Optimal Investment and Liability Ratio Policies when There Is Regime Switching

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Abstract

We consider an insurer who faces an external jump-diffusion risk that is negatively correlated with the capital returns in the financial market. We assume not only the financial market but also the risk process depend on the regime of the economy. The insurer selects investment and liability ratio policies continuously to maximize its expected utility of terminal wealth. We obtain explicit solutions for optimal investment and liability ratio policies for logarithmic, power and exponential utility functions.

Key Words: jump diffusion; Markov chain; risk management; stochastic control; utility maximization.

1 Introduction

1.1 Economic Motivations

The 2007-2008 financial crisis and economic recession almost put the global financial system on the brink of collapse. With the accurate total cost of the financial crisis being incalculable, economists estimated a conservative number of the loss in the U.S. would be $6 trillion to $14 trillion, or equivalently,

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$50,000 to $120,000 for every U.S. household (See Atkinson et al. (2013)). To further illustrate how severe this financial crisis is, we review the staggering case of American International Group, Inc. (AIG), once one of the largest and most successful insurance companies in the world. AIG’s stock price was traded at as high as over $50 per share in February 2008 before the financial crisis, but dropped to less than $2 per share in September 2008 when AIG was deep in the crisis. To prevent the financial system from breakdown, the U.S. government took over AIG through an initial rescue of $85 billion in September 2008, the largest bailout amount in the U.S. history. According to record, the total amount of rescue in the AIG case is over $182 billion (See Sjostrom (2009) for more statistical and economic numbers).

Apart from the huge impact on the industry and the financial markets, this financial crisis also brought intense discussions and research to the academic field. Many academics investigate the influence of complicated financial products, such as credit default swaps (CDS), on capital structure, debate over monetary policies, government intervention and regulation of the markets, discuss how to regulate systemic risk, and among others. In short, a lot of work has been done regarding the 2007-2008 financial crisis in economics and finance. But among all these research, few of them targeted quantitatively on investment and risk management problems, not to mention propose new generalizations and improvements on mathematical models with regards to the financial crisis.

1.2 Review of Investment-Consumption Models and Reinsurance Models

The study on investment/consumption problems started with the seminal paper Merton (1969) in which dynamic programming was applied to find explicit optimal investment and consumption policies. Karatzas et al. (1986) provided a more general and rigorous analysis to Merton’s problem, including arguments for whether the positive constraint of consumption is active, different scenarios for the natural payments, and conditions under which the value function is finite. Karatzas et al. (1991) revisited the same problem in an incomplete market using martingale and duality methods. Many early contributions to investment/consumption problems can be found in the monograph Sethi (1997). Browne (1995) further extended Merton’s framework by assuming that investors are subject to an external risk process (modeled by a
diffusion process). He then sought to obtain optimal investment policies for investors under two criteria: maximizing exponential utility and minimizing the probability of ruin. Following the same vein, Yang and Zhang (2005) and Wang et al. (2007) used a jump-diffusion process to model investors’ external risk. They both obtained explicit optimal investment strategies, but used different methods: the former by solving the associated HJB equations and the latter through martingale approach. In optimal investment problems with an external risk process, e.g., above mentioned Browne (1995), Wang et al. (2007), Yang and Zhang (2005), investors rather leave their external risk processes uncontrolled, which means the risk process is independent of investment decisions. Zou and Cadenillas (2014b) incorporated an external risk (which can be insured against through the purchase of insurance policies) into Merton’s model and considered consumption, investment and insurance problems when there is regime switching in the economy.

In the insurance industry, a commonly used risk management tool for insurers is reinsurance. In a typical reinsurance problem, an insurer manages its risk exposure by controlling reinsurance strategies under certain objectives. In discrete time, risk can be modeled by a compound Poisson process, also called Cramér-Lundberg Model (e.g., Schmidli (2001)). While in continuous time, diffusion process (with or without jumps) is often used to model risk process (See, for instance, Hojgaard and Taksar (1998), Taksar (2000)). Regarding common reinsurance types used in the literature, there are proportional reinsurance (see Hojgaard and Taksar (1998)), step loss reinsurance (see Kaluszka (2001)) and among others. Academics also study reinsurance problems under various objectives, such as mean-variance criterion in Kaluszka (2001), maximizing expected utility of running reserve in Hojgaard and Taksar (1998), minimizing the probability of ruin in Schmidli (2001), maximizing expected utility of terminal wealth in Liu et al. (2013).

1.3 Review of the AIG Case

As pointed out in Stein (2012, Chapter 6), one major mistake AIG made is ignore the negative correlation between its liabilities and the capital returns. Such correlation is also ignored in investment/consumption problems with external risk process and reinsurance problems with investment. To overcome this drawback, Stein (2012, Chapter 6) proposed a diffusion model for AIG’s risk process that is negatively related to the stock price process. He then found optimal liability ratio for AIG when its objective is to maximize the
expected logarithmic utility of terminal wealth. Zou and Cadenillas (2014a) added investment as a control and obtained optimal investment and liability ratio strategies under HARA, CARA and quadratic utility functions using the martingale method.

As agreed by most economists, the trigger of the 2007-2008 financial crisis is the crash of the housing market. But back at that time, most individual investors, companies, financial institutions and banks did not seriously consider the business cycles in the U.S. housing market and made their financial decisions based on the false prediction of the housing price index. In the AIG case, AIG Financial Products Corp. (AIGFP), AIG’s subsidiary, significantly underestimated the risk of writing CDS backed by mortgage payments. To manager the risk arose from business cycles, regime switching models should be applied (e.g., Sotomayor and Cadenillas (2009), Zhou and Yin (2004), Zou and Cadenillas (2014b) and the references therein for more details). Bauerle and Rieder (2004) considered portfolio optimization problems in a regime switching market under the utility maximization criterion. In Sotomayor and Cadenillas (2009), they further assumed the utility function is also regime dependent and obtained explicit consumption and investment policies. Zhou and Yin (2004) also studied Merton’s problem in a regime switching model but under Markowitz’s mean-variance criterion. Regime switching models can also be found in reinsurance problems, see, for instance, Liu et al. (2013) and Zhuo et al. (2013).

