

# AUTOMORPHISMS OF TOROIDAL LIE ALGEBRAS AND THEIR CENTRAL QUOTIENTS

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Abstract. We describe the structure of the group of automorphisms of Lie algebras of the form  $\mathfrak{g} \otimes_k R$ , when  $R$  has trivial Picard group. We also look at central quotients of the universal covering algebra of  $\mathfrak{g} \otimes_k R$  as well as conjugacy questions.

**Introduction.** Throughout this work  $k$  will denote a field of characteristic zero,  $\mathfrak{g}$  a finite dimensional split simple Lie algebra over  $k$ , and  $R$  a commutative associative and unital algebra over  $k$ . Unless mention to the contrary in what follows *all algebras, algebra homomorphisms, and tensor products are over  $k$* . If  $A$  is an algebra, its group of automorphisms will be denoted by  $\text{Aut } A$ .

Our main interest is the Lie algebra  $\mathfrak{g}(R) := \mathfrak{g} \otimes_k R$ , central quotients of its universal cover, and the corresponding groups of automorphisms. We emphasize one last time that we are viewing  $\mathfrak{g}(R)$  as a *Lie algebra over  $k$*  (in general infinite dimensional), and *not* as a Lie algebra over  $R$  (free of finite rank). The best known example comes from  $R = k[t, t^{-1}]$  and one is then within affine Kac-Moody theory.

The contents of the paper are as follows. In Proposition 1 we give a precise description of the group  $\text{Aut } \mathfrak{g}(R)$  under certain assumptions on  $R$ . This is followed by two remarks drawing comparisons with the analogous result for groups, and with the description of  $\text{Aut } \mathfrak{g}(R)$  due to Peterson and Kac in the case of Laurent polynomials. Following in the spirit of [BM], in Propositions 4 and 5 we relate the structure of the group of automorphisms of a perfect Lie algebra  $\mathfrak{l}$  to that of its universal covering algebra  $\tilde{\mathfrak{l}}$  and its central quotients. As a consequence we obtain the existence of a canonical group isomorphism  $\text{Aut } \mathfrak{g}(R) \simeq \text{Aut } \widetilde{\mathfrak{g}(R)}$  (Corollary 7). When combined with Proposition 1, this last yields the structure of the group of automorphisms for a large class of interesting infinite dimensional Lie algebras. A particularly important case is when  $R$  is the  $k$ -algebras of Laurent polynomials in finitely many variables. Then the  $\widetilde{\mathfrak{g}(R)}$  are the toroidal Lie algebras defined in [MRY]. Finally, in Proposition 9 we extend the conjugacy theorem of  $\mathfrak{g}(R)$  to  $\widetilde{\mathfrak{g}(R)}$ .

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Fix once and for all a split Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ , as well as a base  $\Pi = \{\alpha_1 \dots \alpha_l\}$  of the corresponding root system  $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ . We begin by describing four subgroups of  $\text{Aut } \mathfrak{g}(R)$  that arise in a natural fashion.

- 1 Let  $G$  be the simply connected Chevalley-Demazure group scheme corresponding to  $\mathfrak{g}$ . The group  $G(R)$  of  $R$  points of  $G$  acts on  $\mathfrak{g}(R)$  via the adjoint representation. The kernel of this action is the group  $Z(R)$  of  $R$  points of the centre  $Z$  of  $G$ . The resulting subgroup  $G(R)/Z(R)$  of  $\text{Aut } \mathfrak{g}(R)$ , will be denoted by  $\text{Ad } G(R)$ . (Note that this group is in general smaller than the group of  $R$  points of the adjoint group  $\mathbf{Ad } G$ ).
- 2 A copy of the group  $\text{Aut } \Pi$  of automorphisms of  $\Delta$  that stabilize  $\Pi$ . This group lives inside  $\text{Aut } \mathfrak{g}$ , and lifts to  $\text{Aut } \mathfrak{g}(R)$  by extension of the base ring.
- 3  $H(R) \simeq \text{Hom}(Q, R^\times)$  where  $Q$  is the root lattice of  $(\mathfrak{g}, \mathfrak{h})$  (the  $R$  points of the Cartan subgroup  $H$  of the adjoint group  $\mathbf{Ad } G$  corresponding to  $\mathfrak{h}$ ). If  $\lambda \in \text{Hom}(Q, R^\times)$ , then  $\lambda$  viewed as an element of  $H(R)$  acts on each of the spaces  $\mathfrak{g}^\alpha \otimes_k R$  with  $\alpha \in Q$ , as multiplication by  $\lambda(\alpha)$ . Note that  $\lambda$  is completely determined by its value on the elements of  $\Pi$ . Thus  $H(R) \simeq R^{\times l}$  where  $l$  is the rank of  $\mathfrak{g}$ .
- 4 A copy of the group  $\text{Aut } R$  of automorphisms of the  $k$ -algebra  $R$  acting on  $\mathfrak{g}(R)$  via  $\rho(x \otimes r) = x \otimes \rho(r)$ .

