Chapter 3: Vectors in 2-space and 3-space

Section 3.1: Introduction to vectors

Definition: A geometric vector is a directed line segment (arrow) in 2-
or 3-space. The direction of the arrow specifies the direction of the vector
and the length of the arrow specifies the magnitude of the vector.

Example:

Definition: Vectors with the same direction and magnitude are called
equivalent. If $\vec{v}$ and $\vec{w}$ are equivalent, we write $\vec{v} = \vec{w}$.

Example:

Definition: If $\vec{v}$ and $\vec{w}$ are two vectors, then the sum $\vec{v} + \vec{w}$ is the vector
determined by positioning the initial point of $\vec{w}$ to coincide with the terminal
point of \( \vec{v} \). The vector \( \vec{v} + \vec{w} \) is represented by the arrow from the initial point of \( \vec{v} \) to the terminal point of \( \vec{w} \).

**Example:**

**Note:** \( \vec{v} + \vec{w} = \vec{w} + \vec{v} \)

**Definition:** The vector of length zero is the zero vector, denoted \( \vec{0} \), and \( \vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v} \).

**Definition:** If \( \vec{v} \) is a nonzero vector, the negative of \( \vec{v} \), denoted \( -\vec{v} \), is the vector with the same magnitude as \( \vec{v} \) but is oppositely directed, and \( \vec{v} + (-\vec{v}) = \vec{0} \).

**Definition:** If \( \vec{v} \) and \( \vec{w} \) are two vectors, then the difference of \( \vec{w} \) from \( \vec{v} \) is defined by \( \vec{v} - \vec{w} = \vec{v} + (-\vec{w}) \).

**Example:**

**Alternate definition:** To find \( \vec{v} - \vec{w} \), position \( \vec{v} \) and \( \vec{w} \) with initial points coinciding. The vector from the terminal point of \( \vec{w} \) to the terminal point of \( \vec{v} \) is the vector \( \vec{v} - \vec{w} \).
Definition: If $\vec{v}$ is a nonzero vector, $k$ is a nonzero scalar, then the product (scalar multiple) $k\vec{v}$ is the vector whose length is $|k|$ times the length of $\vec{v}$ and whose direction is the same as $\vec{v}$ if $k > 0$, and opposite direction if $k < 0$. $k\vec{v} = \vec{0}$ if $k = 0$ or $\vec{v} = \vec{0}$.

Example:

Note: Vectors that are scalar multiples of each other are parallel.

Definition: Vectors written in a rectangular coordinate system such as the plane are said to have an **analytic representation**. If $\vec{v}$ is a vector in the plane with its initial point at the origin then the coordinates $(v_1, v_2)$ of the terminal point of $\vec{v}$ are called the components of $\vec{v}$, $\vec{v} = (v_1, v_2)$.

Example:

Note: If $P$ is the point $(v_1, v_2)$ and $O$ denotes the origin then we can also write $\vec{v} = \overrightarrow{OP}$.

Definition: In a rectangular coordinate system, two vectors $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ are equivalent if and only if $v_1 = w_1$, $v_2 = w_2$. 
**Definition:** The sum, difference and scalar multiple of vectors in a rectangular coordinate system can be carried out in terms of components.

Sum: $\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2)$
Difference: $\vec{v} - \vec{w} = (v_1 - w_1, v_2 - w_2)$
Scalar multiple: $k\vec{v} = (kv_1, kv_2)$

**Example:**

**Definition:** We can use the same definitions for vectors in 3-space. The coordinates of the terminal point when the vector is positioned at the origin are $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$. Then $\vec{v}$ and $\vec{w}$ are equivalent if and only if $v_1 = w_1$, $v_2 = w_2$, $v_3 = w_3$, $\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$ and $k\vec{v} = (kv_1, kv_2, kv_3)$, $k$ scalar.

**Example:**

**Definition:** A coordinate system in 3-space is right handed if a screw pointed in the positive direction on the z-axis would be advanced if the x-axis were rotated 90° towards the y-axis. A coordinate system is left handed if the screw would be retracted.

**Example:**
Definition: To translate a vector from the coordinate system \((x, y)\) to a new coordinate system \((x', y')\) whose origin \(O'\) is at the point \((x, y) = (k, l)\) then the translation equations for the components of vector \(\vec{v}\) are \(x' = x - k, \ y' = y - l\).

Example:
Section 3.2: Norm of a vector; vector arithmetic

**Theorem 3.2.1: Properties of vector arithmetic** If \( \vec{u}, \vec{v}, \vec{w} \) are vectors in 2- or 3-space, \( k, l \) scalars, then

1. \( \vec{u} + \vec{v} = \vec{v} + \vec{u} \)
2. \( (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \)
3. \( \vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u} \)
4. \( \vec{u} + (-\vec{u}) = \vec{0} \)
5. \( k(l\vec{u}) = (kl)\vec{u} \)
6. \( k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v} \)
7. \( (k + l)\vec{u} = k\vec{u} + l\vec{u} \)
8. \( 1\vec{u} = \vec{u} \)

**Proof:** See text.

**Definition:** The **length** or **norm** of a vector \( \vec{u} \) is denoted by \( \|\vec{u}\| \) and calculated by

- 2-space: \( \|\vec{u}\| = \sqrt{u_1^2 + u_2^2} \)
- 3-space: \( \|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2} \)

and for any scalar \( k \), \( \|k\vec{u}\| = |k|\|\vec{u}\| \).