1.4 Contributions

Motivated by the infamous AIG case mentioned above, we propose a regime switching model which addresses two major mistakes AIG made in the financial crisis. We consider an insurer whose external risk (liabilities) can be modeled by a jump-diffusion process and suppose the insurer can control the risk process. We assume the insurer makes investment decisions in a financial market which consists of a riskless asset and a risky asset. Following Stein (2012, Chapter 6), we also assume the insurer’s risk process is negatively correlated with the price process of the risky asset. In our model, both the financial market and the risk process depend on the regime of the economy. The objective of the insurer is to select the proportion of wealth invested in the risky asset and the liability ratio (which is defined as total liabilities over wealth) to maximize its expected utility of terminal wealth.

As far as we are concerned, this is by far the first paper studying in-
vestment and liability ratio problems when there is regime switching in the economy. We also successfully obtain optimal investment and liability ratio policies in explicit forms for logarithmic utility, power utility and exponential utility. Stein (2012, Chapter 6) considered a similar problem under the same criterion, but in a much simpler way compared with our work. First, he did not consider regime switching in the model. Second, the insurer does not control investment decisions and the risk is modeled by a diffusion process without jumps. Last, the only utility function considered in Stein (2012, Chapter 6) is logarithmic function. Different from investment/consumption problems with regime switching, e.g., Bauerle and Rieder (2004), Sotomayor and Cadenillas (2009), Zhou and Yin (2004), our model incorporates an external risk process. Our research also differs from recent work in reinsurance problems in several directions. For instance, in Liu et al. (2013), the insurer’s risk process is governed by a continuous diffusion process (without jumps) and is assumed to be independent of the price process of the securities. In Zhuo et al. (2013), investment is not included and they only provided numerical solutions.

This paper is organized as follows. In Section 2, we describe the regime switching model and formulate the problem. In Section 3, we develop the associated Hamilton-Jacobi-Bellman equation and provide the corresponding verification theorem. In Section 4, we obtain explicit solutions of optimal investment and liability ratio policies for logarithmic utility, power utility and exponential utility functions. In Section 5, we present an economic analysis. Section 6 concludes our work.

2 The Regime-switching Model

We consider a model that presents observable regime switching features. The regime of the economy is represented by an observable, continuous and stationary Markov chain \( \epsilon = \{ \epsilon_t, 0 \leq t \leq T \} \) with finite state space \( S = \{1, 2, \ldots, S\} \), where \( T > 0 \) is the terminal time and \( S \in \mathbb{N}^+ \) is the number of regimes in the economy. We assume the Markov Chain \( \epsilon \) has a strongly irreducible generator \( Q = (q_{ij})_{S \times S} \), where \( \sum_{j \in S} q_{ij} = 0 \) for all \( i \in S \).

In the financial market, there are two trading assets, namely, a riskless asset and a risky asset. The price processes of the riskless asset and the risky asset are represented by \( P_0 \) and \( P_1 \), respectively, which satisfy the Markov-
modulated stochastic differential equations:

\[ dP_0(t) = r_{\epsilon(t)} P_0(t) dt, \]
\[ dP_1(t) = P_1(t) (\mu_{\epsilon(t)} dt + \sigma_{\epsilon(t)} dW^{(1)}(t)), \]

where \( t \in [0, T] \) and the initial conditions are \( P_0(0) = 1 \) and \( P_1(0) > 0 \). The coefficients \( r_i, \mu_i, \sigma_i, i \in \mathcal{S} \), are all positive constants, and \( W^{(1)} \) is a standard one-dimensional Brownian motion defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

We assume insurers are subject to a controllable external risk (liabilities). Following Wang et al. (2007), we further assume the unit risk (risk per dollar amount of liabilities) is modelled by a jump-diffusion process

\[ dR(t) = a_{\epsilon(t)} dt + b_{\epsilon(t)} d\bar{W}(t) + \gamma_{\epsilon(t)} dN(t), \]

where \( \bar{W} \) is another standard one-dimensional Brownian motion and \( N \) is a Poisson process with constant intensity \( \lambda > 0 \). For all \( i \in \mathcal{S} \), the coefficients \( a_i, b_i \) and \( \gamma_i \) are positive constants. As discussed in Stein (2012, Chapter 6), we assume the risk process \( R \) is negatively correlated with the capital gains in the financial market. Hence we can rewrite \( \bar{W} \) as

\[ \bar{W}(t) = \rho W^{(1)}(t) + \sqrt{1 - \rho^2} W^{(2)}(t), \]

where \( \rho < 0 \), and \( W^{(2)} \) is a standard Brownian motion defined on \((\Omega, \mathcal{F}, \mathbb{P})\) independent of \( W^{(1)} \). We assume the unit premium (per dollar amount of liabilities) at time \( t \) is \( p_{\epsilon(t)} \), where \( p_i > 0 \) for all \( i \in \mathcal{S} \). Therefore, insurers’ unit profit (loss if being negative) over the time period \((t, t+dt)\) is \( p_{\epsilon(t)} dt - dR(t) \).

Denote the insurer’s total liabilities at time \( t \) by \( L(t) \). Then the dynamics of the insurer’s total profit is given by

\[ L(t) \left( p_{\epsilon(t)} dt - dR(t) \right). \]

Following Sotomayor and Cadenillas (2009), we assume the Brownian motions \( W^{(1)} \) and \( W^{(2)} \), the Poisson process \( N \) and the Markov chain \( \epsilon \) are mutually independent. We take the \( \mathbb{P} \)-augmented filtration generated by \( W^{(1)}, W^{(2)}, N \) and \( \epsilon \) as our filtration \( \{ \mathcal{F}_t \}_{0 \leq t \leq T} \).

Remark 2.1 The above model for risk process can be understood as a limiting process of the classic Cramér-Lundberg model in continuous time, see, e.g., Taksar (2000), Wang et al. (2007), Zhuo et al. (2013).
Remark 2.2 We assume the coefficients satisfy \( \mu_i > r_i > 0 \) and \( p_i > a_i > 0 \) for all \( i \in S \). Such assumption is reasonable, and is also in accordance with the financial numbers in real life. This is due to the well accepted conclusion that extra uncertainty must be compensated by extra return.

