1_{\Delta N_t = 1}] = \lambda \Delta t E[I_t(L_t W_t)]$, where the second equality comes from the independence assumption of $B$, $N$ and $L$.

For an investor with triplet strategies $u_t := (\pi_t, c_t, I_t)$, the corresponding wealth process $W$ is given by

\[
dW_t = \left(rW_t + (\mu - r)\pi_t W_t - c_t - \lambda(1 + \theta)E[I_t(L_t W_t)]\right)dt \\
+ \sigma \pi_t W_t dB_t - (L_t W_t - I_t(L_t W_t))dN_t,
\]

with the initial condition $W_0 = x > 0$.

We define performance function by

\[ J(x; u) := E_x \left[ \int_0^\infty e^{-\delta t} U(c_t)dt \right], \]

\[1\text{Due to such assumption, we may write } L(t) \text{ as } L \text{ when there is no confusion on time } t. \]

\[2\text{Although we do not use the notation } W^u, W \text{ is a controlled process and the dependence of } W \text{ on } u \text{ is evident.} \]
where \( E_x \) means taking conditional expectation given \( W_0 = x \), \( \delta > 0 \) is the discount factor and utility function \( U \) is \( C^2(0, \infty) \), strictly increasing and concave, and satisfies the linear growth condition: \( \exists K > 0 \) such that

\[
U(y) \leq K(1 + y), \quad \forall y > 0.
\]

Besides, we denote \( U(0) = \lim_{y \to 0^+} U(y) \).

Define the bankruptcy time as

\[
\Theta := \inf \{ t \geq 0 : W_t \leq 0 \}
\]

and impose \( W_t = 0 \) for all \( t \geq \Theta \). Then \( \forall t \geq \Theta \), we can set \( u_t = 0 \).

With the help of \( \Theta \), we can rewrite the performance function as

\[
J(w; u) = E_x \left[ e^{-\delta \int_0^\Theta U(c_t) dt} \right] + \frac{U(0)}{\delta} E_x[e^{-\delta \Theta}].
\]

We denote \( \mathcal{A} \) as the set of all admissible controls and define \( \mathcal{A} \) as follows: \( \forall u = (\pi, c, I) \in \mathcal{A} \), \( u \) is progressively measurable with respect to the filtration \( (\mathcal{F}_t)_{t \geq 0} \) and satisfies the following conditions, \( \forall t \geq 0 \)

\[
E_x \left[ \int_0^t \pi_s^2 ds \right] < +\infty,
\]

\[
E_x \left[ \int_0^t c_s ds \right] < +\infty \text{ and } c_t \geq 0,
\]

\[
E_x \left[ \int_0^\Theta e^{-\delta t} U^+(c_t) dt \right] < +\infty,
\]

\( \{I\}_{t \geq 0} \) is left continuous on time \( t \),

and \( I_t \in \mathcal{I} := \{ I : 0 \leq I(y) \leq y, \forall y \geq 0 \} \).

**Problem 2.1** Find an admissible control \( u^* \in \mathcal{A} \) such that

\[
V(x) := \sup_{u \in \mathcal{A}} J(x; u) = J(x; u^*).
\]

\( V \) is called the value function and \( u^* \) optimal control or optimal policy.
3 Verification Theorems

\( \forall u = (\pi, c, I) \in \mathcal{A}_0 := (-\infty, \infty) \times [0, \infty) \times I \) and \( \psi : (0, +\infty) \to \mathbb{R} \) in \( C^2(0, +\infty) \), define the operator \( \mathcal{L}_u \) by

\[
\mathcal{L}_u(\psi(x)) := (r x + (\mu - r) \pi x - c - \lambda (1 + \theta) E[I(Lx)]) \psi'(x) + \frac{1}{2} \sigma^2 \pi^2 x^2 \psi''(x) - \delta \psi(x).
\]

**Theorem 3.1** Suppose \( U(0) \) is finite. Let \( v \in C^2(0, \infty) \) be an increasing and concave function such that \( v(0) = \frac{U(0)}{\delta} \). If \( v \) satisfies the following Hamilton-Jacobi-Bellman equation

\[
\sup_{u \in \mathcal{A}_0} \{ \mathcal{L}_u v(x) + U(c) + \lambda E[v(x - Lx + I(Lx)) - v(x)] \} = 0,
\]

for \( \forall w > 0 \) and \( u^* = (\pi^*, c^*, I^*) \) defined by

\[
u^*_t := \arg \sup_{u \in \mathcal{A}_0} \left( \mathcal{L}_u v(W^*_t) + U(c_t) + \lambda E[v(W^*_t - L_t W^*_t + I_t(L_t W^*_t)) - v(W^*_t)] \right)
\]
is admissible, then \( u^* \) is optimal control to Problem 2.1 and \( v \) is the corresponding value function.

**Proof.** Define \( f(t, x) := e^{-\delta t} (v(x) - \frac{U(0)}{\delta}) \). Applying generalized Ito’s formula to \( f(t, W_t) \), we obtain

\[
f(t, W_t) = f(0, W_0) + \int_0^t e^{-\delta s} (\mathcal{L}_u v(W_s) + U(0)) ds + \int_0^t e^{-\delta s} \sigma \pi_s W_s v'(W_s) dB_s + \sum_{0 < s \leq t} e^{-\delta s} (v(W_s) - v(W_{s-})),
\]

where the last summation term can be rewritten as

\[
\sum_{0 < s \leq t} e^{-\delta s} (v(W_s) - v(W_{s-})) = \int_0^t e^{-\delta s} \lambda (v(W_s) - L_s W_s + I_s) - v(W_s)) ds
\]

\[
+ \int_0^t e^{-\delta s} (v(W_{s-}) - L_s W_{s-} + I_s) - v(W_{s-})) dM_s,
\]

where \( M_t := N_t - \lambda t \) is a martingale. Since on the given interval \([0, t]\), Poisson process \( N \) has up to countable jump points, we can replace \( W_{s-} \) by \( W_s \) in the Riemann integral.
Let $0 < a < W_0 = x < b < \infty$ and define two stopping times $\tau_a := \inf\{t \geq 0 : W_t \leq a\}$ and $\tau_b := \inf\{t \geq 0 : W_t \geq b\}$. Take $\tau = \tau_a \wedge \tau_b$, then $\tau$ is also a stopping time.

