AFFINE KAC-MOODY LIE ALGEBRAS AS TORSORS OVER THE PUNCTURED LINE

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Abstract. We interpret and develop a theory of loop algebras as torsors (principal homogeneous spaces) over $\operatorname{Spec}(k[t,t^{-1}])$. As an application, we recover Kac's realization of affine Kac-Moody Lie algebras.

Introduction. There is a beautiful construction of Victor Kac's, realizing affine Kac-Moody Lie algebras over the complex numbers as (twisted) loop algebras. The construction gives explicit generators for the algebras, which are then shown to satisfy the relations corresponding to the affine Cartan matrix at hand.

In this short note, we propose to look at loop algebras in a completely different way. The basic idea is to view loop algebras as algebras over a ring of Laurent polynomials R, all of which become isomorphic after an étale covering $R \to S$. Thus loop algebras become torsors over $\operatorname{Spec}(R)$ under the group of automorphisms of the algebra at hand. Since this point of view applies to arbitrary algebras and base fields, we are able to obtain some rather general new results about loop algebras in **8** and **10**. As an application, we show how to recover from this Kac's original result. This is done in **11.**

We begin with a review of loop algebras in **1**, and then recall Kac's construction in **2**. This is followed by some abstract results on algebraic groups (**3** through **5**) which are later needed. The description of loop algebras as torsors is given in **7**.

0. Conventions and notation. Throughout this note k will denote a field. If G is a k-group and $X = \operatorname{Spec}(R)$ an affine k-scheme, the X-group $G \times_{\operatorname{Spec}(k)} X$ will be denoted by G_X or G_R . For an X-group F the Cěch cohomology $\check{H}^1(X_{\text{ét}}, F)$ on the étale site of X wil be denoted simply by $H^1(X,F)$. For terminology and results about schemes and principal homogeneous spaces (torsors), the reader is referred to [DG] Ch. 3.4, [Jtz], [Mln] Ch. 3, and [SGA1].

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We fix once and for all a positive integer m, and denote by $^-\colon \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ the canonical map. For convenience we henceforth set

$$R = k[z^m, z^{-m}] \text{ and } S = k[z, z^{-1}].$$

1. Loop algebras. Let A be an algebra over k of a certain "type" (eg. associative, Lie, Jordan, etc) which we assumed comes equiped with a $\mathbb{Z}/m\mathbb{Z}$-grading Σ. Thus

$$A = \bigoplus_{0 \leq i < m} A_{\bar{\imath}}$$

where the $A_{\bar{\imath}}$ are subspaces of A such that $A_{\bar{\imath}_1} A_{\bar{\imath}_2} \subset A_{\bar{\imath}_1 + \bar{\imath}_2}$. We then define the *loop algebra* of A with respect to the given grading Σ by

$$L(A, \Sigma) := \bigoplus_{i \in \mathbb{Z}} A_{\bar{\imath}} \otimes z^i \subset A \otimes_k k[z, z^{-1}] := A \otimes_k S.$$

If the grading is trivial, namely if $A_{\bar{0}} = A$, then $L(A, \Sigma) \simeq A \otimes_k R \simeq A \otimes_k S$. Loop algebras isomorphic as k-algebras to $A \otimes_k S$ are said to be *trivial.*

The most interesting loop algebras come from automorphisms of finite order when the base field k contains primitive m-th roots of unity. Fix such a root ξ, and let σ be an automorphism of A of period m. (Eventually we want to compare loop algebras coming from automorphisms of different order. Because of this, is important that we work with periods and not orders). Then A decomposes as the direct sum of eigenspaces $A_{\bar{\imath}}$ where σ acts on $A_{\bar{\imath}}$ as scalar multiplication by ξ^i. We thus obtain a $\mathbb{Z}/m\mathbb{Z}$-grading Σ of A as above. Conversely, any such grading Σ comes from a period m automorphism σ of A. The resulting loop algebra will be denoted by $L(A, \sigma)$, or simply by $L(\sigma)$ if A is understood from the context. Note that $L(\mathrm{id}_A)$ is trivial.

It is easy to verify that up to isomorphism, the k-algebra $L(\sigma)$ is independ on the choice of period m of σ, as well as of the primitive m-th root of unity ξ. Note also that loop algebras are in a natural way algebras of the same type as A (eg. loop algebras of a Lie algebra are Lie algebras etc.), and are viewed as such in what follows.

Here is the most remarkable application of loop algebras.

Theorem 2. *Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$, and let $^-$ denote the canonical map from $\mathrm{Aut}(\mathfrak{g})$ onto the group $\mathrm{Out}(\mathfrak{g})$ of outer automorphisms of $\mathfrak{g}$. Then.*
- (i) *$L(\sigma) \simeq L(\overline{\sigma})$ for all $\sigma \in \mathrm{Aut}(\mathfrak{g})$.*
- (ii) *Let $\hat{\mathfrak{l}}$ be an affine Kac-Moody Lie algebra, and let $\mathfrak{l}$ be its derived algebra modulo its centre. There exists $\mathfrak{g}$ and $\pi \in \mathrm{Out}(\mathfrak{g})$ as above such that $\mathfrak{l} \simeq L(\pi)$.*
- (iii) *In (ii) above, $\mathfrak{g}$ is unique up to isomorphism, and π unique up to conjugacy in $\mathrm{Out}(\mathfrak{g})$.*

Proof. Parts (i) and (ii) are due to Kac (see [Kac] Theorem 8.3]). For (iii) one needs the conjugacy theorem of Peterson and Kac [PK] □

Loop algebras are thus concrete realizations of the affine Kac-Moody Lie algebras (in fact, there are no known example of realizations of Kac-Moody Lie algebras aside from the finite and affine case). We propose now to give new insight into this theorem, as well as obtain new results about loop algebras in general, by interpreting such algebras as torsors over the punctured line. To this end, we begin with some results on the cohomology of algebraic groups.

The following result shows that the classical vanishing of H^1 theorems of Steinberg and of Borel and Springer, hold for certain semisimple group schemes over Dedekind domains.

Proposition 3. *Let D be a Dedekind domain and K its field of quotients. Set $X = \mathrm{Spec}\,(D)$. Let G be a quasisplit reductive connected X-group, B a Borel subgroup of G, and T a maximal torus of B.*

- (i) *The canonical map $H^1(X, T) \to \mathrm{Ker}\,(H^1(X, G) \to H^1(K, G_K))$ is surjective. In particular, this kernel is trivial whenever $H^1(X, T)$ is trivial.*
- (ii) *Assume G is semisimple of either adjoint or simply connected type. If all connected étale coverings of X have trivial Picard group, then $H^1(X, T)$ is trivial.*
- (iii) *Assume G and X are as in (ii). If K is of cohomological dimension 1, then $H^1(X, G)$ is trivial.*
- (iv) *Assume that X and K are as in (iii). If G is semisimple then the canonical map $H^1(X, \mathbf{Aut}(G)) \to H^1(X, \mathbf{Out}(G))$ is bijective.*

Proof. (i) For G semisimple, this is the extent of Satz 3.2 in [Hrd]. Here is another proof based on an idea (used in [C-TO] in the case of a base field) more in tune with the spirit of this note. A torsor Y on the kernel in question is rationally trivial, i.e. it admits a section over a non-empty Zariski open of X. Consider the exact sequence.

$$1 \to B \to G \to G/B \to 1$$

as well as the contracted product $Y \times^G G/B$. It is clear that the structure morphism $\kappa : Y \times^G G/B \to X$ admits a rational section. Since κ is proper and X is one dimensional and regular, this section extends to all of X ([EGA] Ch. II Cor. 7.3.6). By [DG] Ch.III 4.4.6 the G-torsor Y comes from B, thence from T given that $H^1(X, \mathrm{rad}^{\mathrm{u}}(B)) = \{0\}$ ([SGA3] XXVI Cor. 2.2)

(ii) By [SGA3] XXIV 3.13-3.15 and [Hrd] 1.4 there exists a finite family of connected étale coverings X_i/X of X such that

$$T = \prod_i R_{X_i/X}(\mathbb{G}_m).$$

where the $R_{X_i/X}$ are Weil restrictions. By Shapiro's lemma one has

$$H^1(X, T) = \prod_i H^1(X_{i\,\text{ét}}, \mathbb{G}_m) = \prod_i \mathrm{Pic}\,(X_i) = \{0\}.$$

(iii) By classical results of Steinberg and of Borel and Springer $H^1(K, G_K)$ is trivial. (See [BS] 8.2, [Stb], and [JPS1] Ch. 3.1 and 3.2). Now (iii) follows from (i) and (ii)

(iv) Recall from [SGA3] XXV the existence of a split exact sequence of X-groups

$$1 \to \mathbf{Ad}(G) \to \mathbf{Aut}(G) \to \mathbf{Out}(G) \to 1$$

The group $\mathbf{Out}(G)$ is a finite constant twisted group, and admits a section $s : \mathbf{Out}(G) \to \mathbf{Aut}(G)$ whose image consists of those elements of $\mathbf{Aut}(G)$ that stabilize both B and T . Passing to cohomology yields

$$H^1(X, \mathbf{Ad}(G)) \to H^1(X, \mathbf{Aut}(G)) \to H^1(X, \mathbf{Out}(G)).$$

Let Y be an X-torsor under $\mathbf{Out}(G)$, and consider the groups ${}_Y\mathbf{Ad}(G)$ (resp.${}_Y\mathbf{Ad}(B)$, ${}_Y\mathbf{Ad}(T)$) obtained from $\mathbf{Ad}(G)$ (resp. $\mathbf{Ad}(B)$, $\mathbf{Ad}(T)$) by twisting by Y. Being a form of $\mathbf{Ad}(G)$, the group ${}_Y\mathbf{Ad}(G)$ is semisimple and of adjoint type as well. It is also quasisplit by means of $({}_Y\mathbf{Ad}(B), {}_Y\mathbf{Ad}(T))$. By (iii) then, $H^1(X, {}_Y\mathbf{Ad}(G))$ is trivial. From this it follows that the map in question is injective. The surjectivity is clear because the existence of the section s. □

Note. A priori ${}_Y\mathbf{Ad}(G)$ is a sheaf of groups on X. That it is an affine and smooth scheme over X follows from descent. That its geometric fibers are reductive and connected follows from the analogous properties for $\mathbf{Ad}(G)$. Thus ${}_Y\mathbf{Ad}(G)$ *is* a reductive group in the sense of [SGA3]. Along similar lines ${}_Y\mathbf{Ad}(B)$ is a Borel subgroup ...

The following result shows that the assumption on étale coverings made in part (ii) of Proposition 3, holds in a crucial case.

Proposition 4. (P. Gille) *Assume k is of characteristic 0. Every connected finite étale covering of* $\mathrm{Spec}\,(k[t, t^{-1}])$ *has trivial Picard group.*

Proof. Let $Y \to X := \mathrm{Spec}\,(k[t, t^{-1}])$ be one such covering. Fix an element $x \in X(k)$. From [SGA1] Exp. IX Th. 6.3.1 together with a Theorem of Grauert-Remmert ([SGA4] Exp. XI Th. 4.3), as well as from [SGA1] Exp. XIII Cor. 2.12, it follows that the fundamental group $\Pi_1(X, x)$ is the semidirect product $(\mathrm{inv}\lim \mu_n(k_s)) \rtimes \mathrm{Gal}(k_s/k)$, where the μ_n come from the Kummer coverings. There thus exists a positive integer m, a finite Galois field extension L/k containing a primitive m-th root of unity, and a subgroup Γ of $\mu_m(L) \rtimes \mathrm{Gal}(L/k)$, such that $Y = Y_0/\Gamma$ where Y_0 is the k–variety defined by the morphism $Y_0 = X_L \to \mathrm{Spec}\,(L) \to \mathrm{Spec}\,(k)$. As the morphism $Y_0 \to Y$ is a Galois covering, the beginning of the Hochschild–Serre spectral sequence $E_2^{p,q} = H^p(\Gamma, H^q(Y_{0\,\text{ét}}, \mathbb{G}_m)) \Longrightarrow H^{p+q}(Y, \mathbb{G}_m)$ yields an exact sequence

$$0 \to H^1(\Gamma, H^0(Y_0, \mathbb{G}_m)) \to H^1(Y, \mathbb{G}_m) \to H^1(Y_{0\,\text{ét}}, \mathbb{G}_m)^\Gamma.$$

Since $H^1(Y_{0\,\text{ét}}, \mathbb{G}_m) = \mathrm{Pic}\,(Y_0) = \{0\}$, we get an isomorphism $H^1(\Gamma, H^0(Y_0, \mathbb{G}_m)) \simeq \mathrm{Pic}\,(Y)$. One has an exact sequence of Γ–modules (and of $\mu_m(L) \rtimes \mathrm{Gal}(L/k)$–modules)

$$0 \to L^\times \to H^0(Y_0, \mathbb{G}_m) \to \mathbb{Z} \to 0,$$

where $\mathbb{Z}$ has trivial Γ–action. By Hilbert's theorem 90 one has $H^1(\Gamma, L^\times) = \{0\}$, and as Γ is finite, one also has $H^1(\Gamma, \mathbb{Z}) = \{0\}$. Thus $H^1(\Gamma, H^0(Y_0, \mathbb{G}_m)) = \{0\}$ and therefore $\mathrm{Pic}\,(Y) = \{0\}$ as desired. □

Part (i) of the next result is an easy, but useful generalization of a result of [C-TO].

Proposition 5. *Let k be an infinite perfect field and let G be a smooth connected linear algebraic k-group. Let X be a nonempty open subscheme of* $\mathrm{Spec}\,(k[t]) = \mathbb{A}^1$. *Then.*

(i) *The canonical map $H^1(X,\, G_X) \to H^1(k(t),\, G_{k(t)})$ has trivial kernel.*

(ii) *$H^1(X, G_X)$ is trivial in the following three cases.*

- (a) *If $H^1(k(t), G_{k(t)})$ is trivial.*
- (b) *If k is algebraically closed and of characteristic 0.*
- (c) *If k is algebraically closed and G is reductive.*

Proof. Let us begin by showing that every Zariski G-torsor over X (i.e. a torsor that can be trivialized by a Zariski covering of X) extends to $\mathbb{A}^1$ (this much holds for any group scheme G). Indeed. The underlying space of X is obtained by removing a finite set F of points from the affine line. Let now Y be a Zariski G-torsor over X. Let U be the intersection of a finite number of non empty open subschemes of X that cover X, and over which Y is trivial. Finally, let Z be the trivial G-torsor over the open subscheme of $\mathbb{A}^1$ corresponding to $U \cup F$. Then by gluing Y with Z along U we obtain a torsor over $\mathbb{A}^1$ as desired.

(i) By reasoning as in Théorème 2.1 of [C-TO] we may assume that G is reductive and connected. A Theorem of Nisnevich [Nsn] then shows that the kernel of the map in question is comprised precisely of Zariski G-torsors over X. But we have seen that any such torsor extends to the full affine line. Now (i) follows from [C-TO] Corollaire 2.3 which asserts that (i) does hold for $\mathbb{A}^1$.

(ii) By the theorems of Steinberg and Borel-Springer mentioned above, $H^1(k(t), G_{k(t)})$ is trivial under the assumptions of either (b) or (c). Thus (b) and (c) reduce to (a), and this last holds by (i). □

Remark 6. Assume that m is invertible in k. Then the finite covering $\mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R)$ is étale and in fact Galois, with Galois group $\Gamma \simeq \mathbb{Z}/m\mathbb{Z}$ (see **7** infra). Assume now that we are under one of the cases of Proposition 5 (ii). Then the usual non-abelian cohomology $H^1(\Gamma,\, G(S))$ vanishes. If $k = \mathbb{C}$ and G is semisimple, we recover the observation made in [Kac] §8.9.

7. Loop algebras as torsors. We return now to our general setup of **1** and consider an arbitrary k-algebra A, together with $\mathbb{Z}/m\mathbb{Z}$-grading Σ. For convenience we henceforth denote $\mathrm{Spec}\,(R) := \mathrm{Spec}\,(k[t^m, t^{-m}])$ by X and $\mathrm{Spec}\,(S) := \mathrm{Spec}\,(k[t, t^{-1}])$ by Y

The loop algebra $L(\Sigma) = L(A, \Sigma)$ is naturally an R-algebra, and it is not hard to show that $L(\Sigma) \otimes_R S \simeq A \otimes_k S$ as S-algebras (see [ABP]). In other words, $L(\Sigma)$ is an S/R-form of $A \otimes_k S$. Since S/R is a faithfully flat and finitely presented (fppf), $L(\Sigma)$ is an $\mathbf{Aut}(A_X)$-torsor over X. In this context, $\mathbf{Aut}(A_X)$ stands for the sheaf of groups on the flat site of X that attaches to X'/X, the group of $\mathcal{O}(X')$-algebra automorphism of $A \otimes_k \mathcal{O}(X')$ of the same type as A.

The isomorphism class of the $R-$algebra $L(\Sigma)$ is thus an element of $H^1(X_{\text{fppf}}, \mathbf{Aut}(A_X))$ that we denote by $L^1(\Sigma)$. To say that $L^1(\Sigma_1) = L^1(\Sigma_2)$ is to say that $L(\Sigma_1)$ and $L(\Sigma_2)$ are isomorphic as *algebras over* R. It is clear then that $L(\Sigma_1)$ and $L(\Sigma_2)$ are a fortiori isomorphic as algebras over k. The converse does not hold in general, but as we shall see in **10** and **11** , it does hold in some very interesting cases.

Next we assume that the base field k contains a primitive m-th root of unity ξ, and go on to describe explicitly how to construct $L^1(\sigma)$ from σ. The finite covering $Y \to X$ is étale and in fact Galois. Its Galois group Γ can be identifyed with $\mathbb{Z}/m\mathbb{Z}$ where $1 + m\mathbb{Z}$ acts on S via $t \to \xi t$. This leads to a natural action of Γ on the abstract group of automorphisms $\text{Aut}A \otimes_k S := \mathbf{Aut}(A_X)(Y)$ of the $S-$algebra $A \otimes_k S$. The map $u : n + m\mathbb{Z} \mapsto \sigma^{-n}$ is clearly a 1-cocycle in $Z^1(\Gamma, \mathbf{Aut}(A_X)(Y)) = Z^1((Y/X)_{\text{fppf}}, \mathbf{Aut}(A_X))$, and a straightforward computation shows that the form corresponding to u is precisely $L(\sigma)$. To obtain then $L^1(\sigma)$ we simply follow u along the canonical maps $Z^1((Y/X)_{\text{fppf}}, \mathbf{Aut}(A_X)) \to H^1((Y/X)_{\text{fppf}}, \mathbf{Aut}(A_X)) \subset H^1(X_{\text{fppf}}, \mathbf{Aut}(A_X))$

Kac's theorem states that for complex simple Lie algebras, the isomorphism class of a covering algebra $L(\sigma)$ depends only on the outer part of σ. In addition, if σ is inner then $L(\sigma)$ is trivial. The following two Propositions show this to be a particular instance of a rather general picture.

Proposition 8. *Let k be an infinite perfect field. Assume that the k-algebra A is finite dimensional, and that the linear algebraic k-group $\mathbf{Aut}(A)$ is smooth and connected. If either of the conditions of Proposition 5 (ii) hold, then all loop algebras of A are trivial.*

Proof. By **7**, it suffices to show that $H^1(X_{\text{fppf}}, \mathbf{Aut}(A_X))$ is trivial. Since $\mathbf{Aut}(A_X)$ is smooth, we may replace the fppf by the étale topology. Now apply Proposition 5 (ii). □

Example 9. If A is the associative unital algebra of n by n matrices with entries in an algebraically closed field k, then all its covering algebras are trivial. Indeed. $\mathbf{Aut}(A) \simeq \mathbf{PGL}_n$.

Proposition 10. *Let k be an algebraically closed field of characteristic* 0*. Assume that the k-algebra A is finite dimensional, and that the linear algebraic k-group $\mathbf{Aut}(A)$ coincides with the group of automorphisms $\mathbf{Aut}(G)$ of some semisimple algebraic k-group G. Set $X = \text{Spec}\, k[t, t^{-1}]$. Then there exists a canonical bijection between the following four sets:*

(1) $H^1(X, \mathbf{Aut}(G_X))$
(2) $H^1(X, \mathbf{Out}(G_X))$
(3) *Conjugacy classes of the (abstract) finite group* $\text{Out}(G) := \mathbf{Out}(G)(k)$.
(4) *Isomorphism classes in k-alg of loop algebras of A.*

In particular, all these sets are finite.

Proof. (1) $\simeq$ (2). This was established in Theorem 3 (iv).
(2) $\simeq$ (3). Let E be the set of continuous group homomorphisms from the fundamental group $\Pi(X, x)$ into $\text{Out}(G)$. Clearly $\text{Out}(G)$ acts on E by conjugation. Since $\mathbf{Out}(G_X)$ is finite and constant, $H^1(X, \mathbf{Out}(G_X))$ can be computed as the quotient set $E/\text{Out}(G)$ [SGA1]. Now from Proposition 4 we know that $\Pi(X, x) \simeq \text{inv lim}\, \mathbb{Z}/n\mathbb{Z}$. It follows that the elements of E can be identified in a natural way with elements of $\text{Out}(G)$, and that then two

elements of E are equivalent under the action of $\mathrm{Out}(G)$ if and only if the corresponding elements of $\mathrm{Out}(G)$ are conjugate.

(3) $\simeq$ (4). In view of the explicit construction of $L^1(\sigma)$ given in **7**, we have no choice but to assign to $L(\sigma)$ the conjugacy class of $\overline{\sigma}^{-1}$ Here $\sigma \in \mathrm{Aut}(A) = \mathbf{Aut}(A)(k) = \mathbf{Aut}(G)(k) = \mathrm{Aut}(G(k))$, and $^- : \mathrm{Aut}(G(k)) \to \mathrm{Out}(G)$ is the canonical map. This process is surjective and we claim it factors through isomorphisms in k-alg: Consider two automorphisms σ_1 and σ_2 of A of period m_1 and m_2, and assume the corresponding loop algebras $L(\sigma_1)$ and $L(\sigma_2)$ are isomorphic as $k-$algebras. We claim that $\overline{\sigma}_1^{-1}$ and $\overline{\sigma}_2^{-1}$ are conjugate elements of $\mathrm{Out}(G)$. (Of course all is well if our isomorphism was over R. See **7**). To establish the claim we reason as in in [ABP]. For completeness, we outline how this argument goes.

Since neither the k-isomorphism classes of the two loop algbras, nor the claim depends on the choice of period, we may assume that both automorphisms have the same period m. Set $t = z^m$, so $X = \mathrm{Spec}\,(R)$. Fix a $k-$algebra isomorphism ψ between $L(\sigma_1)$ and $L(\sigma_2)$. Then ψ induces an automorphism ψ_Z of the centroid Z of $L(\sigma_1)$. In our case $Z \simeq R$, and there are two "types" of automorphisms of Z: those that do not interchange t^m and t^{-m}, and those which do . The upshot of this is that $L(\sigma_1)$ in is isomorphic as an $R-$algebra to either $L(\sigma_2)$ or $L(\sigma_2^{-1})$ depending on the type of ψ_Z. From this and the correspondence (1) $\simeq$ (2) $\simeq$ (3) above, it follows that $\overline{\sigma}_1^{-1}$ is conjugate in $\mathrm{Out}(G)$ to either $\overline{\sigma}_2^{-1}$ or $\overline{\sigma}_2$. But $\mathrm{Out}(G)$, being a subgroup of the group of automorphisms of the corresponding Dynkin diagram, has the property that all its elements are conjugate to their inverses. The claim follows.

We thus have a surjective map from (4) to (3) sending the isomorphism class of $L(\sigma)$ to the conjugacy class of $\overline{\sigma}^{-1}$. To see this is injective note that if $\overline{\sigma}_1^{-1}$ and $\overline{\sigma}_2^{-1}$ are conjugate, then the correspondence (1) $\simeq$ (2) $\simeq$ (3) yields that the loop algebras $L(\sigma_1)$ and $L(\sigma_2)$, when viewed as torsors, induce the same element of $H^1(X, \mathbf{Aut}(G_X))$. But this means that $L(\sigma_1)$ and $L(\sigma_2)$ are isomorphic as $R-$algebras and, a fortiori, as $k-$algebras. □

We can now recover the parts of Theorem 2 concerning loop algebras.

Corollary 11. *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over an algebraically closed field k of characterisitic* 0*. Let σ_1 and σ_2 be two automorphisms of $\mathfrak{g}$ of finite order. For $L(\sigma_1)$ to be isomorphic to $L(\sigma_2)$ as algebras over k, it is necessary and sufficient that $\overline{\sigma}_1$ and $\overline{\sigma}_2$ be conjugate in* $\mathrm{Out}(\mathfrak{g})$.

Proof. Let G be the simply connected Chevalley-Demazure group corresponding to $\mathfrak{g}$. Then $\mathbf{Aut}(G_X) \simeq \mathbf{Aut}(\mathfrak{g}_X)$ with $\mathbf{Out}(G)$ corresponding to $\mathbf{Out}(\mathfrak{g})$ ([SGA3] XXV). □

Remark 12 It is natural to ask if Proposition 10 holds for symmetrizable Kac-Moody Lie algebras. We look at this problem in [ABP], and make good progress by using essentially what amounts to Remark 6, in conjunction with the Gantmacher-like decomposition of automorphisms described in [KW].

It would be interesting to know if the answer to this question can be had by purely cohomological methods (as the finite dimensional case here as well as [Pzl1] and [Pzl2] seem to suggest is possible). The abstract construction of section **7** applies, but the real difficulty of course comes when one tries to recreate Proposition 3 for the various group schemes attached to Kac-Moody algebras (See [Tts]). This appears to be a very difficult question, but in the affine case at least, progress should be possible.

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