At time \( t \), an insurer selects \( \pi(t) \), fraction of wealth invested in the risky asset, and liability ratio \( \kappa(t) \), defined as the ratio of total liabilities over wealth. Define control \( u := \{(\pi(t), \kappa(t))\}_{t \in [0, T]} \). For any control \( u \), we denote \( X^u(t) \) as the insurer’s wealth (surplus) at time \( t \), and thus \( L(t) = \kappa(t)X^u(t) \).

Based on the above model setting, we have

\[
dX^u(t) = r\epsilon(t)(1-\pi(t))X^u(t)dt + \frac{\pi(t)X^u(t)}{P_1(t)}dP_1(t) + \kappa(t)X^u(t)(p\epsilon(t)dt - dR(t)).
\]

Hence the dynamics of \( X^u(t) \) is given by

\[
\frac{dX^u(t)}{X^u(t)} = (r\epsilon(t) + (\mu\epsilon(t) - r\epsilon(t))\pi_t + (p\epsilon(t) - a\epsilon(t))\kappa_t)dt - \gamma\epsilon(t)\kappa_t dN_t
\]

(2.1)

\[+(\sigma\epsilon(t)\pi_t - \rho b\epsilon(t)\kappa_t)dW^{(1)}_t - \sqrt{1 - \rho^2 b\epsilon(t)\kappa_t}dW^{(2)}_t,
\]

with \( X^u(0) = x > 0 \).

We denote \( A_{x,i} \) as the set of all admissible controls under the initial conditions \( X^u(0) = x \) and \( \epsilon(0) = i \), where \( x > 0 \) and \( i \in S \). For all \( u \in A_{x,i} \), \( u \) is a predictable process with respect to the filtration \( \{\mathcal{F}_t\} \), and satisfies

\[
\int_0^T (\pi(t))^2 dt \leq \infty, \quad \int_0^T (\kappa(t))^2 dt \leq \infty,
\]

and

\[0 \leq \kappa(t) \leq \frac{1}{\gamma\epsilon(t)}, \text{ for all } t \in [0, T].\]

Since the coefficients of all \( dt \), \( dW^{(1)} \), \( dW^{(2)} \) and \( dN \) terms are bounded for every \( u \in A_{x,i} \), by Oksendal and Sulem (2005, Theorem 1.19), there exists a unique solution \( X^u_t \) to SDE (2.1) such that

\[E[|X^u_t|^2] < \infty \quad \text{for all } t \in [0, T].\]

We define the criterion functional \( J \) by

\[J(x, i; u) := E_{x,i}[U(X^u(T))].\]
where the utility function $U$ is strictly increasing and concave, and satisfies the linear growth condition. Notation $E_{x,i}$ means conditional expectation given $X^u(0) = x$ and $\epsilon(0) = i$ under the actual measure $P$.

We then formulate optimal investment and liability ratio problem as follows.

**Problem 2.1** Select an admissible control $u^* = (\pi^*, \kappa^*)$ that attains the value function $V$, defined by

$$V(x, i) := \sup_{u \in A_{x,i}} J(x, i; u).$$

The control $u^*$ is called an optimal control or optimal strategies of investment and liability ratio.

To apply stochastic control method to solve Problem 2.1, we consider a modified version of Problem 2.1 (See, e.g., Bauerle and Rieder (2004), Fleming and Soner (1993, Chapter III) for details)

$$V(t, x, i) := \sup_{u \in A_{t,x,i}} E_{t,x,i}[U(X_t^u)],$$

where $A_{t,x,i}$ is defined similarly as $A_{x,i}$ except for the starting point of time being $t$ instead of 0. The notation $E_{t,x,i}$ means conditional expectation under $X_t^u = x$ and $\epsilon(t) = i$.

### 3 The Analysis

Let $\psi(t, x, i)$ be a $C^{1,2}$ function for all $i \in S$. Define operator $\mathcal{L}_i^u$ by

$$\mathcal{L}_i^u \psi := \psi_t(t, x, i) + [r_i + (\mu_i - r_i)\pi + (p_i - a_i)\kappa]x\psi_x(t, x, i) + \frac{1}{2}[(\sigma_i \pi - \rho b_i \kappa)^2 + (1 - \rho^2)b_i^2 \kappa^2]x^2\psi_{xx}(t, x, i),$$

where $\pi \in \mathbb{R}$ and $0 \leq \kappa < \frac{1}{\gamma_i}$.

**Theorem 3.1** Suppose $v(\cdot, \cdot, i) \in C^{1,2}$ and $v(t, \cdot, i)$ be an increasing and concave function for all $t \in [0, T]$ and $i \in S$. If $v(t, x, i)$ satisfies the Hamilton-Jacobi-Bellman equation

$$(3.2)$$

$$\sup_{u \in \mathbb{R} \times [0, \frac{1}{\gamma_i})} \left\{ \mathcal{L}_i^u v(t, x, i) + \lambda[v(t, (1 - \gamma_i \kappa)x, i) - v(t, x, i)] \right\} = -\sum_{j \in S} q_{ij} v(t, x, j)$$

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and the boundary condition

\[ v(T, x, i) = U(x) \]

for every \( x > 0, i \in \mathcal{S} \), and the control \( u^* = (\pi^*, \kappa^*) \) defined by

\[ u^* = \arg \sup_{u \in \mathbb{R} \times [0, 1]} \left\{ \mathcal{L}_i^u v(t, x, i) + \lambda [v(t, (1 - \gamma_t \kappa)x, i) - v(t, x, i)] \right\} \]

is admissible, then \( u^* \) is optimal control to Problem 2.1 and \( v(t, x, i) \) is the associated value function.

Proof. \( \forall u \in \mathcal{A}_{t,x,i} \), by applying Markov-modulated Ito’s formula (see, e.g., Sotomayor and Cadenillas (2009)), we obtain

\[ v(\theta, X^u_{\theta}, \epsilon_{\theta}) = v(t, X^u_t, \epsilon_t) + \int_t^\theta \left( \mathcal{L}^{u}_{\epsilon(s)} v(s, X^u_s, \epsilon_s) + \sum_{j \in \mathcal{S}} q_{\epsilon(s), j} v(s, X^u_s, j) \right) ds \]

\[ + \int_t^\theta X^u_s \epsilon_x(s, X^u_s, \epsilon_s) (\sigma_{\epsilon(s)} \pi_s - \rho b_{\epsilon(s)} \kappa_s) dW^{(1)}_s \]

\[ - \int_t^\theta X^u_s \epsilon_x(s, X^u_s, \epsilon_s) \sqrt{1 - \rho^2 b_{\epsilon(s)} \kappa_s} dW^{(2)}_s \]

\[ + \int_t^\theta \left( v(s, (1 - \gamma_{\epsilon(s)} \kappa_s) X^u_{s-}, \epsilon_s) - v(s, X^u_{s-}, \epsilon_s) \right) dN_s + m^v_\theta, \]

where \( m^v \) is a square-integrable martingale and \( m^v_0 = 0 \).

For \( u \in \mathcal{A}_{t,x,i}, X^u_s, \pi_s \) and \( \kappa_s \) are all bounded \( \mathbb{P} \)-a.s. for all \( s \in [t, T] \). By assumption, \( v_x \) is bounded on \( [t, T] \) as well. Hence we have

\[ E_{t,x,i} \left[ \int_t^\theta X^u_s \epsilon_x(s, X^u_s, \epsilon_s) (\sigma_{\epsilon(s)} \pi_s - \rho b_{\epsilon(s)} \kappa_s) dW^{(1)}_s \right] = 0, \]

\[ E_{t,x,i} \left[ \int_t^\theta X^u_s \epsilon_x(s, X^u_s, \epsilon_s) \sqrt{1 - \rho^2 b_{\epsilon(s)} \kappa_s} dW^{(2)}_s \right] = 0. \]

The function \( v(s, (1 - \gamma_{\epsilon(s)} \kappa_s) X^u_{s-}, \epsilon_s) - v(s, X^u_{s-}, \epsilon_s) \) is left continuous and bounded, thus

\[ E_{t,x,i} \left[ \int_t^\theta \left( v(s, (1 - \gamma_{\epsilon(s)} \kappa_s) X^u_{s-}, \epsilon_s) - v(s, X^u_{s-}, \epsilon_s) \right) dM_s \right] = 0, \]

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where $M_t$, defined as $M_t = N_t - \lambda t$, is the compensated Poisson process of $N$, and then a true martingale under measure $\mathbb{P}$.

Hence, taking conditional expectation for $v(\theta, X^u_{\theta}, \epsilon_{\theta})$ yields

$$E_{t,x,i}[v(\theta, X^u_{\theta}, \epsilon_{\theta})] = v(t, x, i) + \int_t^\theta \left( \mathcal{L}^u_{s} v(s, X^u_{s}, i) + \lambda [v(s, (1 - \gamma_i \kappa)x_{s}, i) - v(s, X^u_{s}, i)] \right) ds + \int_t^\theta \sum_{j \in S} q_{t(s),j} v(s, X^u_{s}, j) ds,$$

which directly implies the HJB equation (3.2). Then $v$ defined in Theorem 3.1 is the value function to modified Problem 2.1. Given $u^*$ is admissible, $u^*$ is optimal control to Problem 2.1 (See, e.g., Fleming and Soner (1993, Chapter III), Oksendal and Sulem (2005, Chapter 3) for analysis). □

4 Construction of Explicit Solutions

In this section, we obtain explicit solutions to Problem 2.1 in a regime switching market. Our strategy is to conjecture a strictly increasing and strictly concave candidate for the value function and then obtain candidate for optimal control. We next verify that such candidate control is admissible and then indeed optimal.

To obtain candidate for optimal control, we separate the optimization problem in the HJB equation (3.2) into two sub optimization problems

$$\max_{\pi \in \mathbb{R}} \left[ xv_x(t, x, i)(\mu_i - r_i)\pi + \frac{1}{2}x^2 v_{xx}(t, x, i)(\sigma^2_i \pi^2 - 2\rho \sigma_i b_i \pi \kappa) \right]$$

for investment portfolio $\pi$, and

$$\max_{\kappa \in [0, \frac{1}{\gamma_i}]} \left\{ xv_x(t, x, i)(p_i - a_i)\kappa + \frac{1}{2}x^2 v_{xx}(t, x, i)(-2\rho \sigma_i b_i \pi \kappa + b_i^2 \kappa^2) + \lambda v(t, (1 - \gamma_i \kappa)x_{i}) \right\}$$

for liability ratio $\kappa$.

Under the assumption that $v(t, \cdot, i)$ is strictly increasing and strictly concave, we obtain the candidate of $\pi^*$ as

$$\pi^* = -\frac{v_x(t, x, i)(\mu_i - r_i)}{xv_{xx}(t, x, i)\sigma^2_i} + \rho \frac{b_i}{\sigma_i} \kappa^*,$$  

(4.4)
where the candidate of $\kappa^*$ satisfies
\begin{equation}
 x v_{xx}(t, x, i) (1 - \rho^2) b_i^2 \kappa^* - \lambda \gamma_i v_x(t, (1 - \gamma_i \kappa^*) x, i) \\
 + v_x(t, x, i) (p_i - a_i + \frac{b_i}{\sigma_i} (\mu_i - r_i)) = 0.
\end{equation}

(4.5)

To guarantee the equation (4.5) has a unique solution, we impose a technical condition
\begin{equation}
p_i - a_i + \frac{b_i}{\sigma_i} (\mu_i - r_i) > \lambda \gamma_i \text{ for all } i \in \mathcal{S}.
\end{equation}

(4.6)

To obtain the value function $v$ and optimal control $u^* = (\pi^*, \kappa^*)$ in explicit forms, we consider three utility functions
1. $U(x) = \ln(x), \ x > 0$,
2. $U(x) = \frac{1}{\alpha} x^\alpha, \ x > 0$, where $\alpha < 1$ and $\alpha \neq 0$,
3. $U(x) = -\frac{1}{\alpha} e^{-\alpha x}, \text{ where } \alpha > 0$.

4.1 $U(x) = \ln(x), \ x > 0$

In this case, we find the solution to HJB equation (3.2) is
\[ v(t, x, i) = \ln(x) + g(t, i), \]
where $g(t, i)$ will be determined below.

We obtain $v_x(t, x, i) = \frac{1}{x}$ and $v_{xx}(t, x, i) = -\frac{1}{x^2}$, then
\begin{equation}
\pi^* = \frac{\mu_i - r_i}{\sigma_i^2} + \frac{b_i}{\sigma_i} \kappa^*,
\end{equation}

(4.7)

and
\begin{equation}
A_i (\kappa^*)^2 - B_i \kappa^* + C_i = 0,
\end{equation}

(4.8)

with
\begin{equation}
A_i := (1 - \rho^2) b_i^2 \gamma_i, \\
B_i := (1 - \rho^2) b_i^2 + \gamma_i \left( p_i - a_i + \frac{b_i}{\sigma_i} (\mu_i - r_i) \right), \\
C_i := p_i - a_i + \frac{b_i}{\sigma_i} (\mu_i - r_i) - \lambda \gamma_i.
\end{equation}

(4.9)
Lemma 4.1 If the technical condition (4.6) holds, or equivalently, \( C_i > 0 \) for all \( i \in S \), then there exists a unique solution in \([0, \frac{1}{\gamma_i}]\) to the equation (4.8).

Proof. We calculate the discriminant of the quadratic equation system (4.8)

\[
\Delta_i := B_i^2 - 4A_iC_i = (b_i^2(1 - \rho^2) - \gamma_i(C_i + \lambda \gamma_i))^2 + 4\lambda b_i^2(1 - \rho^2)\gamma_i^2 > 0.
\]

Therefore there are two solutions to the equation (4.8), and one of them is given by

\[
\kappa_i^+ = \frac{B_i + \sqrt{\Delta_i}}{2A_i} > \frac{1}{\gamma_i},
\]

while the other is given by

\[
\kappa_i^- = \frac{B_i - \sqrt{\Delta_i}}{2A_i}.
\]

(4.10)

For all \( i \in S \), \( A_i > 0 \). Along with the assumption \( C_i > 0 \) and the result \( \kappa_i^+ > \frac{1}{\gamma_i} > 0 \), we obtain \( \kappa_i^- > 0 \).

To show \( \kappa_i^- < \frac{1}{\gamma_i} \), it is equivalent to show that

\[
\Delta_i > \left( \frac{B_i - 2A_i}{\gamma_i} \right)^2 = (\gamma_i(C_i + \lambda \gamma_i) - (1 - \rho^2)b_i^2)^2.
\]

Recall the definition of \( \Delta_i \), the above inequality is always satisfied.

Denote \( \kappa^*(x, i) := \kappa_i^- \) as the unique solution in \([0, \frac{1}{\gamma_i}]\) to the equation (4.8) when \( C_i > 0 \) for all \( i \in S \).

By substituting candidate strategies \( \pi^* \) and \( \kappa^* \), given by (4.7) and (4.10), into the HJB equation (3.2), we obtain the following system of linear differential equations:

\[
(4.11) \quad g_i(t, i) + \sum_{j \in S} q_{ij} g(t, j) + \Pi_i = 0
\]

with boundary condition

\[
g(T, i) = 0 \text{ for all } i \in S.
\]

In (4.11), \( \Pi_i \) is defined as \( \Pi_i := r_i + \frac{(\mu_i - r_i)^2}{2\sigma_i^2} + \left(p_i - a_i + \rho \frac{b_i}{\sigma_i}(\mu_i - r_i)\right)\kappa^* + \lambda \ln(1 - \gamma_i\kappa^*) - \frac{1}{2}(1 - \rho^2)b_i^2(\kappa^*)^2.
\]
Theorem 4.1 When \( U(x) = \ln(x) \), \( u^* = (\pi^*, \kappa^*) \), where
\[
\pi^*(t) = \frac{\mu_i(t) - r_i(t)}{\sigma_i^2} + \frac{b_i(t)}{\sigma_i^2} \kappa^*(t),
\]
and \( \kappa^*(t) \) is the unique solution to the equation
\[
A_i(t)(\kappa^*(t))^2 - B_i(t) \kappa^*(t) + C_i(t) = 0,
\]
is optimal control to Problem 2.1.

Proof. First notice that the ODE system (4.11) has a unique solution (See Bauerle and Rieder (2004)) and then \( v(t, x, i) = \ln(x) + g(t, i) \) is the value function to modified Problem 2.1. By Lemma 4.1, there exists a unique solution in \([0, \frac{1}{\gamma_i(t)}) \) to the equation (4.8). The boundedness of \( \pi^* \) and \( \kappa^* \) implies that both are square integrable in \([0, T] \). Hence, \( u^* = (\pi^*, \kappa^*) \) defined above is admissible, and then optimal control to Problem 2.1.

4.2 \( U(x) = \frac{1}{\alpha}x^\alpha, x > 0, \) where \( \alpha < 1 \) and \( \alpha \neq 0 \)

In this case, the utility function is given by \( U(x) = \frac{1}{\alpha}x^\alpha \), where \( \alpha < 1, \alpha \neq 0 \). This utility function is of constant relative risk aversion (CRRA) type and the relative risk aversion coefficient is \( 1 - \alpha \).

The solution to the HJB (3.2) is given by
\[
v(t, x, i) = \frac{1}{\alpha}x^\alpha \cdot \hat{g}(t, i),
\]
where \( \hat{g}(t, i) > 0 \) for all \( i \in S \) will be determined below.