In the interval $[0, t \wedge \tau]$, both $v'$ and $W$ are bounded, so $\forall s \in [0, t \wedge \tau]$, $|e^{-\delta s}v'(W_s)| \leq M$ for some positive constant $M$ and $W_s \leq b$. Hence, by the definition of admissible control, we have

$$E_x\left[\int_0^{t \wedge \tau} (e^{-\delta s}\pi_s W_s v'(W_s))^2 ds\right] \leq M^2 b^2 \sigma^2 E_x\left[\int_0^{t \wedge \tau} (\pi_s)^2 ds\right] < \infty,$$

which implies

$$E_x\left[\int_0^{t \wedge \tau} e^{-\delta s} \pi_s W_s v'(W_s) dB_s\right] = 0.$$

Furthermore, integrand $e^{-\delta s} (v(W_{s-} - L_s W_{s-} + I_s) - v(W_{s-}))$ is left continuous and bounded in $[0, t \wedge \tau]$, so by Theorem 11.4.5 in Shreve [11],

$$E_x\left[\int_0^{t \wedge \tau} e^{-\delta s} (v(W_{s-} - L_s W_{s-} + I_s) - v(W_{s-})) dM_s\right] = 0.$$

Therefore, by taking conditional expectation and applying the HJB equation (2), we obtain

$$E_x[f(t \wedge \tau, W_{t \wedge \tau})] \leq v(x) - \frac{U(0)}{\delta} - E_x\left[\int_0^{t \wedge \tau} e^{-\delta s} (U(c_s) - U(0)) ds\right].$$

Let $a \downarrow 0$ and $b \uparrow \infty$, then $\tau_a \to \Theta$ and $\tau_b \to \infty$, so $\tau \to \Theta$, and $t \wedge \tau \to \Theta$ when $t \to \infty$. Along with the continuity of $f$, we get, when $a \downarrow 0$, $b \uparrow \infty$, $t \to \infty$

$$f(t \wedge \tau, W_{t \wedge \tau}) \to f(\Theta, 0) = e^{-\delta \Theta} \left(v(0) - \frac{U(0)}{\delta}\right) = 0.$$

Taking limit for the above inequality yields

$$0 \leq v(x) - \frac{U(0)}{\delta} - E_x\left[\int_0^{\Theta} e^{-\delta s} (U(c_s) - U(0)) ds\right].$$

Since utility function $U(x)$ satisfies the linear growth condition, it follows

$$E_x\left[\int_0^{\Theta} e^{-\delta s} (U(c_s) - U(0)) ds\right] \leq K_1 E_x\left[\int_0^{\Theta} c_s ds\right] < \infty, \quad K_1 > 0.$$
Therefore, due to the finiteness of $U(0)$, we have

\[ v(x) \geq E_x \left[ \int_0^{\Theta} e^{-\delta s}(U(c_s) - U(0)) ds \right] + \frac{U(0)}{\delta} \]

\[ = E_x \left[ \int_0^{\Theta} e^{-\delta s}U(c_s) ds \right] + \frac{U(0)}{\delta} \left( E[e^{-\delta \Theta}] - 1 \right) + \frac{U(0)}{\delta} \]

\[ = E_x \left[ \int_0^{\Theta} e^{-\delta s}U(c_s) ds \right] + \frac{U(0)}{\delta} E[e^{-\delta \Theta}] = J(x; u), \quad \forall x > 0, u \in \mathcal{A} \]

and the equality will be achieved when $u = u^*$.

The previous verification theorem depends on the assumption that $U(0)$ is finite, while $U(x)$ is a strictly increasing function and so $U(0) \neq +\infty$. The next verification theorem deals with the case when $U(0) = -\infty$.

**Theorem 3.2** Suppose $v(0) = \frac{U(0)}{\delta} = -\infty$. Theorem 3.1 still holds.

**Proof.** Define $g(t, x) := e^{-\delta t}v(x)$. By following a similar argument, we obtain

\[ E_x [g(t \wedge \tau, W_{t \wedge \tau})] = v(w) + E_x \left[ \int_0^{t \wedge \tau} e^{-\delta s} \mathcal{L}_{u_s} v(W_s) ds \right] \]

\[ + E_x \left[ \int_0^{t \wedge \tau} \lambda e^{-\delta s} E (v(W_s - L_sW_s - I_s) - v(W_s)) ds \right] \leq v(w) - E_x \left[ \int_0^{t \wedge \tau} e^{-\delta s}U(c_s) ds \right]. \]

Since we assume $E_x \left[ \int_0^{\Theta} e^{-\delta t}U^+(c_t) dt \right] < +\infty$ and $U(x)$ satisfies the linear growth condition, we have $E_x \left[ \int_0^{t \wedge \tau} e^{-\delta s}U(c_s) ds \right] < +\infty$. So the above inequality can be rearranged as

\[ v(w) \geq E_x [g(t \wedge \tau, W_{t \wedge \tau})] + E_x \left[ \int_0^{t \wedge \tau} e^{-\delta s}U(c_s) ds \right]. \]

From the assumption that $v$ is increasing on $(0, \infty)$ and $v(0) = \frac{U(0)}{\delta}$, we get

\[ E_x [g(t \wedge \tau, W_{t \wedge \tau})] \geq E_x \left[ e^{-\delta(t \wedge \tau)} \frac{U(0)}{\delta} \right] = E_x \left[ \int_{t \wedge \tau}^{\infty} e^{-\delta s}U(0) ds \right]. \]
By letting \( a \downarrow 0, b \uparrow \infty \) and \( t \to \infty \) and applying the dominated convergence theorem and monotone convergence theorem, we obtain

\[
v(w) \geq E_x \left[ \int_0^\infty e^{-\delta s} U(0) ds \right] + E_x \left[ \int_0^\Theta e^{-\delta s} U(c_s) ds \right] = J(w; u),
\]

and the equality holds when \( u = u^* \). \( \square \)

### 4 Explicit Solutions for HARA Utility Function

In this section, we obtain explicit solutions to the HJB equation (2) for three specific HARA utility functions. To such purpose, we first conjecture candidates for the value function and optimal control, then verify our conjecture.

We rewrite the HJB equation (2) as follows

\[
rx' + \max_{\pi} \left[ (\mu - r)\pi x' + \frac{1}{2} \sigma^2 \pi^2 x^2 v''(x) \right] + \max_c \left[ U(c) - cv'(x) \right] + \lambda \max I \left[ E(v(x - Lx + I(Lx))) - (1 + \theta) E(I(Lx)) v'(x) \right] = (\delta + \lambda) v(x).
\]

By the assumption that \( v' > 0, v'' < 0 \), we obtain candidates for optimal investment and consumption as follows

\[
\pi^* := \arg \sup_{\pi} \left[ (\mu - r)\pi x' + \frac{1}{2} \sigma^2 \pi^2 x^2 v''(x) \right] = -\frac{(\mu - r)v'(x)}{\sigma^2 x v''(x)},
\]

\[
c^* := \arg \sup_c \left[ U(c) - cv'(x) \right] = (U')^{-1}(v'(x)).
\]

The following lemma and theorem provide the criteria for optimal insurance \( I^* \).

**Lemma 4.1**

(a) \( I^*(y_0) = 0 \), where \( y_0 = l_0 x, l_0 \in (0, 1) \) if and only if

\[
(1 + \theta)v'(x) \geq v'(x - y_0) = v'((1 - l_0)x).
\]

(b) \( 0 < I^*(y_0) < y_0 \), where \( y_0 = l_0 x, l_0 \in (0, 1) \) if and only if

\[
(1 + \theta)v'(x) = v'(x - y_0 + I^*(y_0)).
\]
Proof. We use the notation $y_0 := l_0 x$ for $l_0 \in (0, 1)$ in what follows. First, we show that $I^*(y_0) \neq y_0, \forall y_0$. Assume to the contrary that $\exists y_0$ such that $I^*(y_0) = y_0$. We consider $I(y) := I^*(y) - \epsilon i(y), y = lx, l \in (0, 1)$, where $i(y) = 1$ when $l_0 - \eta < l \leq l_0 + \eta$ and 0 otherwise. Here we take $\epsilon$ and $\eta$ small enough such that $I \in I$. Since $I^*$ is the maximizer, we have

$$E[v(x-Lx+I(Lx))-v(x-Lx+I^*(Lx))] - (1+\theta)v'(x)E[I(Lx)-I^*(Lx)] \leq 0.$$ 

By applying Taylor’s expansion and dividing by $\epsilon$ on both hand sides of the inequality, we shall get

$$(1 + \theta)v'(x)\frac{E[i(y)]}{\epsilon} \leq E[v'(x-Lx+I^*(Lx))i(y)], \text{ when } \epsilon \to 0^+.$$ 

Taking $\eta \to 0^+$ and using the mean value theorem for integration, we obtain

$$(1 + \theta)v'(x) \leq v'(x - y_0 + I^*(y_0)) = v'(x),$$

which does not hold since $v'(x) > 0$ and $\theta > 0$.

To prove the sufficient condition in case (a), we consider $I'(y) := I^*(y) + \epsilon i(y)$ where $i(y)$ has the same definition as above. Then a similar analysis gives

$$(1 + \theta)v'(x) \geq v'(x - y_0).$$

To prove the sufficient condition in case (b), notice that for small enough $\epsilon$ and $\eta$, both $I(y)$ and $I'(y)$ are in the set of $I$, so we have $(1 + \theta)v'(x) \leq v'(x - y_0 + I^*(y_0))$ and $(1 + \theta)v'(x) \geq v'(x - y_0 + I^*(y_0))$ at the same time and then obtain equality.

To prove the necessary condition in case (a), we assume to the contrary that $I^*(y_0) > 0$. Then by the sufficient condition in case (b), we have

$$(1 + \theta)v'(x) = v'(x - y_0 + I^*(y_0)) < v'(x - y_0),$$

which is a contradiction to the given condition.

The above argument also applies to proof of the necessary condition in case (b).

Theorem 4.1

(a) Optimal insurance is no insurance $I^*(Lx) = 0$ when

$$(1 + \theta)v'(x) \geq v'\left((1 - \text{ess sup}(L))x\right).$$

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(b) Optimal insurance is deductible insurance \( I^*(Lx) = (Lx - d)^+ \) when there exists \( d \in (0, x) \) such that

\[(1 + \theta)v'(x) = v'(x - d). \quad (6)\]

Proof. Case(a): Assume there exists \( y_0 = l_0x, l_0 \in (0, 1) \) such that

\[0 < I^*(y_0) < y_0.\]

Then by Lemma 4.1, we have

\[(1 + \theta)v'(x) = v'(x - y_0 + I^*(y_0)).\]

Define \( N := \{ \omega \in \Omega : L(\omega) > \text{ess sup} (L) \} \). It is evident that \( P(N) = 0 \). If \( l_0 \leq \text{ess sup} (L) \), we obtain \( (1 - \text{ess sup} L) x < x - y_0 + I^*(y_0) \) and then

\[v'((1 - \text{ess sup} L) x) > v'(x - y_0 + I^*(y_0)) = (1 + \theta)v'(x),\]

which is a contradiction to the condition (5). Hence \( I^*(Lx) = 0 \) on set \( N^c \).

Next we need to show that if two insurance policies \( I_1 \) and \( I_2 \) only differ on a negligible set, then \( h(I_1) = h(I_2) \), where \( h(I) := E[v(x - Lx + I(Lx))] - (1 + \theta)v'(x)E[I(Lx)] \). This result is evident since the integration of a bounded function over a negligible set is zero.

Therefore, \( I^*(Lx) = 0 \) almost surely.

Case(b): First, by the strict concavity of \( v \), if such \( d \) exists, it is also unique.

\[\forall 0 < l_0 \leq \frac{d}{x} < 1, \text{ we have } v'((1 - l_0) x) \leq v'(x - d) = (1 + \theta)v'(x).\]

Then by Lemma 4.1, \( I^*(l_0x) = 0 = (l_0x - d)^+ \).

\[\forall \frac{d}{x} < l_0 < 1, \text{ we have } v'((1 - l_0) x) > v'(x - d) = (1 + \theta)v'(x),\]

which implies \( 0 < I^*(l_0x) < l_0 \).

By Lemma 4.1, we get

\[(1 + \theta)v'(x) = v'(x - l_0x + I^*(l_0x)) = v'(x - d).\]
Then recall the strict monotonicity of \( v \), we obtain
\[
I^*(l_0x) = l_0x - d = (l_0x - d)^+, \text{ and } I^*(Lx) = (Lx - d)^+.
\]

To complete the proof, we need to show that either (5) or (6) will hold. Assume the condition (5) does not hold, then
\[
v'(x) < (1 + \theta)v'(x) < v'((1 - \text{ess sup } L)x) \leq v'(0),
\]
where \( v'(0) := \lim_{x \to 0^+} v'(x) \). Since \( v' \) is continuous and strictly decreasing on \([0, x]\), there exists a unique \( d \in (0, x) \) such that \((1 + \theta)v'(x) = v'(x - d)\).
\[\square\]

4.1 \( U(y) = \ln(y), \ y > 0 \)

When \( U(y) = \ln(y) \), \( \ln(y) \leq 1 + y, \forall y > 0 \), so we can take \( K = 1 \) to guarantee the linear growth condition will be hold. In this case, a solution to the HJB equation (2) is given by
\[
v(x) = \frac{1}{\delta} \ln(\delta x) + A,
\]
where \( A \) is constant.

