Linearization, Taylor polynomials, hyperbolic functions

Material covered: Chapter 3, Sections 3.10, 3.11

Linearization

**Definition:** For values of $x$ near $a$, the linear or tangent line approximation of $f$ at $a$ is given by

$$f(x) \approx f(a) + f'(a)(x - a)$$

where we think of $a$ as fixed, so that $f(a)$ and $f'(a)$ are constant. In other words, this is a linear function that approximates $f(x)$ well near $a$. 
**Definition:** If \( y = f(x) \), where \( f \) is a differentiable function, then the differential of \( x \), \( dx \), can be thought of as an independent variable, while the differential of \( y \), \( dy \), is given in terms of \( dx \) as

\[
dy = f'(x)dx
\]

**Note:** There is a difference between \( \Delta y \), the amount that the curve \( y = f(x) \) rises or falls in the \( y \)-direction, and the differential \( dy \), which is the amount that the tangent line rises or falls.
Taylor polynomials

**Idea:** We want to approximate a function $f(x)$ near a point $a$ using a polynomial of degree $n$

**Definition:** The Taylor polynomial of degree $n$ approximating $f(x)$ for $x$ near $a$ is given by

$$f(x) \approx P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^n(a)}{n!}(x - a)^n$$
Special case: Maclaurin polynomials are Taylor polynomials about $x = 0$.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n$$
Hyperbolic functions

Definition of hyperbolic functions:

\[
\begin{align*}
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\tanh x &= \frac{\sinh x}{\cosh x} \\
\csch x &= \frac{1}{\sinh x} \\
\sech x &= \frac{1}{\cosh x} \\
\coth x &= \frac{\cosh x}{\sinh x}
\end{align*}
\]
Hyperbolic identities:

\[
\begin{align*}
\sinh(-x) & = -\sinh x \\
\cosh(-x) & = \cosh x \\
\cosh^2 x - \sinh^2 x & = 1 \\
1 - \tanh^2 x & = \text{sech}^2 x \\
\sinh(x + y) & = \sinh x \cosh y + \cosh x \sinh y \\
\cosh(x + y) & = \cosh x \cosh y + \sinh x \sinh y
\end{align*}
\]
Derivatives of hyperbolic functions:

\[
\begin{align*}
\frac{d}{dx} \sinh x &= \cosh x \\
\frac{d}{dx} \cosh x &= \sinh x \\
\frac{d}{dx} \tanh x &= \operatorname{sech}^2 x \\
\frac{d}{dx} \operatorname{csch} x &= -\operatorname{csch} x \coth x \\
\frac{d}{dx} \operatorname{sech} x &= -\operatorname{sech} x \tanh x \\
\frac{d}{dx} \coth x &= -\operatorname{csch}^2 x
\end{align*}
\]
Inverse hyperbolic functions:

\[ y = \sinh^{-1} x \iff \sinh y = x \]
\[ y = \cosh^{-1} x \iff \cosh y = x, \text{ and } y \geq 0 \]
\[ y = \tanh^{-1} x \iff \tanh y = x \]

Inverse hyperbolic functions and natural logarithms:

\[ \sinh^{-1} x = \ln \left( x + \sqrt{x^2 + 1} \right), x \in \mathbb{R} \]
\[ \cosh^{-1} x = \ln \left( x + \sqrt{x^2 - 1} \right), x \geq 1 \]
\[ \tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right), -1 < x < 1 \]

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Derivatives of inverse hyperbolic functions:

\[
\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1 + x^2}} \\
\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}} \\
\frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2} \\
\frac{d}{dx} \text{csch}^{-1} x = -\frac{1}{|x|\sqrt{1 + x^2}} \\
\frac{d}{dx} \text{sech}^{-1} x = -\frac{1}{x\sqrt{1 - x^2}} \\
\frac{d}{dx} \coth^{-1} x = \frac{1}{1 - x^2}
\]