Next, we obtain the candidate for optimal control
\[
(4.12) \quad \pi^* = \frac{\mu_i - r_i}{(1 - \alpha)\sigma_i^2} + \frac{b_i}{\sigma_i^2} \kappa^*,
\]
and
\[
(4.13) \quad (\alpha - 1)(1 - \rho^2)b_i^2 \kappa^* - \lambda \gamma_i(1 - \gamma_i \kappa^*)^{\alpha - 1} + p_i - a_i + \rho \frac{b_i}{\sigma_i}(\mu_i - r_i) = 0.
\]

Lemma 4.2 If the condition (4.6) holds, then there exists a unique solution in \([0, \frac{1}{\gamma_i}) \) to the equation (4.13).
Proof. Let $\phi_i := 1 - \gamma_i\kappa^*$ and define $\hat{B}_i$ and $\hat{C}_i$ by

\begin{align}
\hat{B}_i := & \frac{(\alpha - 1)(1 - \rho^2)b_i^2}{\lambda \gamma_i^2}, \\
\hat{C}_i := & -\frac{1}{\lambda \gamma_i} \left[ p_i - a_i + \rho \frac{b_i}{\sigma_i}(\mu_i - r_i) \right] - \hat{B}_i.
\end{align}

(4.14)

Then to show there exists a unique solution in $[0, \frac{1}{\gamma_i}]$ to the equation (4.13), we only need to prove that the following equation has a unique solution in $(0, 1]$

$$\phi_i^{\alpha - 1} + \hat{B}_i \phi_i + C_i = 0.$$  

Consider $\hat{h}_i(x) := x^{\alpha - 1} + \hat{B}_i x + C_i$. At the two end points, we have

$$\hat{h}_i(0) = \lim_{x \to 0^+} \hat{h}_i(x) = +\infty,$$

$$\hat{h}_i(1) = 1 - \frac{1}{\lambda \gamma_i} \left[ p_i - a_i + \rho \frac{b_i}{\sigma_i}(\mu_i - r_i) \right] < 0,$$

where the above inequality comes from the condition (4.6). Furthermore, we have $\hat{h}_i'(x) = (\alpha - 1)x^{\alpha - 2} + B_i < 0$ and $\hat{h}_i(x)$ is continuous in $(0, 1)$, which together give the desired result. $\square$

By plugging candidate control into the HJB equation (3.2), we obtain

\begin{equation}
\hat{g}_t(t, i) + \sum_{j \in S} q_{ij} \hat{g}_t(t, j) + \alpha \hat{\Pi}_i \hat{g}_t(t, i) = 0
\end{equation}

with the boundary condition

$$\hat{g}(T, i) = 1 \text{ for all } i \in S.$$  

Here $\hat{\Pi}_i$ is defined by $\hat{\Pi}_i := r_i + \frac{2(\mu_i - r_i)^2}{2(1 - \alpha)\sigma_i^2} + \left( p_i - a_i + \rho \frac{b_i}{\sigma_i}(\mu_i - r_i) \right) \kappa^* + \lambda[(1 - \gamma_i\kappa^*)^\alpha - 1] - \frac{1}{2}(1 - \alpha)(1 - \rho^2)b_i^2(\kappa^*)^2$.

We remark that the above ODE system has a unique solution. Furthermore, to verify our conjecture that $v(t, \cdot, i)$ is strictly increasing and concave, we need to show $\hat{g}(t, i)$ is strictly positive for all $i \in S$. The lemma below provides the proof for $\hat{g}(t, i) > 0$.

Lemma 4.3 The function $\hat{g}(t, i)$, which is the unique solution to the system (4.15), is strictly positive.
Proof. Using Ito’s formula for Markov-modulated process, we obtain

$$\hat{g}(T, \epsilon_T) = \hat{g}(t, \epsilon_t) + \int_t^T \hat{g}_t(s, \epsilon_s)ds + \int_t^T \sum_{j \in S} q_{\epsilon_s,j} \hat{g}(s, \epsilon_s)ds + m^\hat{g}_T,$$

where $m^\hat{g}$ is a square integrable martingale with $E[m^\hat{g}_t] = 0$ for all $t \in [0, T]$.

Taking conditional expectation and using the equation (4.15), we get

$$E_{t,x,i}[\hat{g}(T, \epsilon_T)] = \hat{g}(t, \epsilon_t) - E_{t,x,i} \left[ \int_t^T \alpha \hat{\Pi}(s) \hat{g}(s, \epsilon_s)ds \right],$$

which is equivalent to (Recall the boundary condition $\hat{g}(T, i) = 1$)

$$\hat{g}(t, i) = 1 + E_{t,x,i} \left[ \int_t^T \alpha \hat{\Pi}(s) \hat{g}(s, \epsilon_s)ds \right].$$

Solving the above equation yields

$$\hat{g}(t, i) = E_{t,x,i} \left[ \exp \left\{ \int_t^T \alpha \hat{\Pi}(s)ds \right\} \right].$$

Hence, the positiveness of $\hat{g}(t, i)$ follows. \(\square\)

From the construction of $\hat{g}(t, i)$ and Lemma 4.3, $v(t, x, i) = \frac{1}{\alpha} x^\alpha \cdot \hat{g}(t, i)$ is the associated value function to the modified Problem 2.1. Thanks to Lemma 4.2, $u^* = (\pi^*, \kappa^*)$, with $\pi^*$ and $\kappa^*$ given by (4.12) and (4.13), is admissible. Hence Theorem 4.2 follows accordingly.

**Theorem 4.2** When $U(x) = \frac{1}{\alpha} x^\alpha$, where $\alpha < 1$ and $\alpha \neq 0$, optimal control to Problem 2.1 is $u^* = (\pi^*, \kappa^*)$, where

$$\pi^*(t) = \frac{\mu(t) - r(t)}{(1 - \alpha)\sigma^2(t)} + \rho \frac{b(t)}{\sigma(t)} \kappa^*(t),$$

and $\kappa^*(t)$ is the unique solution to the equation

$$(\alpha - 1)(1 - \rho^2)b^2(t)\kappa^*(t) - \lambda \gamma(t)(1 - \gamma(t)\kappa^*(t))^{\alpha-1} + p(t) - a(t) + \rho \frac{b(t)}{\sigma(t)} (\mu(t) - r(t)) = 0.$$
4.3 \( U(x) = -\frac{1}{\alpha} e^{-\alpha x} \), where \( \alpha > 0 \)

In this subsection, we consider exponential utility function, which belongs to the class of constant absolute risk aversion (CARA) utility functions. We find the solution to the HJB (3.2) is of the form

\[
v(t, x, i) = -\frac{1}{\alpha} e^{\gamma(T-t)x + \tilde{g}(t, i)}
\]

where \( \tilde{g}(t, i) \) will be determined below.