Differentiating \( v(x) \) yields \( v'(x) = \frac{1}{\delta x} \) and \( v''(x) = -\frac{1}{\delta^2 x^2} \). Then candidates for optimal portfolio and consumption are given by
\[
\pi^* = \frac{\mu - r}{\sigma^2} \text{ and } c^* = \delta x.
\]

Solving \((1 + \theta)v'(x) = v'(x - d)\) gives \( d = \frac{\theta}{1+\theta} x \in (0, x) \). Therefore, by Theorem 4.1, candidate for optimal insurance is
\[
I^*(Lx) = (Lx - d)^+ = (L - \frac{\theta}{1+\theta})^+ x.
\]

By plugging candidate strategies into the HJB equation (2), we obtain
\[
\frac{r}{\delta} + \frac{(\mu - r)^2}{2\delta^2} - 1 + \frac{\lambda}{\delta} \Lambda = \delta A,
\]
where \( \Lambda := E[\ln(1 - L + (L - \frac{\theta}{1+\theta})^+)] - (1 + \theta)E(L - \frac{\theta}{1+\theta})^+ \).

So constant \( A \) is determined by
\[
A = \frac{1}{\delta^2} \left[ r - \delta + \frac{(\mu - r)^2}{2\sigma^2} + \lambda \Lambda \right]. \tag{7}
\]
Proposition 4.1 The function defined by
\[ v(x) := \begin{cases} \frac{1}{\delta} \ln(\delta x) + A, & x > 0 \\ -\infty, & x = 0 \end{cases} \]
where \( A \) is given by (7), is the value function of Problem 2.1. Furthermore, the control given by
\[ u_t^* := \left( \frac{\mu - r}{\sigma^2}, \delta W_t^*, (L_t - \frac{\theta}{1 + \theta})^+ W_t^* \right) \]
is optimal control to Problem 2.1.

Proof. By the definition of \( v \), we have \( v(0) = \frac{U(0)}{\delta} = -\infty \) and \( v(x) \) is strictly increasing and concave on \((0, \infty)\). By the construction of \( A \), \( v \) satisfies the HJB equation (2). Therefore, by Theorem 3.2, \( v(x) \) is the value function of Problem 2.1.

Under the control \( u^* \) given above, the SDE (1) becomes
\[ dW_t^* = (r - \delta + \frac{(\mu - r)^2}{\sigma^2})W_t^* dt - \lambda(1 + \theta)E[I_t^*] dt + \frac{\mu - r}{\sigma} W_t^* dB_t \\
- (L_t - (L_t - \frac{\theta}{1 + \theta})^+) W_t^* dN_t. \]

Since we cannot solve the above SDE explicitly, we then drop the premium part and consider an upper bound process \( Y \) of \( W^* \) with dynamics
\[ \frac{dY_t}{Y_t} = (r - \delta + \frac{(\mu - r)^2}{\sigma^2}) dt + \frac{\mu - r}{\sigma} dB_t - d \left( \sum_{i=1}^{N(t)} (L_{\tau_i} - (L_{\tau_i} - \frac{\theta}{1 + \theta})^+) \right), \]
where \( \tau_i \) is the hitting time of the \( i \)-th jump of Poisson process \( N \).

Given \( Y_0 = W_0^* = x \), we solve to obtain \( Y \)
\[ Y_t = xe^{(r-\delta+\frac{(\mu-r)^2}{2\sigma^2})t+\frac{\mu-r}{\sigma}B_t} \prod_{i=1}^{N(t)} \left( 1 - L_{\tau_i} + (L_{\tau_i} - \frac{\theta}{1 + \theta})^+ \right). \]
Apparently, we have $W^*_t \leq Y_t$, $\forall t \geq 0$. So we obtain

$$E_x \left[ \int_0^t c^*_s ds \right] = \delta E_x \left[ \int_0^t W^*_s ds \right] \leq \delta E_x \left[ \int_0^t Y_s ds \right]$$

$$= x \delta \int_0^t e^{(r-\delta + \frac{(\mu-r)^2}{\sigma^2})s} \lambda s E(1 - L + (L - \frac{\theta}{1+\theta})^+)ds$$

$$\leq x \delta \lambda \int_0^t s e^{(r-\delta + \frac{(\mu-r)^2}{\sigma^2})s} ds < +\infty,$$

and

$$E_x \left[ \int_0^\infty e^{-\delta t} \ln^+(c^*_t) dt \right] \leq E_x \left[ \int_0^\infty e^{-\delta t} \ln^+(\delta Y_t) dt \right]$$

$$\leq E_x \left[ \int_0^\infty e^{-\delta t} (|\ln(\delta x)| + |r - \delta + \frac{(\mu-r)^2}{\sigma^2}| t + \frac{\mu-r}{\delta} |B_t| + \sum_{i=1}^{N(t)} |\ln(1 - L_{r_i} + (L_{r_i} - \frac{\theta}{1+\theta})^+)|) dt \right].$$

Since $B(t)$ is normally distributed with mean 0 and variance $t$, we get $E_x[|B_t|] = \sqrt{\frac{2t}{\pi}}$. Due to the independence assumption of $N$ and $L$, we have

$$E_x \left[ \sum_{i=1}^{N(t)} |\ln(1 - L_{r_i} + (L_{r_i} - \frac{\theta}{1+\theta})^+)| \right] = \lambda t E_x |\ln(1 - L + (L - \frac{\theta}{1+\theta})^+)|$$

$$\leq \lambda t \ln(1 + \theta).$$

Therefore, we can claim

$$E_x \left[ \int_0^\infty e^{-\delta t} \ln^+(c^*_t) dt \right] \leq \frac{1}{\delta} |\ln(\delta w)| + \frac{1}{\delta^2} |r - \delta + \frac{(\mu-r)^2}{\sigma^2}|$$

$$+ \int_0^\infty e^{-\delta t} \left( \sqrt{\frac{2}{\pi}} \frac{\mu-r}{\sigma} t^{\frac{3}{2}} + \lambda \ln(1 + \theta) t \right) dt$$

$$= \text{constant} + \frac{\mu-r}{\sqrt{2\delta^3 \sigma}} + \frac{\lambda \ln(1 + \theta)}{\delta^2} < +\infty.$$