Note that the elements of the first three of these groups are in fact  $R$ -linear, and therefore automorphisms of  $\mathfrak{g}(R)$  as a Lie algebra over  $R$ .

**Proposition 1.** *Let  $\text{Aut}_R \mathfrak{g}(R)$  be the subgroup of  $\text{Aut } \mathfrak{g}(R)$  generated by  $\text{Ad } G(R)$ ,  $H(R)$ , and  $\text{Aut } \Pi$ . Assume that the  $k$ -algebra  $R$  is an integral domain with trivial Picard group, and that there exist a maximal ideal  $\mathfrak{m} \subset R$  such that  $R/\mathfrak{m} \simeq k$  (existence of a rational point in  $\text{Spec } (R)$ ). Then*

- (i)  $\text{Aut } \mathfrak{g}(R) = \text{Aut}_R \mathfrak{g}(R) \rtimes \text{Aut } R$ .
- (ii) *The elements of  $\text{Aut } \mathfrak{g}(R)$  which are  $R$ -linear are precisely those of  $\text{Aut}_R \mathfrak{g}(R)$ .*

The most common examples of such algebras are local domains or noetherian unique factorization domains with rational points. The point is that under the above assumption one has a conjugacy theorem for regular maximal  $k$ -diagonalizable subalgebras of  $\mathfrak{g}(R)$  ([Pzl1]. See 8 below for definitions. See also Remark 11). As we shall presently see, the structure of  $\text{Aut } \mathfrak{g}(R)$  is then very easy to obtain by reasoning along standard lines.

*Proof.* (i) We begin by showing that the four subgroups in question do generate  $\text{Aut } \mathfrak{g}(R)$ . Let  $\sigma \in \text{Aut } \mathfrak{g}(R)$ . By [Pzl1] there exists an element of  $\text{Ad } G(R)$  that takes  $\sigma(\mathfrak{h})$  onto  $\mathfrak{h}$ , and we may thus assume that  $\sigma$  stabilizes  $\mathfrak{h}$ . A standard argument then shows that for all  $\alpha \in \Delta$  we have  $\sigma(\mathfrak{g}^\alpha \otimes R) = \mathfrak{g}^{\sigma^*(\alpha)} \otimes R$ , where  $\sigma^*$  is the transpose inverse of the restriction of  $\sigma$  to  $\mathfrak{h}$ . With the aid of  $\text{Ad } G(k)$  and  $\text{Aut } \Pi$  (see [Bbk] Ch. 8 §5) we may thus assume that  $\sigma$  fixes  $\mathfrak{h}$  pointwise, hence that  $\sigma$  stabilizes  $\mathfrak{g}^\alpha \otimes R$  for all  $\alpha \in \Delta$ .

For each  $1 \leq i \leq \ell$  define functions  $\sigma_i^+$ ,  $\sigma_i^0$ , and  $\sigma_i^-$  from  $R$  to  $R$  by

$$\sigma(e_i \otimes r) = e_i \otimes \sigma_i^+(r), \quad \sigma(\alpha_i^\vee \otimes r) = \alpha_i^\vee \otimes \sigma_i^0(r), \quad \sigma(f_i \otimes r) = f_i \otimes \sigma_i^-(r).$$

We have  $\alpha_i^\vee \otimes 1 = \sigma(\alpha_i^\vee \otimes 1) = \sigma([e_i \otimes 1, f_i \otimes 1]) = \alpha_i^\vee \otimes \sigma_i^+(1)\sigma_i^-(1)$ . In particular,  $1 = \sigma_i^+(1)\sigma_i^-(1)$  so that both  $\sigma_i^+(1)$  and  $\sigma_i^-(1)$  are units of  $R$  inverse of each other. By

multiplying by an element of  $H(R)$  we may assume that  $\sigma_i^+(1) = \sigma_i^-(1) = 1$ . In other words,  $\sigma$  fixes  $e_i$ ,  $\alpha_i^\vee$ , and  $f_i$  for all  $1 \leq i \leq \ell$ . We claim that then  $\sigma \in \text{Aut } R$ .