For the above solution, we calculate that

\[
\begin{align*}
\frac{\partial v}{\partial t}(t, x, i) &= \left[ \alpha r_i e^{\gamma(T-t)} + \tilde{g}_t(t, i) \right] v(t, x, i), \\
\frac{\partial v}{\partial x}(t, x, i) &= -\alpha e^{\gamma(T-t)} v(t, x, i), \\
\frac{\partial^2 v}{\partial x^2}(t, x, i) &= \alpha^2 e^{2\gamma(T-t)} v(t, x, i).
\end{align*}
\]

Hence, we obtain the candidate for \( \pi^* \)

\[
(4.16) \quad \pi^* = e^{-r_i(T-t)} \frac{\mu_i - r_i}{\alpha x \sigma^2_i} + \rho \frac{b_i}{\sigma_i} \kappa^*.
\]

Apparently, in this case, it is more convenient to use the actual amount instead of the proportion as the control of investment, see, e.g., Browne (1995), Wang et al. (2007), Yang and Zhang (2005). We then define \( \theta(t) \) as the amount of money invested in the risky asset \( P_1 \) and \( L(t) \) as the total liabilities at time \( t \). By definition, we have \( \theta(t) = \pi(t) X^u(t) \) and \( L(t) = \kappa(t) X^u(t) \).

Thus, by (4.16) and (4.5), we obtain the candidate for \( \theta^* \)

\[
(4.17) \quad \theta^* = e^{-r_i(T-t)} \frac{\mu_i - r_i}{\alpha x \sigma^2_i} + \rho \frac{b_i}{\sigma_i} L^*,
\]

and the candidate for \( L^* \), which satisfies

\[
(4.18) \quad \lambda \gamma_i e^{\hat{A}_i L^*} + \hat{B}_i L^* - \hat{C}_i = 0,
\]

where

\[
\begin{align*}
\hat{A}_i &:= \alpha \gamma_i e^{\gamma(T-t)}, \\
\hat{B}_i &:= \alpha e^{\gamma(T-t)} b_i^2 (1 - \rho^2), \\
\hat{C}_i &:= p_i - a_i + \rho b_i \frac{\mu_i - r_i}{\sigma_i}.
\end{align*}
\]
Lemma 4.4 If the condition (4.6) holds, then there exists a unique solution to the equation (4.18).

**Proof.** Define \( \tilde{h}_i(x) := \lambda \gamma_i \tilde{A}_i e^{\tilde{A}_i x} + \tilde{B}_i > 0, \)

since \( \lambda > 0, \gamma_i > 0, \tilde{A}_i > 0 \) and \( \tilde{B}_i > 0 \) for all \( i \in S \). If the condition (4.6) is satisfied, we can obtain \( \tilde{h}_i(0) = \lambda \gamma_i - \tilde{C}_i < 0 \). Since both \( \tilde{A}_i \) and \( \tilde{B}_i \) are positive, \( \tilde{h}_i(x) \) must be positive when \( x \) is large enough. Therefore, the desired conclusion is obtained. \( \square \)

Next, we rewrite the HJB equation (3.2) as follows

\[
(4.19) \quad \tilde{g}_t(t,i) + \sum_{j \in S} q_{ij} \exp \left\{ -\alpha x e^{(r_j-r_i)(T-t)} \right\} e^{\tilde{g}(t,j)-\tilde{g}(t,i)} + \tilde{\Pi}_i = 0,
\]

where \( \tilde{\Pi}_i := -\alpha e^{r_i(T-t)}[\mu_i - r_i] \theta^* + (p_i - a_i) L^* + \frac{1}{2} \alpha^2 e^{2r_i(T-t)}[\sigma_i^2(\theta^*)^2 - 2p_i \sigma_i \theta^* L^* + b_i^2(L^*)^2] + \lambda (\exp (\alpha \gamma_i e^{r_i(T-t)} L^*) - 1). \)

Let \( \tilde{q}_{ij} := q_{ij} \exp \left\{ -\alpha x e^{(r_j-r_i)(T-t)} \right\} \) and \( \Phi(t,i) := \exp \{ \tilde{g}(t,i) \} \). Then equation (4.19) becomes

\[
\Phi(t,i) + \sum_{j \in S} \tilde{q}_{ij} \Phi(t,j) + \tilde{\Pi}_i \Phi(t,i) = 0,
\]

which, similar to the system (4.15), bears a unique solution. Hence, there exists a unique solution \( \tilde{g}(t,i) \) to the system (4.19).

Since the value function is well defined even when \( x \leq 0 \), we remove the constraint \( \kappa(t) < \frac{1}{\gamma(t)} \) from the conditions of the admissible set \( A_{x,i} \), which is used Subsection 4.1 and 4.2 to guarantee \((1-\kappa(t)\gamma(t)) > 0 \) (so \( \ln(1-\kappa(t)\gamma(t)) \) and \((1-\kappa(t)\gamma(t))^{\alpha-1} \), where \( \alpha < 1 \), are well defined).