Besides, it’s obvious that

$$E_x \left[ \int_0^t \sigma^2(\pi^*_s)^2 ds \right] = \frac{(\mu-r)^2 t}{\sigma^2} < +\infty,$$
and $I^*$ is left continuous in time satisfying

$$0 \leq I^*_t(L_t x) = (L_t - \frac{\theta}{1+\theta})^+ x \leq L_t x.$$ 

Therefore, we conclude $u^*_t := (\frac{\mu - r}{\sigma^2}, \delta W^*_t, (L_t - \frac{\theta}{1+\theta})^+ W^*_t)$ is indeed an admissible control, and thus optimal control to Problem 2.1.

In Proposition 4.1, we obtain the value function $v$ in explicit expression with $A$ determined by (7), which depends on the distribution of $L$. In the next two example, we calculate $A$ in closed form.

### 4.1.1 Loss proportion $L$ is a constant

If $L = l \in (0, \frac{\theta}{1+\theta}]$, then

$$A = \frac{1}{\delta^2} \left[ r - \delta + \frac{(\mu - r)^2}{2\sigma^2} + \lambda \ln(1 - l) \right].$$

If $L = l \in (\frac{\theta}{1+\theta}, 1)$, then

$$A = \frac{1}{\delta^2} \left[ r - \delta + \frac{(\mu - r)^2}{2\sigma^2} - \lambda \ln(1 + \theta) - \lambda l(1 + \theta) + \lambda \theta \right].$$

### 4.1.2 Loss proportion $L$ is uniformly distributed on $(0, 1)$

In this example, a straightforward calculus gives

$$E[\ln(1 - L + (L - \frac{\theta}{1+\theta})^+)] = -\frac{\theta}{1+\theta},$$

$$E(L - \frac{\theta}{1+\theta})^+ = \int_{\frac{\theta}{1+\theta}}^{1} (l - \frac{\theta}{1+\theta}) dl = \frac{1}{2(1+\theta)^2}.$$

Therefore, constant $A$ will be

$$A = \frac{1}{\delta^2} \left[ r - \delta + \frac{(\mu - r)^2}{2\sigma^2} - \frac{\lambda(1 + 2\theta)}{2(1+\theta)} \right].$$
4.2 \( U(y) = -y^\alpha, \alpha < 0 \)

Similarly, we can take \( K = 1 \) regarding the linear growth condition when \( U(y) = -y^\alpha \). In this scenario, a solution to the HJB equation (2) has the form

\[
\tilde{v}(x) = -\tilde{A}^{1-\alpha}x^\alpha, \; \tilde{A} > 0,
\]

from which we obtain

\[
\tilde{v}'(x) = -\alpha \tilde{A}^{1-\alpha}x^{-(1-\alpha)} \quad \text{and} \quad \tilde{v}''(x) = \alpha(1-\alpha)\tilde{A}^{1-\alpha}x^{-(2-\alpha)}.
\]

We then find candidates for optimal portfolio and consumption as follows

\[
\pi^* = -\frac{\mu - r}{\sigma^2 x} \tilde{v}'(x) = \frac{\mu - r}{(1-\alpha)\sigma^2},
\]

\[
c^* = \left(U'\right)^{-1}(\tilde{v}') = \frac{x}{\tilde{A}} > 0.
\]

Solving \((1+\theta)\tilde{v}'(x) = \tilde{v}'(x - d)\) gives unique \( d = (1-\gamma)x \in (0, x) \), where \( \gamma := (1+\theta)^{-\frac{1}{1-\alpha}} \in (0, 1) \). Therefore, by Theorem 4.1, the uniform expression of optimal insurance is

\[
I^*(Lx) = (Lx - d)^+ = (L + \gamma - 1)^+x.
\]

From the HJB equation (2), we can solve to get \( \tilde{A} \) as

\[
\tilde{A} = \frac{1 - \alpha}{\delta - \alpha r - \frac{\alpha(\mu - r)^2}{2(1-\alpha)\sigma^2} + \lambda(1 - \Lambda')}.
\]

where \( \Lambda' := E(1 - L + (L + \gamma - 1)^+)^\alpha - \alpha(1+\theta)E(L + \gamma - 1)^+ \).

Assume the cumulative distribution function of \( L \) is \( F_L \), then

\[
1 - \Lambda' = 1 - \int_0^{1-\gamma}(1 - l)^\alpha dF_L - \int_{1-\gamma}^1 \gamma^\alpha dF_L + \alpha(1+\theta) \int_{1-\gamma}^1 (l + \gamma - 1)dF_L
\]

\[
\geq 1 - \gamma^\alpha F_L(1 - \gamma) - \gamma^\alpha(1 - F_L(1 - \gamma)) + \alpha(1+\theta)(1 - F_L(1 - \gamma))
\]

\[
\geq 1 - \gamma^\alpha + \alpha\gamma(1 + \theta).
\]

So in order to guarantee the existence of a strictly increasing and concave value function, we impose the following condition for the discount rate \( \delta \)

\[
\delta > \alpha r + \frac{\alpha(\mu - r)^2}{2(1-\alpha)\sigma^2} - \lambda(1 - \gamma^\alpha + \alpha\gamma(1+\theta)).
\]

Due to the technical condition (9) above, we have \( \tilde{A} > 0 \).
Proposition 4.2 The function defined by
\[ \tilde{v}(x) := \begin{cases} -\tilde{A}^{1-\alpha}x^{\alpha}, & x > 0 \\ -\infty, & x = 0 \end{cases} \]
where \( \tilde{A} \) is given by (8), is the value function of Problem 2.1. Furthermore, the control \( u^* \) given by
\[ u^*_t := \left( \frac{\mu - r}{(1-\alpha)\sigma^2} \right) A_t^{(L_t + \gamma - 1)^+} W_t^* \]
is optimal control to Problem 2.1.

Proof. By definition, \( \tilde{v}(0) = V(0) = -\infty \) and \( \tilde{v}(x) \) is a strictly increasing and concave function. By the construction of \( \tilde{A} \), \( \tilde{v} \) satisfies the HJB equation (2). Therefore, by Theorem 3.2, \( \tilde{v}(x) \) is the value function of Problem 2.1.