**Example:**
Section 3.3: Dot product; projections

Definition: If \( \vec{u}, \vec{v} \) are vectors in 2- or 3-space and \( \theta \) is the angle between \( \vec{u} \) and \( \vec{v} \) then the dot product or Euclidean inner product \( \vec{u} \cdot \vec{v} \) is defined by

\[
\vec{u} \cdot \vec{v} = \begin{cases} 
\|\vec{u}\|\|\vec{v}\| \cos(\theta) & \vec{u} \neq 0 \text{ and } \vec{v} \neq 0 \\
0 & \vec{u} = 0 \text{ or } \vec{v} = 0 
\end{cases}
\]

Using analytic representation,

2-space: \( \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 \)

3-space: \( \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3 \)

Note: The dot product can be used to determine the angle between two vectors:

\[
\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}
\]

Example:
**Theorem 3.3.1:** Let \( \vec{u}, \vec{v} \) be vectors, then

1. \( \vec{v} \cdot \vec{v} = ||\vec{v}||^2; \ ||\vec{v}|| = (\vec{v} \cdot \vec{v})^{1/2} \)

2. If \( \vec{u}, \vec{v} \) are nonzero vectors, \( \theta \) is the angle between them, then
   - \( \theta \) is acute if and only if \( \vec{u} \cdot \vec{v} > 0 \)
   - \( \theta \) is obtuse if and only if \( \vec{u} \cdot \vec{v} < 0 \)
   - \( \theta = \pi/2 \) if and only if \( \vec{u} \cdot \vec{v} = 0 \)

**Proof:** See text.

**Definition:** Two vectors \( \vec{u}, \vec{v} \) are orthogonal if they are perpendicular to each other. That is, \( \vec{u} \) and \( \vec{v} \) are orthogonal if and only if \( \vec{u} \cdot \vec{v} = 0 \), which we denote \( \vec{u} \perp \vec{v} \).

**Example:**

**Theorem 3.3.2: Properties of the dot product** If \( \vec{u}, \vec{v}, \vec{w} \) are vectors in 2- or 3-space, \( k \) a scalar, then

1. \( \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \)

2. \( \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \)

3. \( k(\vec{u} \cdot \vec{v}) = (k\vec{u}) \cdot \vec{v} \)

4. \( \vec{v} \cdot \vec{v} > 0 \) if \( \vec{v} \neq 0 \), and \( \vec{v} \cdot \vec{v} = 0 \) if \( \vec{v} = 0 \)

**Proof:** See text
Theorem 3.3.3: If $\vec{u}$ and $\vec{a} \neq \vec{0}$ are vectors in 2- or 3-space then the vector component of $\vec{u}$ along $\vec{a}$ is

$$proj_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a}$$

and the vector component of $\vec{u}$ orthogonal to $\vec{a}$ is

$$\vec{u} - proj_{\vec{a}} \vec{u} = \vec{u} - \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a}$$

That is, we can express the vector $\vec{u}$ in terms of another vector $\vec{a}$ where $\vec{u}$ has components that are parallel to $\vec{a}$ and components that are perpendicular to $\vec{a}$.

Example:

Definition: The length of the vector component of $\vec{u}$ along $\vec{a}$ is given by

$$\|proj_{\vec{a}} \vec{u}\| = \frac{|\vec{u} \cdot \vec{a}|}{\|\vec{a}\|}$$

and if $\theta$ is the angle between $\vec{u}$ and $\vec{a}$ then

$$\|proj_{\vec{a}} \vec{u}\| = \|\vec{u}\| \cos(\theta)$$

Definition: The distance between a point $P_0(x_0, y_0)$ and the line $ax + by + c = 0$ is

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

Example:
Section 3.4: Cross product

**Definition:** If \( \vec{u} = (u_1, u_2, u_3) \), \( \vec{v} = (v_1, v_2, v_3) \) are vectors in 3-space then the **cross product** \( \vec{u} \times \vec{v} \) is given by

\[ \vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \]

**Notes:**

1. The cross product produces a vector that is perpendicular to \( \vec{u} \) and \( \vec{v} \) (recall the dot product produces a scalar)

2. The cross product is only defined for vectors in 3-space

**Alternate definition** You can calculate the cross product using the matrix

\[
\begin{bmatrix}
  u_1 & u_2 & u_3 \\
  v_1 & v_2 & v_3
\end{bmatrix}
\]