Indeed. For  $r \in R$  we have  $\alpha_i^\vee \otimes \sigma_i^0(r) = \sigma(\alpha_i^\vee \otimes r) = \sigma([e_i \otimes r, f_i \otimes 1]) = \alpha_i^\vee \otimes \sigma_i^+(r)$ . Similarly for  $\sigma_i^-$ . We conclude that  $\sigma_i^+ = \sigma_i^0 = \sigma_i^-$ , and henceforth denote this common value by  $\sigma_i$ . For  $r$  and  $s$  in  $R$  we have  $\alpha_i^\vee \otimes \sigma_i(rs) = \sigma(\alpha_i^\vee \otimes rs) = \sigma([e_i \otimes r, f_i \otimes s]) = [e_i \otimes \sigma_i(r), f_i \otimes \sigma_i(s)] = \alpha_i^\vee \otimes \sigma_i(r)\sigma_i(s)$ . Thus  $\sigma_i(rs) = \sigma_i(r)\sigma_i(s)$ . Since in addition  $\sigma_i$  fixes  $k$  pointwise (because  $\sigma_i^0$  does) and is clearly additive, it follows that  $\sigma_i$  is an endomorphism of the  $k$ -algebra  $R$ . By considering  $\sigma^{-1}$ , one concludes that  $\sigma_i$  is in fact an automorphism of  $R$ .

Assume next that  $1 \leq i < j \leq \ell$  are such that  $A_{i,j} \neq 0$ . For  $r \in R$  we then have  $e_j \otimes \sigma_j(r) = \sigma([\alpha_i^\vee \otimes A_{j,i}^{-1}r, e_j \otimes 1]) = [\alpha_i^\vee \otimes A_{j,i}^{-1}\sigma_i(r), e_j \otimes 1] = e_j \otimes \sigma_i(r)$ . Since the Cartan matrix  $A$  is indecomposable it follows that all the  $\sigma_i$ 's coincide, and hence that  $\sigma \in \text{Aut } R$  as claimed. We have thus shown that  $\text{Aut}_R \mathfrak{g}(R)$  and  $\text{Aut } R$  generate  $\text{Aut } \mathfrak{g}(R)$ .

Let  $\sigma \in \text{Aut}_R \mathfrak{g}(R) \cap \text{Aut } R$ . Then  $\sigma$  is  $R$ -linear and fixes  $\mathfrak{g} \otimes 1$ . Thus  $\sigma = 1$ . Next we show that  $\text{Aut } R$  normalizes each of the three subgroups generating  $\text{Aut}_R \mathfrak{g}(R)$ . Let  $\mu \in \text{Aut } R$  and identify  $\mu$  with an element of  $\text{Aut } \mathfrak{g}(R)$  as above. Then  $\mu$  acts naturally (as an automorphism) of  $G(R)$  and  $\mu \text{Ad } \mathbf{P} \mu^{-1} = \text{Ad } \mu(\mathbf{P})$ . (The action of  $\mu$  on  $G(R) = \text{Hom}(k[G], R)$  is given by  $\mu(\mathbf{P}) = \mu \circ \mathbf{P}$ . Extending  $\mu$  to  $\text{Aut } R(\epsilon)$  (dual numbers) gives by restriction the action of  $\mu$  on  $\mathfrak{g}(R) = \text{Ker}(G(R(\epsilon)) \rightarrow G(R))$ . The formula in question now follows from the definitions. See [DG] II §4.4) This shows that  $\text{Aut } R$  normalizes  $\text{Ad } G(R)$ . Similarly considerations apply to  $H(R)$ . Since  $\mu$  clearly commutes with  $\text{Aut } \Pi$  the proof of (i) is complete.