Under the optimal control \( u^* \), the corresponding wealth process is
\[
dW_t^* = (r + \frac{(\mu - r)^2}{2(1-\alpha)(1-\alpha)\sigma^2} - \frac{1}{A_t}) W_t^* dt - \lambda (1 + \theta) E[I_t] dt + \frac{\mu - r}{(1-\alpha)\sigma} W_t^* dB_t \\
- (L_t - (L_t + \gamma - 1)^+) W_t^* dN_t.
\]

Similarly, we consider an upper bound process \( Z \) of \( W^* \) defined by
\[
\frac{dZ_t}{Z_t} = \left( r + \frac{(\mu - r)^2}{2(1-\alpha)^2\sigma^2} \right) dt + \frac{\mu - r}{(1-\alpha)\sigma} dB_t.
\]

Given \( Z_0 = W^*_0 = x \), the above SDE has a unique solution
\[
Z_t = x \exp \left( \left( r + \frac{(1 - 2\alpha)(\mu - r)^2}{2(1 - \alpha)^2\sigma^2} \right) t + \frac{\mu - r}{(1-\alpha)\sigma} B_t \right).
\]

In the next step, we verify candidate control is admissible.
\[
E_x \left[ \int_0^t c_s^* ds \right] \leq \frac{1}{A} E_x \left[ \int_0^t Z_s ds \right] < +\infty.
\]
\[
E_x \left[ \int_0^\infty e^{-\delta t} U^+(c_t^*) dt \right] = E_x \left[ \int_0^\infty e^{-\delta t} \left( \frac{c_t^*}{A} \right)^+ dt \right] = 0.
\]
Furthermore,

\[ E_x \left[ \int_0^t \sigma^2(\pi_s^*)^2ds \right] = \frac{(\mu - r)^2 t}{(1 - \alpha)^2 \sigma^2} < +\infty. \]

\( I^* \) is left continuous in time and satisfies

\[ 0 \leq I^*_t(L_t x) = (L_t + \gamma - 1^+) x \leq L_t x. \]

So \( u^*_t := \left( \frac{\mu - r}{(1 - \alpha)^2 \sigma^2}, W_t^* / \tilde{A}, (L_t + \gamma - 1^+) W_t^* \right) \) is an admissible control and then optimal control to Problem 2.1. □

4.2.1 Loss proportion \( L \) is a constant

If \( L = l \in [0, 1 - \gamma] \), then

\[ \tilde{A} = \frac{1 - \alpha}{\delta - \alpha r - \frac{\alpha(\mu - r)^2}{2(1 - \alpha)^2 \sigma^2} + \lambda(1 - (1 - l)^\alpha)}. \]

If \( L = l \in (1 - \gamma, 1] \), then

\[ \tilde{A} = \frac{1 - \alpha}{\delta - \alpha r - \frac{\alpha(\mu - r)^2}{2(1 - \alpha)^2 \sigma^2} + \lambda(1 - \gamma^\alpha + \alpha(1 + \theta)(l + \gamma - 1))}. \]

4.2.2 Loss proportion \( L \) is uniformly distributed on \( (0, 1) \)

In this example, we calculate

\[ E(L + \gamma - 1^+) = \int_{1-\gamma}^1 (l + \gamma - 1) dl = \frac{1}{2} \gamma^2, \]

and

\[ E(1 - L + (L + \gamma - 1^+)^\alpha) = \begin{cases} 1 + \alpha \gamma^{1+\alpha} & \text{if } \alpha \neq -1, \\ \frac{1 + \alpha \gamma^{1+\alpha}}{1 + \ln(\gamma)} & \text{if } \alpha = -1. \end{cases} \]

Therefore, we obtain when \( \alpha \neq -1 \),

\[ \tilde{A} = \frac{1 - \alpha}{\delta - \alpha r - \frac{\alpha(\mu - r)^2}{2(1 - \alpha)^2 \sigma^2} + \lambda \left[ \frac{\alpha}{1 + \alpha} (1 - \gamma^{1+\alpha}) + \frac{1}{2} \alpha(1 + \theta) \gamma^2 \right]}, \]

and when \( \alpha = -1 \),

\[ \tilde{A} = \frac{1 - \alpha}{\delta - \alpha r - \frac{\alpha(\mu - r)^2}{2(1 - \alpha)^2 \sigma^2} + \lambda \left[ \ln(\gamma) + \frac{1}{2} \alpha(1 + \theta) \gamma^2 \right]}. \]
4.3 $U(y) = y^\alpha$, $0 < \alpha < 1$

When $U(y) = y^\alpha$, the linear growth condition is satisfied for $K = 1$. We find a solution to the HJB equation (2) given by

$$\bar{v}(x) = \bar{A}^{1-\alpha} x^\alpha, \bar{A} > 0.$$  

Then we obtain candidates for optimal investment and consumption as follows

$$\pi^* = \frac{\mu - r}{(1-\alpha)\sigma^2} > 0, \text{ and } c^* = \frac{x}{\bar{A}} > 0.$$  

From the equation $(1 + \theta)\bar{v}'(x) = \bar{v}'(x - d)$, we can solve to obtain $d = (1 - \gamma)x$, where $\gamma = (1 + \theta)^{-\frac{1}{1-\alpha}}$, so candidate for optimal insurance is

$$I^*(Lx) = (Lx - d)^+ = (L + \gamma - 1)^+ x.$$  

From the HJB equation (2), we shall get $\bar{A} = \tilde{A}$, where $\tilde{A}$ is given by (8). Since $\alpha > 0$ in this case, we have $1 - \Lambda' > 0$. Therefore, the technical condition we need here is

$$\delta > \alpha r + \frac{\alpha (\mu - r)^2}{2(1-\alpha)\sigma^2},$$  

which guarantees $\bar{A} > 0$.

**Proposition 4.3** The function defined by $\bar{v}(x) = \bar{A}^{1-\alpha} x^\alpha$, $x \geq 0$, where $\bar{A} = \bar{A}$, is the value function of Problem 2.1. Furthermore, the control $u^*$ given by

$$u_t^* := \left(\frac{\mu - r}{(1 - \alpha)\sigma^2}, \frac{W_t^*}{\bar{A}}, (L_t + \gamma - 1)^+ W_t^*\right)$$  

is the corresponding optimal control to Problem 2.1.

**Proof.** $U(0) = 0$ is finite and $v(0) = \frac{U(0)}{\delta} = 0$. By definition, $\bar{v}$ is a strictly increasing and concave function. Besides, the construction of $\bar{A}$ ensures that $\bar{v}$ satisfies the HJB equation (2). Therefore, according to Theorem 3.1, $\bar{v}(x)$ is the value function of Problem 2.1.

We use the same upper bound process $Z_t$ defined in the proof Proposition 4.2 to verify candidate control is admissible. By following the same argument, we get

$$E_x \left[ \int_0^t c^*_s ds \right] < \infty.$$
Besides,

\[ \mathbb{E}_x \left[ \int_0^\infty e^{-\delta t} U^+(c_t^*) dt \right] \leq \mathbb{E}_x \left[ \int_0^\infty e^{-\delta t} \left( \frac{Z_t}{A} \right)^\alpha dt \right] \\
= \frac{x^\alpha}{A^\alpha} \int_0^\infty e^{-\left( \delta - \alpha r - \frac{\alpha(\mu-r)^2}{2(1-\alpha)^2} \right) t} dt \\
= \frac{x^\alpha}{A^\alpha} \frac{1}{\delta - \alpha r - \frac{\alpha(\mu-r)^2}{2(1-\alpha)^2}} < \infty. \]

Furthermore,

\[ \mathbb{E}_x \left[ \int_0^t \sigma^2 (\pi_s^*)^2 ds \right] = \frac{(\mu-r)^2 t}{(1-\alpha)^2 \sigma^2} < +\infty \]

and left continuous \( I^* \) also satisfies

\[ 0 \leq I^*_t(L_t x) = (L_t + \gamma - 1)^+ x \leq L_t x. \]

Therefore, \( u^*_t := \left( \frac{\mu-r}{(1-\alpha)\sigma^2}, \frac{W^*_t}{A}, (L_t + \gamma - 1)^+ W^*_t \right) \) is an optimal policy to Problem 2.1. \( \square \)

### 4.3.1 Loss proportion \( L \) is a constant

In this case, \( \bar{A} \) has the exactly same formula as \( \tilde{A} \) in the previous section.

### 4.3.2 Loss proportion \( L \) is uniformly distributed on \((0, 1)\)

In this example, we find \( \bar{A} \) given by

\[ \bar{A} = \frac{1 - \alpha}{\delta - \alpha r - \frac{\alpha(\mu-r)^2}{2(1-\alpha)^2} + \lambda \left[ \frac{\alpha}{1+\alpha} (1 - \gamma^{1+\alpha}) + \frac{1}{2} \alpha (1 + \theta) \gamma^2 \right]} . \]

### 5 Economical Analysis

In Section 5.1, we analyze how market parameters and investors’ risk aversion affect optimal investment, consumption and insurance strategies. In Section 5.2, we discuss the impact of insurance on the value function and consumption.
5.1 Impact of market and risk aversion on optimal control

According to the results obtained in Section 4, we can rewrite optimal investment proportion $\pi^*$ in the following uniform expression

$$\pi^*_t = \frac{\mu - r}{(1 - \alpha)\sigma^2},$$

where $\alpha = 0$ when $U(y) = \ln(y)$.

From this expression, we can conclude that all investors will allocate more proportion of their capital to the risky asset when economy is booming, in which the expected excess return over variance is higher than that when economy is in recession. Since $\pi^*$ is positively related to the risk aversion tolerance $\alpha$, investors with higher risk tolerance (less risk-averse investors) will spend proportionally more on the risky asset comparing with those with lower risk tolerance.

Optimal consumption rate $c^*$ is proportional to the wealth process $W^*$ for all three utility functions and the proportion (consumption to wealth ratio) is given by

$$\frac{c^*(t)}{W^*_t} = \delta,$$

when $U(y) = \ln(y)$;

$$\frac{c^*(t)}{W^*_t} = \frac{1}{\lambda},$$

when $U(y) = -y^\alpha, \alpha < 0$;

$$\frac{c^*(t)}{W^*_t} = \frac{1}{\lambda},$$

when $U(y) = y^\alpha, 0 < \alpha < 1$.

In order to analyze how risk aversion affects optimal consumption to wealth ratio, we conduct a numerical analysis. We take $\mu = 0.07, \sigma = 0.3, r = 0.05, \theta = 0.25, \lambda = 0.1, \delta = 0.15$. Since the expected value of a random variable uniformly distributed on $(0, 1)$ is $0.5$, so we consider two cases: loss proportion $L$ is constant $L = 0.5$ and $L$ is uniformly distributed on $(0, 1)$. We find optimal ratio in those two cases almost coincides, so in both Figure 1 and Figure 2, we only draw for the case of $L = 0.5$, but depict the difference of optimal ratio between two cases.

From these two graphs, we conclude that optimal consumption to wealth ratio has a positive correlation with $\alpha$, so a less risk-averse investor will consume proportionally more compared with a more risk-averse investor. Mathematically, when $\alpha$ approaches 0, we observe that the optimal consumption
to wealth ratio will converge to 0.15, which is $\delta$ (optimal consumption to wealth ratio when $\alpha = 0$).

For all three HARA utility functions considered in Section 4, optimal insurance can be written as $I_t^* = (L_tW_t^* - d_t)^+$ with deductible $d_t$ being proportional to wealth $W_t^*$. We have such proportion given as

$$\kappa := \frac{d_t}{W_t^*} = 1 - (1 + \theta)^{-\frac{1}{1-\alpha}},$$

where $\alpha = 0$ for logarithm utility function $U(y) = \ln(y)$. 

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Notice that the deductible/wealth ratio is a positive constant, independent of time $t$, which implies the demand for insurance drops when the investor’s wealth is increasing. The deductible/wealth ratio $\kappa$ only depends on premium loading $\theta$ and risk aversion $\alpha$. Since
\[
\frac{\partial \kappa}{\partial \theta} = \frac{1}{1 - \alpha} (1 + \theta)^{-\frac{2}{1 - \alpha}} > 0,
\]
the increase of premium loading $\theta$ will lead to the decrease of insurance demand.

As
\[
\frac{\partial \kappa}{\partial \alpha} = \ln(1 + \theta)(1 - \alpha)^{-2}(1 + \theta)^{-\frac{1}{1 - \alpha}} > 0,
\]
less risk-averse investors will spend less proportion of their wealth on insurance.

