Induction, sequences, limits and continuity

Material covered: eClass notes on induction, Chapter 11, Section 1 and Chapter 2, Sections 2.2 - 2.5

Induction

Principle of mathematical induction: Let $P(n)$ be a statement (proposition) about the natural number n. Suppose that we can show two things:

1. $P(n_0)$ is true for some natural number n_0
2. $P(k+1)$ is true whenever $P(k)$ is true for each $k \geq n_0$

Then $P(n)$ is true for every natural number $n \geq n_0$.

Sequences

Definition: A sequence is a list of numbers written in a definite order:

$$a_1,\ a_2,\ a_3, \ldots a_n, \ldots$$

which can also be written as

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}$$

We can picture a sequence two ways:

1. by plotting its terms on the real number line
2. by plotting its terms on a graph

The big question: What happens to a_n as $n \to \infty$?

Definition: A sequence $\{a_n\}$ has the limit L, written

$$\lim_{n \to \infty} a_n = L \text{ or } a_n \to L \text{ as } n \to \infty$$

if for every $\epsilon > 0$ there is a corresponding integer N such that

$$\text{if } n > N \text{ then } |a_n - L| < \epsilon$$

In other words, a sequence has the limit L if the value of a_n is within a small distance ϵ from L.

Definition: If the sequence $\{a_n\}$ has the limit L, we say the sequence converges. Otherwise, we say the sequence diverges.

Squeeze theorem for sequences: To determine whether the sequence $\{b_n\}$ has a limit, if we can find two sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

then

$$\lim_{n\to\infty} b_n = L$$

In other words, the sequence $\{b_n\}$ is "squeezed" between $\{a_n\}$ and $\{c_n\}$ as n gets large.

Theorem:

$$\text{If } \lim_{n\to\infty} |a_n| = 0, \text{ then } \lim_{n\to\infty} a_n = 0$$

Definition: A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$.
A sequence $\{a_n\}$ is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$.
A sequence is called monotonic if it is either increasing or decreasing.

Definition: A sequence $\{a_n\}$ is bounded above if there is a number M such that $a_n \leq M$ for all $n \geq 1$.
A sequence $\{a_n\}$ is bounded below if there is a number m such that $m \leq a_n$ for all $n \geq 1$.
A sequence is bounded if it is bounded above and below (all terms are between some minimum and maximum numbers).

Monotonic sequence theorem: Every bounded, monotonic sequence is convergent.

Limit of a function

Informal definition: Suppose a function $f(x)$ is defined for all x near a, except perhaps at the point $x = a$. We define the limit of the function $f(x)$ as x approaches a, written

$$\lim_{x \to a} f(x)$$

to be a number L (if one exists) such that $f(x)$ is as close to L as we please whenever x is sufficiently close to a (but $x \neq a$). If L exists, we write

$$\lim_{x \to a} f(x) = L$$

Definition: The right hand limit of $f(x)$ as x approaches a is equal to L if $f(x)$ gets close to L as x gets close to a from the right $(x > a)$, written

$$\lim_{x \to a^+} f(x) = L.$$

The left hand limit of $f(x)$ as x approaches a is equal to L if $f(x)$ gets close to L as x gets close to a from the left $(x < a)$, written

$$\lim_{x \to a^-} f(x) = L.$$

Note:

$$\lim_{x \to a} f(x) = L \text{ iff } \lim_{x \to a^+} f(x) = L \text{ and } \lim_{x \to a^-} f(x) = L.$$

That is, the limit exists if $\lim_{x \to a} f(x) = L$ and L is a finite number. Otherwise the limit does not exist, either because L is not finite, or $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$.

Infinite limits: Let f be a function defined on both sides of a, except possibly at a itself, then

$$\lim_{x \to a} f(x) = \infty$$

mean that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to, but not equal to, a.

Limits at infinity: If $f(x)$ gets arbitrarily close to a finite number L when x gets sufficiently large, then we write

$$\lim_{x \to \infty} f(x) = L$$

Definition: The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if either

$$\lim_{x \to a} f(x) = \pm\infty, \quad \lim_{x \to a^+} f(x) = \pm\infty \text{ or } \lim_{x \to a^-} f(x) = \pm\infty.$$

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \text{ or } \lim_{x \to -\infty} f(x) = L$$

Useful fact: If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided the limit exists.

Theorem: If $f(x) \leq g(x)$ when x is near a and the limits of f and g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

Squeeze theorem for limits of functions: If $f(x) \leq g(x) \leq h(x)$ when x is near a and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

That is, if $g(x)$ is "squeezed" between $f(x)$ and $h(x)$ near x, and $f(x)$ and $h(x)$ have the same limit L at a, then g must also have the limit L at a.

Formal definition of a limit: We define $\lim_{x \to a} f(x)$ to the number L (if one exists) such that for any $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if

$$|x - a| < \delta \text{ and } x \neq a, \text{ then } |f(x) - L| < \epsilon$$

There are equivalent statements for left and right hand limits, and infinite limits.

Continuity

Definition: A function f is continuous at $x = a$ is f is defined at $x = a$ and

$$\lim_{x \to a} f(x) = f(a).$$

That is, $f(x)$ is as close to $f(a)$ as we please, provided x is close to a, and we require that the behaviour of a function *near* a point be consistent with its behaviour *at* a point.
If a function is not continuous at $x = a$, we say it is discontinuous.

Note: The definition of continuity implies three conditions that must be met:

1. $f(a)$ must be defined
2. $\lim_{x \to a} f(x)$ must exist
3. $\lim_{x \to a} f(x) = f(a)$

Definition: A function f is continuous from the right at a number a if $\lim_{x \to a^+} f(x) = f(a)$.
A function f is continuous from the left at a number a if $\lim_{x \to a^-} f(x) = f(a)$.

Definition: A function f is continuous on an interval if it is continuous at every point in the interval. If f is defined only on one side of an endpoint, we define continuity at the endpoint using the appropriate one sided limit.

Theorem: If

1. f is continuous at b, and
2. $\lim_{x\to a} g(x) = b$,

then

$$\lim_{x\to a} f(g(x)) = f(b) \Leftrightarrow \lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)).$$

That is, the limit can be moved through a function if the two conditions are met.

Continuity of composite functions: If

1. g is continuous at a, and
2. f is continuous at $g(a)$,

then the composite function $(f \circ g)(x) = f(g(x))$ is continuous at a.

Intermediate value theorem: If $f(x)$ is continuous on a closed interval $[a, b]$, and if N is a real number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = N$.