Definite integrals, Fundamental theorem of calculus, Indefinite integrals

Material covered: Chapter 5, Sections 5.2 - 5.4

Definite integrals

Idea: We can extend the idea of finding the area under a curve using any point in the subinterval $[x_{i-1}, x_i]$, not just the right, left or midpoint of the subinterval. That is,

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $x_i^* \in [x_{i-1}, x_i]$.
The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

is called a Riemann sum.

Definition: The definite integral of f from a to b is defined as

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

If the limit exists, we say f is integrable on $[a, b]$.

Terminology: $\int$ is called an integral sign
$f(x)$ is called the integrand
a is called the lower limit of integration
b is called the upper limit of integration
dx indicates the independent variable is x

Comparison properties of the definite integral:

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Note: When $f(x)$ is positive for some x values and negative for others, and $a < b$, then $\int_a^b f(x)dx$ is the sum of the areas above the x-axis, counted positively, and the areas below the x-axis, counted negatively.

Properties of the definite integral:

1. $\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b cdx = c(b-a)$, c constant
4. $\int_a^b [cf(x) + g(x)]dx = c\int_a^b f(x)dx + \int_a^b g(x)dx$, c constant
5. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$
6. $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$
7. If f is even, then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$
8. If f is odd, then $\int_{-a}^a f(x)dx = 0$

Fundamental Theorem of Calculus (FTC)

Idea: There is a fundamental relationship between differentiation and integration: they are inverse operations! That is, integration "undoes" differentiation and differentiation "undoes" integration.

FTC, Part I: If f is continuous on $[a, b]$, then

$$F(x) = \int_a^x f(t)dt$$

with $a \leq x \leq b$. Further, if F is differentiable on $[a, b]$ then

$$F'(x) = f(x) \Leftrightarrow \frac{d}{dx} \int_a^x f(t)dt = f(x)$$

Recall: A function F is an antiderivative of f if $F'(x) = f(x)$

FTC, Part II: If f is continuous on $[a, b]$. then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f.

Indefinite integrals

Definition: A general antiderivative that looks like the definite integral without the limits of integration is called the indefinite integral:

$$\int f(x)dx = F(x) \Leftrightarrow F'(x) = f(x)$$

Note: A *definite* integral evaluates to a number while the *indefinite* integral evaluates to a family of functions.

Common indefinite integrals:

$$\begin{aligned}
\int [cf(x) + g(x)]dx &= c\int f(x)dx + \int g(x)dx \\
\int k dx &= kx + c \\
\int x^n dx &= \frac{x^{n+1}}{n+1} + c \quad (n \neq 1) \\
\int \frac{1}{x} dx &= \ln|x| + c \\
\int a^x dx &= \frac{a^x}{\ln a} + c \\
\int \sin x dx &= -\cos x + c \\
\int \cos x dx &= \sin x + c \\
\int \sec^2 x dx &= \tan x + c \\
\int \csc^2 x dx &= -\cot x + c \\
\int \sec x \tan x dx &= \sec x + c \\
\int \csc x \cot x dx &= -\csc x + c \\
\int \frac{1}{x^2+1} dx &= \tan^{-1} x + c \\
\int \frac{1}{\sqrt{1-x^2}} dx &= \sin^{-1} x + c \\
\int \sinh x dx &= \cosh x + c \\
\int \cosh x dx &= \sinh x + c
\end{aligned}$$