5.2 Impact of insurance on value function

Since optimal investment proportion $\pi^*$ is a constant, independent of wealth, and then independent of consumption and insurance, which is due to our choice of HARA utility functions. The performance function $J$ we consider is the expected value of consumption, so we can analyze how insurance affects optimal consumption by examining the impact of insurance on the value function.

To conduct a comparison analysis, we impose a constraint of no insurance to Problem 2.1. With this constraint, the wealth process is given by
\[
dW_t = (rW_t + (\mu - r)\pi_tW_t - c_t)dt + \sigma\pi_tW_tdB_t - L_tW_tdN_t,
\]
with $W(0) = x$.

We then obtain the associated HJB equation as
\[
\sup_{u \in (-\infty, \infty) \times [0, \infty)} \{G_u v(x) + U(c) + \lambda E[v(x - Lx) - v(x)]\} = 0,
\]
where $\forall v \in C^2(0, \infty)$, the operator $G$ is defined by
\[
G_u(v(x)) := (rx + (\mu - r)\pi x - c)v'(x) + \frac{1}{2}\sigma^2\pi^2x^2v''(x) - \delta v(x).
\]
5.2.1 \( U(y) = \ln(y), \ y > 0 \)

We find the corresponding value function is given by

\[
\hat{v}(x) = \frac{1}{\delta} \ln(\delta x) + \hat{A},
\]

where

\[
\hat{A} = \frac{1}{\delta^2} \left( r - \delta + \frac{(\mu - r)^2}{2\sigma^2} + \lambda E[\ln(1 - L)] \right).
\]

If loss proportion \( L \) is constant and \( L = l \leq \frac{\theta}{1 + \theta} \), then by the analysis in Section 4, we have \( I^* = 0 \). So in this case, \( v(x) = \hat{v}(x) \).

If loss proportion \( L \) is constant and \( L = l > \frac{\theta}{1 + \theta} \), define

\[
g(l) := A - \hat{A} = \frac{\lambda}{\delta^2} \left( \ln\left( \frac{1}{1 + \theta} \right) + \theta - l(1 + \theta) - \ln(1 - l) \right).
\]

Since \( g'(l) > 0 \) for \( l > \frac{\theta}{1 + \theta} \) and \( g\left( \frac{\theta}{1 + \theta} \right) = 0 \), we obtain \( A - \hat{A} > 0 \) and then \( v(x) > \hat{v}(x) \).

If \( L \) is uniformly distributed on \((0, 1)\), then \( A - \hat{A} = \frac{\lambda}{\delta^2} \left( 1 - \frac{1 + 2\theta}{2(1 + \theta)} \right) > 0 \), and so we have \( v(x) > \hat{v}(x) \).

5.2.2 \( U(y) = y^\alpha, \ 0 < \alpha < 1, \ y > 0 \)

In this case, the value function to constrained problem is

\[
\hat{v}(x) = \hat{A}^{1-\alpha} x^\alpha, \ \hat{A} > 0,
\]

where constant \( \hat{A} \) is given by

\[
\hat{A} = \frac{1 - \alpha}{\delta - \alpha r - \frac{\alpha(\mu - r)^2}{2(1 - \alpha)\sigma^2} + \lambda(1 - E(1 - L)^\alpha)}.
\]

Since \( 0 < \alpha < 1 \), the condition (10) also guarantees that \( \hat{A} > 0 \).

If loss proportion \( L \) is constant and \( L = l \leq 1 - \gamma \), then \( I^* = 0 \) and so \( \bar{v}(x) = \hat{v}(x) \).

If loss proportion \( L \) is constant and \( L = l > 1 - \gamma \), define

\[
u(l) := \alpha(1 + \theta)(l + \gamma - 1) + (1 - l)^\alpha - \gamma^\alpha.
\]
Then \( u'(l) = \alpha[(1 + \theta) - (1 - l)^{-(1 - \alpha)}] < 0 \) for \( l > 1 - \gamma \), which implies \( u(l) < u(1 - \gamma) = 0 \), \( \forall l > 1 - \gamma \) and thus \( \bar{A} > \dot{A} > 0 \). So we obtain \( \bar{v}(x) > \dot{v}(x) \).

If \( L \) is uniformly distributed on \((0, 1)\), then

\[
\left[ \frac{\alpha}{1 + \alpha}(1 - \gamma^{1+\alpha}) + \frac{1}{2} \alpha(1 + \theta) \gamma^2 \right] - \left( 1 - \frac{1}{1 + \alpha} \right) = -\frac{\alpha(1 - \alpha)}{2(1 + \alpha)}(1 + \theta)^{-\frac{1+\alpha}{1-\alpha}} < 0,
\]

which implies \( \bar{A} > \dot{A} \) and then \( \bar{v}(x) > \dot{v}(x) \).

Based on the results obtained above, we can claim that investors with option to purchase insurance coverage will achieve a greater value function when no insurance is not optimal. To examine how insurance affects consumption, we can define value function as \( V(t, w) = \sup E[\int_t^\infty e^{-\delta s}U(c_s)ds|W_t = w] \).

By following a similar argument, we can obtain \( v(t, x) \geq \hat{v}(t, x) \), where equality only holds when \( I^* = 0 \). Since utility function \( U \) is strictly increasing, we conclude \( c^*(t) > \hat{c}^*(t) \) as long as \( I^* \neq 0 \).

### 6 Conclusion

This paper considers a risk-averse investor who faces a random insurable risk apart from the risk involved in the investment. The investor wants to maximize her expected utility of consumption in infinite time horizon by choosing investment proportion in the risky asset, consumption rate and insurance policy.

We obtain the value function and optimal strategies in explicit forms for three HARA utility functions. Through an economic analysis, we observe that optimal investment proportion in the risky asset is constant for HARA utility functions, and will be greater when economy is booming. Less risk-averse investors will spend proportionally more on the risky asset compared with more risk-averse investors. Optimal consumption is proportional to wealth, and we find such proportion is positively correlated with the investor’s risk aversion parameter \( \alpha \), which means less risk-averse investors spend a greater proportion of their wealth on consumption. We claim Arrow’s finding for optimal insurance in [1] still holds in continuous time, that is, optimal insurance is a deductible insurance. Furthermore, we conclude
that the increase of wealth, premium loading \( \theta \) or risk aversion parameter \( \alpha \) can contribute to the decrease of insurance demand. According to a comparative analysis, we find having the option to purchase insurance will enable investors to achieve a greater value function, and then to spend more in consumption.

References