(ii) Since the elements of  $\text{Aut}_R \mathfrak{g}(R)$  are  $R$ -linear (ii) follows easily from (i).  $\square$

**Remark 2** In contrast with the Lie algebra case, the study of  $\text{Aut } G(R)$ , specially  $GL_n$  and the other classical groups, has a long history(see [HJW]). The most general results for  $G$  an arbitrary Chevalley group (ommiting some small rank anomalies) are due to Abe ( $R$  a commutative noetherian ring [Abe]) and Chen ( $R$  a commutative domain containing  $\mathbb{Q}$  [Chn]). To compare their results to ours, let us assume that  $G$  is of simply connected type since in this case the group and the algebra have the same group scheme  $\mathbf{Aut } G$  of automorphisms. With the present notation, their results state that  $\mathbf{Aut } G(R)$  is generated by  $N_{G(S)}G(R)$  and  $H(R)$  for some ring extension  $S/R$ . In other words, that some automorphisms of  $G(R)$  can be realized only by conjugating by elements of a larger group  $G(S)$ . What Proposition 1 says is that with the assumptions on  $R$  therein, the extension  $S$  is not necessary. (That this is so in the case of  $R = \mathbb{C}[t, t^{-1}]$  was explicitly shown in [CC] with the aid of Borel -Tits theory.)

In fact the conjugacy approach may shed light into the somehow mysterious nature of  $S$ . Indeed in [Pzl1] conjugacy is translated into a problem about the triviality of line bundles over  $\text{Spec } (R)$  and hence fails if the Picard group of  $R$  is not trivial. But of course, conjugacy does hold if one passes to some nice cover  $S/R$  in the flat topology. Note that this approach would also simplify considerably the calculation of  $\text{Aut } G(R)$  since one can right away assume that one is dealing with an automorphism of  $G(R)$  that stabilizes  $T(k)$ .

Finally, we mention that the group  $\text{Aut } G(R)$  is in general larger than its Lie algebra counterpart  $\text{Aut } \mathfrak{g}(R)$ . Both of them contain a copy of  $\text{Aut}(R)$ , but in addition  $\text{Aut } G(R)$  contains the so called radial automorphisms. These are related to homomorphisms from

$G(R)$  into its centre, and have no nontrivial equivalent in the Lie algebra. (Of course neither the elements of  $\text{Aut}(R)$  nor the radial automorphisms are algebraic).

**Remark 3.** Consider the case  $R = k[t, t^{-1}]$ . The present description of  $\text{Aut } \mathfrak{g}(R)$  is slightly different than the one in [PK], and the resulting correspondence far from trivial to be explicitly written down. As an example we look at the simplest case, namely when  $\mathfrak{g} = \mathfrak{sl}_2$ . Fix an  $\mathfrak{sl}_2$ -triplet  $\{e_1, h_1, f_1\}$  of  $\mathfrak{g}$ . The algebra  $\mathfrak{g}(R)$  is generated by  $e_1$  and  $f_1$  together with  $e_0 := f_1 \otimes t$  and  $f_0 := e_1 \otimes t^{-1}$ . According to [PK], the group  $\text{Aut } \mathfrak{g}(R)$  is generated by  $\text{Ad } G(R)$  as above, together with  $\tilde{H}$ ,  $\text{Aut } \tilde{\Pi}$ , and  $\langle \omega \rangle$  (see below). We begin by showing how the elements of these last three groups are written in terms of the automorphisms of Proposition 1

$\tilde{H} \simeq \text{Hom}(\tilde{Q}, k^\times)$  where  $\tilde{Q}$  is the affine root lattice. For  $\mathfrak{sl}_2$  then, an element of  $\tilde{H}$  is a pair  $\lambda = (\lambda_0, \lambda_1) \in k^\times \times k^\times$  where  $\lambda : e_i \mapsto \lambda_i e_i$  and  $\lambda : f_i \mapsto \lambda_i^{-1} f_i$ . The element  $(\lambda_0, 1)$  corresponds to the element of  $\text{Aut } R$  that takes  $t$  to  $\lambda_0 t$ . On the other hand  $(\lambda_0^{-1}, \lambda_0)$  corresponds to the unique element of  $H(k) \subset H(R)$  that scales the elements of  $e_1 \otimes R$  by  $\lambda_0$ .

$\text{Aut } \tilde{\Pi}$  is the group of automorphisms of the corresponding extended Coxeter-Dynkin diagram. In our case it is the group of order 2 generated by the element  $\pi$  that permutes the  $e_i$ 's and the  $f_i$ 's. Let  $\eta$  be the unique element of  $H(R)$  that scales  $e_1 \otimes R$  by  $-t$ , and let  $\tau \in \text{Ad } G(k) \subset \text{Ad } G(R)$  be the involution that takes a matrix of  $\mathfrak{g}(R)$  into minus its transpose. Then  $\pi = \tau \circ \eta$ .

Finally,  $\omega$  is the Chevalley involution which takes  $e_i$  to  $-f_i$ . An easy calculation shows that  $\omega = \tau \circ \epsilon$  where  $\epsilon$  corresponds to the element of  $\text{Aut } R$  that interchanges  $t$  and  $t^{-1}$ .

As for the reverse description, we see from the above how to write  $\text{Aut } R$  in [PK] terms. We are thus left with  $H(R)$  which the reader can try to do. (The trick here is to observe that the element of  $H(R)$  that acts on  $e_1 \otimes R$  as scalar multiplication by  $t^{2n}$  belongs to  $\text{Ad } G(R)$ , namely conjugation by the diagonal matrix with entries  $t^n$  and  $t^{-n}$ ).

Recall that the universal central extension  $\tilde{\mathfrak{l}} \rightarrow \mathfrak{l}$  of a perfect Lie algebra  $\mathfrak{l}$  is as follows (see [Grl] or [MP] for details). Let  $V = \bigwedge^2 \mathfrak{l} / J$  where  $J$  is the subspace of  $\bigwedge^2 \mathfrak{l}$  spanned by all elements of the form

$$x \wedge [y, z] + y \wedge [z, x] + z \wedge [x, y]$$

with  $x, y, z \in \mathfrak{l}$ . The image of  $x \wedge y$  in  $V$  under the canonical morphisms  $\bigwedge^2 \mathfrak{l} \mapsto V$  will be denoted by  $x \vee y$ . There is a natural Lie algebra structure in the space  $\mathfrak{e} = \mathfrak{l} \oplus V$ , where the bracket is given by

$$[x + v, y + w]_{\mathfrak{e}} = [x, y] + x \vee y.$$

Here  $[x, y]$  denotes the bracket of two elements  $x, y \in \mathfrak{l}$ , and  $v, w \in V$ . Let  $\tilde{\mathfrak{l}}$  be the derived algebra of  $\mathfrak{e}$ . Since  $\mathfrak{l}$  is perfect, given  $x \in \mathfrak{l}$  there exists  $c \in V$  such that  $x + c \in \tilde{\mathfrak{l}}$ . The

canonical map  $\tilde{\mathfrak{l}} \rightarrow \mathfrak{l}$  is thus onto with kernel  $\mathfrak{c} \subset V$ . The resulting central extension

$$(0) \rightarrow \mathfrak{c} \rightarrow \tilde{\mathfrak{l}} \rightarrow \mathfrak{l} \rightarrow (0)$$

of  $\mathfrak{l}$  is *universal* in the sense that there exists a unique morphism from it into any other given central extension of  $\mathfrak{l}$ .

**Proposition 4.** *Let  $\mathfrak{l}$  be a perfect Lie algebra and let  $\tilde{\mathfrak{l}}$  be its universal central extension. Every automorphism  $\theta$  of  $\mathfrak{l}$  admits a unique extension  $\tilde{\theta}$  to an automorphism of  $\tilde{\mathfrak{l}}$ . Furthermore, the map  $\theta \mapsto \tilde{\theta}$  is a group monomorphism.*

*Proof.* Every automorphism  $\theta \in \text{Aut } \mathfrak{l}$  induces an automorphism  $\theta_V$  of  $V$  via  $\theta_V(x \vee y) = \theta(x) \vee \theta(y)$ . It is clear that  $\theta$  extends to an automorphism  $\theta_\epsilon$  of  $\epsilon$  satisfying  $\theta_\epsilon : x + v \mapsto \theta(x) + \theta_V(v)$ . By restriction,  $\theta_\epsilon$  induces an automorphism  $\tilde{\theta}$  of  $\tilde{\mathfrak{l}}$ .

It is clear from the definition that  $\theta \mapsto \tilde{\theta}$  is a group homomorphism. Suppose  $\tilde{\theta} = 1$ . Then for all  $x, y \in \mathfrak{l} \subset \epsilon$  we have

$$\begin{aligned} [x, y] + x \vee y &= [x, y]_\epsilon = \tilde{\theta}([x, y]_\epsilon) = \theta_\epsilon([x, y]_\epsilon) = [\theta_\epsilon(x), \theta_\epsilon(y)]_\epsilon = [\theta(x), \theta(y)]_\epsilon = \\ &= [\theta(x), \theta(y)] + \theta(x) \vee \theta(y) = \theta([x, y]) + \theta(x) \vee \theta(y) \end{aligned}$$

Thus  $\theta$  is the identity on  $[\mathfrak{l}, \mathfrak{l}] = \mathfrak{l}$ , showing that our group homomorphism is injective. Next we show that the lifting of  $\theta$  to  $\tilde{\mathfrak{l}}$  is unique (and hence equal to  $\tilde{\theta}$ ). Let  $\theta_1$  and  $\theta_2$  be two lifts of  $\theta$  to  $\tilde{\mathfrak{l}}$ . Then for  $x \in \tilde{\mathfrak{l}}$  we have

$$\theta_1(x) = \theta_2(x) + x_\mathfrak{c}$$

for some unique  $x_\mathfrak{c} \in \mathfrak{c}$ . Thus for all  $x, y \in \tilde{\mathfrak{l}}$  we have

$$\theta_1([x, y]_{\tilde{\mathfrak{l}}}) = [\theta_2(x) + x_\mathfrak{c}, \theta_2(y) + y_\mathfrak{c}]_{\tilde{\mathfrak{l}}} = [\theta_2(x), \theta_2(y)]_{\tilde{\mathfrak{l}}} = \theta_2([x, y]_{\tilde{\mathfrak{l}}}).$$

Thus  $\theta_1 = \theta_2$  on  $[\tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}] = \tilde{\mathfrak{l}}$ . □

In what follows  $E$  will denote a subspace of  $\mathfrak{c}$ . We look at the structure of the group  $\text{Aut } \tilde{\mathfrak{l}}/E$  of automorphisms of the central quotient<sup>2</sup>  $\tilde{\mathfrak{l}}/E$ . Set

$$\text{Aut}(\mathfrak{l}, E) := \{\theta \in \text{Aut } \mathfrak{l} : \theta_V(E) = E\}.$$

If  $\theta \in \text{Aut}(\mathfrak{l}, E)$ , then  $\tilde{\theta}$  induces an automorphism of  $\tilde{\mathfrak{l}}/E$  that we will denote by  $\tilde{\theta}_E$ .

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<sup>2</sup>For the ring  $R$  of Laurent polynomials the universal central extension  $\widetilde{\mathfrak{g}(R)}$  of  $\mathfrak{g}(R)$  is the derived algebra of an untwisted affine Kac-Moody Lie algebra. The centre of  $\widetilde{\mathfrak{g}(R)}$  is one dimensional and its only nontrivial quotient is therefore  $\mathfrak{g}(R)$  itself. For more general rings  $R$  however, the situation changes. Now the centre of  $\widetilde{\mathfrak{g}(R)}$  may be infinite dimensional and many of the different central quotients of  $\widetilde{\mathfrak{g}(R)}$  are of interest. See [EF],[Ymd] and [ABP] for examples.

**Proposition 5.** *The map  $\theta \mapsto \tilde{\theta}_E$  is a group monomorphism from  $\text{Aut}(\mathfrak{l}, E)$  into  $\text{Aut } \tilde{\mathfrak{l}}/E$ . If  $\mathfrak{l}$  is centreless, this map is an isomorphism.*

*Proof.* That our map is a group monomorphism follows by reducing modulo  $E$  the analogous argument in the previous proof. The only difficulty lies in showing that the map is surjective whenever  $\tilde{\mathfrak{l}}$  is centreless. Let then  $\sigma \in \text{Aut } \tilde{\mathfrak{l}}/E$ . Because  $\tilde{\mathfrak{l}}$  is the universal central extension of  $\tilde{\mathfrak{l}}/E$ , we know that  $\sigma$  admits a (unique) extension  $\tilde{\sigma}$  to  $\tilde{\mathfrak{l}}$ . Now if  $\mathfrak{l}$  is centreless, the centre of  $\tilde{\mathfrak{l}}/E$  is  $\mathfrak{c}/E$ . In this case our  $\sigma \in \text{Aut } \tilde{\mathfrak{l}}/E$  induces an automorphisms of  $\tilde{\mathfrak{l}}/E/(\mathfrak{c}/E) \simeq \mathfrak{l}$  that will be denoted by  $\theta$ . Then both  $\tilde{\theta}$  and  $\tilde{\sigma}$  are extensions of  $\theta$  to  $\tilde{\mathfrak{l}}$ , and therefore equal. Now  $\tilde{\sigma}$  stabilizes  $E$  (since  $E$  is the kernel of  $\tilde{\mathfrak{l}} \rightarrow \tilde{\mathfrak{l}}/E$ ), and therefore so does  $\tilde{\theta}$ . But the restriction of  $\tilde{\theta}$  to  $\mathfrak{c} \subset V$  coincides with  $\theta_V$ . This shows that  $\theta$  is in fact an element of  $\text{Aut}(\mathfrak{l}, E)$ .  $\square$

By taking  $E = \{0\}$  we obtain the following.

**Corollary 6.** *If  $\mathfrak{l}$  is centreless then  $\text{Aut } \mathfrak{l} \simeq \text{Aut } \tilde{\mathfrak{l}}$ .*  $\square$

As a direct consequence of this last we have.

**Corollary 7.** *Let  $R$  be an associative commutative unital  $k$ -algebra. Then  $\text{Aut } \mathfrak{g}(R) \simeq \text{Aut } \widetilde{\mathfrak{g}(R)}$*   $\square$

**8.** Recall that a subalgebra  $\mathfrak{a}$  of a Lie algebra  $\mathfrak{L}$  over  $k$  is said to be  *$k$ -diagonalizable*, if all its elements are  $k$ -diagonalizable : If  $\mathfrak{p} \in \mathfrak{a}$  then  $\text{ad}_{\mathfrak{L}} \mathfrak{p}$ , when viewed as a  $k$ -linear endomorphism of  $\mathfrak{L}$ , is diagonalizable. An abelian  $k$ -diagonalizable subalgebra<sup>3</sup> which is maximal with respect to inclusion, is for simplicity referred to as a MAD<sup>4</sup>.

It is clear that the canonical map  $\pi : \widetilde{\mathfrak{g}(R)} \rightarrow \mathfrak{g}(R)$ , being surjective, maps abelian  $k$ -diagonalizable subalgebras of  $\widetilde{\mathfrak{g}(R)}$  onto abelian  $k$ -diagonalizable subalgebras of  $\mathfrak{g}(R)$ . We will say that an abelian  $k$ -diagonalizable subalgebra of  $\widetilde{\mathfrak{g}(R)}$  is *regular* if its image under  $\pi$  contains a regular element (in the usual sense) of  $\mathfrak{g}(R)$ .

**Proposition 9.** *Assume that  $R$  satisfies the conditions of Proposition 1 above.*

- (a) *The canonical map  $\pi : \widetilde{\mathfrak{g}(R)} \rightarrow \mathfrak{g}(R)$  establishes a bijection between regular MADs of  $\widetilde{\mathfrak{g}(R)}$  and of  $\mathfrak{g}(R)$ .*
- (b) *All regular MADs of  $\widetilde{\mathfrak{g}(R)}$  are conjugate under  $\text{Ad } G(R)$  to  $\mathfrak{h} \oplus \mathfrak{c}$ .*
- (c) *If all MADs of  $\mathfrak{g}(R)$  are regular and conjugate under  $\text{Ad } G(R)$ , then the same is true for  $\widetilde{\mathfrak{g}(R)}$ .*

*Proof.* We begin by showing that  $\mathfrak{h} \oplus \mathfrak{c}$  is a regular MAD of  $\widetilde{\mathfrak{g}(R)}$ . Since for this we will make use of Kassel's description of  $\widetilde{\mathfrak{g}(R)}$ , we briefly recall how this goes ([Ksl] and also [MRY]).

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<sup>3</sup>The abelian assumption is in fact redundant: Any subalgebra of  $\mathfrak{L}$  comprised of  $k$ -diagonalizable elements is necessarily abelian.

<sup>4</sup>Unless  $k = R$  a MAD is not its own normalizer and therefore not a Cartan subalgebra.

Let  $(\Omega_R, d)$  be the module of Kähler differentials of  $R$  over  $k$ , and let  $\bar{\cdot} : \Omega_R \rightarrow \Omega_R/dR$  be the canonical map. Then  $\widetilde{\mathfrak{g}(R)} \simeq \mathfrak{g}(R) \oplus \Omega_R/dR$  with bracket given by

$$(9.1) \quad [x \otimes r, y \otimes s]_{\widetilde{\mathfrak{g}(R)}} = [x, y] \otimes rs + \overline{rds}(x, y)$$

and  $\Omega_R/dR$  central. Here  $x, y \in \mathfrak{g}$ ,  $r, s \in R$ , and  $(\cdot, \cdot)$  is the Killing form of  $\mathfrak{g}$ .

It is clear from (9.1) that  $\mathfrak{h}$  viewed as a subspace of  $\widetilde{\mathfrak{g}(R)}$  is in fact an abelian subalgebra which stabilizes  $\mathfrak{g}(R)$  under the adjoint action, hence that  $\mathfrak{h} \oplus \mathfrak{c}$  is a regular abelian  $k$ -diagonalizable subalgebra of  $\widetilde{\mathfrak{g}(R)}$ . To see that this is maximal note that if  $x + c \in \widetilde{\mathfrak{g}(R)}$  commutes with  $\mathfrak{h} \oplus \mathfrak{c}$  then by (9.1)  $\{0\} = [\mathfrak{h}, x]$  and therefore  $x \in \mathfrak{h} \otimes R$ . Since the eigenvalues of  $\text{ad } x$  are in  $k$ , this last yields  $x \in \mathfrak{h} \otimes k$  as desired. Keeping these observations in mind we now move to the main proof.

(a) Assume  $\mathfrak{a} \subset \widetilde{\mathfrak{g}(R)}$  is a regular MAD. Then  $\pi(\mathfrak{a}) \subset \mathfrak{g}(R)$  is abelian  $k$ -diagonalizable and contains a regular element. By [Pzl1] then,  $\pi(\mathfrak{a})$  is conjugate to  $\mathfrak{h}$  under  $\text{Ad } G(R)$ . In particular,  $\pi(\mathfrak{a})$  is maximal.

Conversely let  $\mathfrak{k}$  be a regular MAD of  $\mathfrak{g}(R)$ . As we have just pointed out, there exists  $\theta \in \text{Aut } \mathfrak{g}(R)$  such that  $\theta(\mathfrak{h}) = \mathfrak{k}$ . Lift  $\theta$  to an automorphism  $\tilde{\theta}$  of  $\widetilde{\mathfrak{g}(R)}$  (Proposition 4). Then

$$\pi^{-1}(\mathfrak{k}) = \pi^{-1}(\theta(\mathfrak{h})) = \tilde{\theta}(\pi^{-1}(\mathfrak{h})) = \tilde{\theta}(\mathfrak{h} \oplus \mathfrak{c})$$

and  $\pi^{-1}(\mathfrak{k})$  is thus a regular MAD as desired.

(b) After identifying  $\text{Ad } G(R)$  with a subgroup of  $\text{Aut } \widetilde{\mathfrak{g}(R)}$  (Proposition 4 or Corollary 7), the result follows from the conjugacy of regular MADs described in (a).

(c) This follows from (b). For examples of rings with this property (notably Laurent polynomial rings) see [Pzl2].  $\square$

**Remark 10.** Note that if  $\mathfrak{k} = \theta(\mathfrak{h})$  is as in (a) above, then  $\mathfrak{k}$  is also an abelian subalgebra of  $\widetilde{\mathfrak{g}(R)}$ . For if  $x \in \mathfrak{g}(R) \subset \widetilde{\mathfrak{g}(R)}$  then  $\tilde{\theta}(x) = \theta(x) + x_{\mathfrak{c}}$  for some unique  $x_{\mathfrak{c}} \in \mathfrak{c}$ . Thus for all  $x, y \in \mathfrak{h}$

$$0 = [x, y]_{\widetilde{\mathfrak{g}(R)}} = \tilde{\theta}([x, y]_{\widetilde{\mathfrak{g}(R)}}) = [\tilde{\theta}(x), \tilde{\theta}(y)]_{\widetilde{\mathfrak{g}(R)}} = [\theta(x) + x_{\mathfrak{c}}, \theta(y) + y_{\mathfrak{c}}]_{\widetilde{\mathfrak{g}(R)}} = [\theta(x), \theta(y)]_{\widetilde{\mathfrak{g}(R)}}.$$

**Remark 11.** The proofs of Propositions 1 and 9 go through as long as all regular MADs of  $\mathfrak{g}(R)$  are conjugate under  $\text{Ad } G(R)$ . We will see in [Pzl2] that this conjugacy result holds even if the integral domain assumption on  $R$  is considerably relaxed. Thus these two results are still valid for this more general class of  $k$ -algebras.

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