By Lemma 4.4, the solution \( L^* \) to the equation (4.18) is finite for all \( i \in S \), which implies \( L^* \) is finite and square integrable on \([0,T]\). Hence, by (4.16), \( \theta^* \) is also finite and square integrable on \([0,T]\). By Oksendal and Sulem (2005) Theorem 1.19, there exists a unique wealth process \( X^{u^*} \) such that \( E[|X^{u^*}(t)|^2] < \infty \) for all \( t \in [0,T] \). In consequence, \( u^* := \frac{1}{X^{u^*}(\theta^*,L^*)} \) is admissible and the theorem below follows.
Theorem 4.3 When utility function is $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$, where $\alpha > 0$, $u^*(t) = \frac{1}{X^*(t)}(\theta^*(t), L^*(t))$ is optimal control to Problem 2.1, where

$$
\theta^*(t) = e^{-r_e(t)(T-t)}\frac{\mu_e(t) - r_e(t)}{\alpha \sigma_e^2(t)} + \frac{b_e(t)}{\sigma_e(t)}L^*(t),
$$

and $L^*(t)$ is the unique solution to the equation

$$
\lambda \gamma_e(t) \exp\{\tilde{A}_e(t)L^*(t)\} + \tilde{B}_e(t)L^*(t) - \tilde{C}_e(t) = 0.
$$

5 Economic Analysis

In this section, we study the impact of the economy and the insurer’s risk attitude on optimal policy. To this purpose, we assume there are two regimes in the economy. Regime 1 represents a bull market, in which the economy is booming. Regime 2 represents a bear market, meaning the economy is in recession. For comparative analysis, we consider HARA utility functions, namely, $U(x) = \frac{1}{\alpha}x^\alpha$, where $\alpha < 1$ ($\alpha = 0$ is associated with the case of logarithmic utility function $U(x) = \ln(x)$). When $\alpha < 0$, insurers are high risk-averse, when $\alpha = 0$, insurers are moderate risk-averse, when $0 < \alpha < 1$, insurers are low risk-averse.

Following Fama and French (1989), we assume $\mu_i > r_i > 0$ and $p_i > a_i > 0$, $i = 1, 2$ (Remark 2.2). French et al. (1987) find that capital returns are higher in a bull market, hence we assume $\mu_1 > \mu_2$ and $r_1 > r_2$. Hamilton and Lin (1996) show that the stock volatility is greater when the economy is in recession, which implies $\sigma_1 < \sigma_2$. Furthermore, we assume $\frac{\mu_1 - r_1}{\sigma_1^2} > \frac{\mu_2 - r_2}{\sigma_2^2}$, as supported by French et al. (1987). In the insurance market, the risk process (claims) is negatively correlated with the stock returns and interest rate, see, e.g., Haley (1993), Norden and Weber (2007). This conclusion leads to the assumption that $a_2 > a_1$, $b_2 > b_1$, and $\gamma_2 > \gamma_1$. When the economy is in recession, the insurance companies charge a higher premium, hence $p_2 > p_1$.

We also notice that the coefficient we choose should satisfy the technical condition (4.6). Based on the above argument, we choose the parameters and list in Table 1.

By Theorem 4.1, we calculate the optimal policy for moderate risk-averse insurers (that is, $\alpha = 0$). For both high risk-averse and low risk-averse insurers, we obtain the corresponding optimal policy through Theorem 4.2. The results are listed in Table 2.
According to the optimal policy obtained in Table 2, we observe that both the optimal investment proportion in the risky asset $\pi^*$ and the optimal liability ratio $\kappa^*$ are increasing functions of the risk aversion parameter $\alpha$. Hence less risk-averse insurers (that is, insurers with large $\alpha$) invest proportionally more in the risky asset and choose a higher liability ratio.

As pointed out in Stein (2012, Chapter 6), a major mistake that contributed significantly to AIG’s sudden collapse is the negligence of the negative correlation between the risk and the capital returns (or equivalently,
\( \rho < 0 \). Hence in the next analysis, we calculate the optimal policy for different values of \( \rho \). We still keep all the other parameters unchanged as in Table 1, but consider \( \rho = -0.9, -0.5, -0.2 \).

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \alpha )</th>
<th>Regime</th>
<th>( \pi^* ) (Investment)</th>
<th>( \kappa^* ) (Liability Ratio)</th>
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Table 3: Impact of \( \rho \) on optimal policies

Based on the results in Table 3, we find the optimal proportion invested in the risky asset \( \pi^* \) is an increasing function of \( \rho \). However, the relation between \( \kappa^* \) and \( \rho \) is more complicated, some show convexity while other show monotonicity.

Furthermore, the dependency of the optimal policy on the regime of the economy is evident. We notice from Table 2 and Table 3 that \( \pi^*_1 > \pi^*_2 \) and \( \kappa^*_1 > \kappa^*_2 \) for all insurers (all \( \alpha \)). This result shows that all insurers take more risk in a bull market by spending a greater proportion on the risky asset and selecting a higher liability ratio.
6 Conclusion

The 2007-2008 financial crisis brought new challenges on risk management to all market participants. Because of taking problematic risk management procedures, AIG collapsed in a few months and almost went bankrupt. There are two major contributors to AIG’s sudden collapse. First, AIG did not pay full attention to the business cycles in the U.S. housing market, which directly caused a significant underestimation of the risk involved in the CDS trading. Second, AIG ignored the negative correlation between its liabilities and the capital gains in the financial market. Such ignorance indicates that AIG was not fully aware of the impact of derivatives trading on its capital structure.

To address these two problems in the AIG case, we set up a regime switching model from an insurer’s perspective and assume not only the financial market but also the insurer’s risk process depend on the regime of the economy. An insurer makes investment decisions in a financial market which consists of a riskless asset and a risky asset, and faces an external risk that is negatively correlated with the price of the risky asset. The insurer wants to maximize its expected utility of terminal wealth by selecting optimal investment proportion in the risky asset and liability ratio simultaneously. We obtain explicit solutions of optimal investment and liability ratio policies when the insurer’s utility is given by logarithmic, power and exponential utility functions.

Through an economic analysis, we find the optimal policy depends on the regime of the economy. To be more specific, all insurers spend a greater proportion in the risky asset and choose a higher liability ratio when the market is in bull regime. We also observe that the optimal proportion invested in the risky asset is increasing with respect to both $\alpha$ and $\rho$. In the meantime, the optimal liability ratio also increases when $\alpha$ rises, but its relation with $\rho$ is undeterminate.

References


Bauerle, N. and Rieder, U., 2004. Portfolio optimization with markov-