First term: determinant of matrix missing first column
Second term: determinant of matrix missing second column
Third term: determinant of matrix missing third column

**Example:**

**Theorem 3.4.1: Relationships involving cross and dot product**

If \( \vec{u}, \vec{v}, \vec{w} \) are vectors in 3-space then
Theorem 3.4.2: Properties of cross products

If \( \vec{u}, \vec{v}, \vec{w} \) are vectors in 3-space, \( k \) scalar, then

1. \( \vec{u} \times (\vec{v} \times \vec{w}) = \vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w} \)
2. \( (\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w} \)

Proof: See text.
Note: The cross product between unit vectors can be summarized as follows:

Alternate definition: The cross product can be calculated as a determinant using unit vectors:

\[
\vec{u} \times \vec{v} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3
\end{vmatrix}
\]

\[
= \vec{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}
\]

More cross product properties:

1. \(\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}\)

2. \(||\vec{u} \times \vec{v}|| = ||\vec{u}|| ||\vec{v}|| \sin(\theta)\)

Theorem 3.4.3: If \(\vec{u}, \vec{v}\) are vectors in 3-space, then \(||\vec{u} \times \vec{v}||\) is equal to the area of the parallelogram determined by \(\vec{u}\) and \(\vec{v}\).

Example:
Note: This has the same area as the parallelogram defined by \( \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3} \)

\textbf{Theorem 3.4.4:} Geometric interpretation of determinants

1. The absolute value of the determinant

\[
\det \begin{bmatrix}
  u_1 & u_2 \\
  v_1 & v_2
\end{bmatrix}
\]

is equal to the area of the parallelogram in 2-space determined by \( \vec{u} = (u_1, u_2), \vec{v} = (v_1, v_2) \).

2. The absolute value of the determinant

\[
\det \begin{bmatrix}
  u_1 & u_2 & u_3 \\
  v_1 & v_2 & v_3 \\
  w_1 & w_2 & w_3
\end{bmatrix}
\]

is equal to the volume of the parallelepiped in 3-space determined by \( \vec{u} = (u_1, u_2, u_3), \vec{v} = (v_1, v_2, v_3), \vec{w} = (w_1, w_2, w_3) \).

Example:
Definition: If \( \vec{u}, \vec{v}, \vec{w} \) are vectors in 3-space, then \( \vec{u} \cdot (\vec{v} \times \vec{w}) \) is called the scalar triple product of \( \vec{u}, \vec{v} \) and \( \vec{w} \) and can be calculated using

\[
\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
\]

Note: The volume of a parallelepiped defined by \( \vec{u}, \vec{v}, \vec{w} \) in 3-space can be calculated using \( V = |\vec{u} \cdot (\vec{v} \times \vec{w})| \).

Theorem 3.4.5: If the vectors \( \vec{u}, \vec{v}, \vec{w} \) have the same initial point, then they lie in the same plane if and only if \( \vec{u} \cdot (\vec{v} \times \vec{w}) = 0 \). That is, the shape determined by \( \vec{u}, \vec{v}, \vec{w} \) has zero volume (e.g., a plane in 3-space). It also means \( \vec{u} \) is orthogonal to \( \vec{v} \times \vec{w} \).
Section 3.5: Lines and planes in 3-space

**Definition:** A nonzero vector that is perpendicular to a specific plane is called a normal vector, denoted $\vec{n}$. The normal vector describes the inclination of the plane.

**Example:**

**Definition:** Given the point $P_0(x_0, y_0, z_0)$ and the vector $\vec{n} = (a, b, c)$, the point normal form of the equation of the plane that passes through the point $P_0$ and has $\vec{n}$ as its normal is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

**Theorem 3.5.1:** If $a, b, c, d$ are constant and $a, b, c$ are not all zero, then the graph of

$$ax + by + cz + d = 0$$

is a plane having the normal vector $\vec{n} = (a, b, c)$

**Example:**
Definition: Let $\mathbf{r}_0 = (x_0, y_0, z_0)$ be a vector from the origin to the point $P_0$ in the plane, $\mathbf{r} = (x, y, z)$ is a vector from the origin to the point $P(x, y, z)$ in the plane, and $\mathbf{n} = (a, b, c)$ is a normal to the plane, then the vector form of the equation of a plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

Example:

Definition: If the line $l$ passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathbf{v} = (a, b, c)$ the parametric equations for $l$ are

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc, \quad -\infty < t < \infty$$

Definition: Let $\mathbf{r}_0 = (x_0, y_0, z_0)$ be a vector from the origin to the point $P_0$ on the line $l$, let $\mathbf{r} = (x, y, z)$ be a vector from the origin to the point $P(x, y, z)$ on the line, and let $\mathbf{v} = (a, b, c)$ be a vector parallel to the line, then the vector form of the equation of a line in 3-space is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

Example:
Theorem 3.5.2: The distance $D$ between a point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example: