LINE BUNDLES AND CONJUGACY THEOREMS FOR TOROIDAL LIE ALGEBRAS

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We link the Picard group of Spec$R$ to the question of conjugacy of maximal abelian diagonalizable subalgebras of $R \otimes \mathfrak{g}$.

Nous faisons le lien entre le groupe de Picard de Spec$R$ et la question de conjugation de sous-algèbres abéliennes maximales diagonalizables de $R \otimes \mathfrak{g}$.

Throughout k will denote a field of characteristic zero. Unless specifically mentioned otherwise *all algebras, tensor products, vector spaces, and schemes are over* k.

One of the central results of classical Lie theory is Chevalley's theorem establishing that all split Cartan subalgebras of a simple finite dimensional Lie algebra $\mathfrak{g}$ are conjugate under its adjoint group. The analogous result for invariant (i.e symmetrizable) Kac-Moody algebras is due to Peterson and Kac (See [PK] and also Ch. 7 of [MP]) . As a consequence of their work one knows that all maximal abelian k-diagonalizable subalgebras of the loop algebra $k[t, t^{-1}] \otimes \mathfrak{g}$ are conjugate (We reserve the terminology "Cartan subalgebra" for nilpotent subalgebras which are self-normalized. See [BP]). Now it is reasonable to expect that conjugacy questions for loop algebras, or more generally for algebras of the form $R \otimes \mathfrak{g}$, can be dealt with in a direct fashion. The following result is a small step in this direction[2].

Theorem 1. *Let $\mathfrak{g}$ be a finite dimensional split simple Lie algebra and $\mathfrak{G}$ its simply connected Chevalley-Demazure group scheme. Let R be an integral domain and $\mathfrak{X} = Spec\, R$ its corresponding scheme. Assume that the Picard group of $\mathfrak{X}$ is trivial and $\mathfrak{X}(k)$ is not empty. Then all regular maximal abelian k-diagonalizable subalgebras of $R \otimes \mathfrak{g}$ are conjugate under $\mathfrak{G}(R)$.*

Let $\mathfrak{g}$, $\mathfrak{G}$, $\mathfrak{X}$, and R be as in the statement of Theorem 1. The residue field of an element x of $\mathfrak{X}$ will be denoted by $k(x)$. For convenience in what follows the group $\mathfrak{G}(k(x))$ will be denoted simply by $\mathfrak{G}(x)$, and the corresponding group homomorphism $\mathfrak{G}(R) \to \mathfrak{G}(x)$ by $\mathtt{P} \mapsto \mathtt{P}(x)$.

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The constructions of the last paragraph above can be repeated, *mutatis mutandi*, if we replace $\mathfrak{G}$ by its Lie algebra functor $\mathfrak{g}(\ )$. Since $\mathfrak{g}$ is finite dimensional we have that $\mathfrak{g}() = \mathrm{Hom}_{k\text{–alg}}(S(\mathfrak{g}^*),\ )$. Thus $\mathfrak{g}(F) = F \otimes \mathfrak{g}$ for any associative commutative unital algebra F. In particular $\mathfrak{g} \simeq \mathfrak{g}(k)$.

Let $f_{\mathrm{reg}} \in S(\mathfrak{g}^*)$ be the polynomial function defining the basic Zariski open dense set of regular elements of $\mathfrak{g}$ (see [Bbk] Ch. VII). Let F as before be an associative commutative unital algebra. Since f_{reg} is defined over k, we can think of it as a polynomial function on the free F-module $\mathfrak{g}(F)$. An element $\mathtt{p}$ of $\mathfrak{g}(F)$ will be said to be *regular*[3] if $f_{\mathrm{reg}}(\mathtt{p})$ is a unit of F , and to be *k-diagonalizable* if $ad\,\mathtt{p}$ is diagonalizable when viewed as a k-linear endomorphism of $\mathfrak{g}(F)$. Finally a subalgebra of $\mathfrak{g}(F)$ will be said to be *regular* if it contains a regular element.

Proposition 1. *Let* $\mathtt{p} \in \mathfrak{g}(R)$. *Then*

(i) *If* $\mathtt{p}$ *is regular then* $\mathtt{p}(x) \in \mathfrak{g}(x)$ *is regular for all* $x \in \mathfrak{X}$.

(ii) *If* $\mathtt{p}$ *is k-diagonalizable then* $ad\,\mathtt{p}(x) \in \mathrm{End}_{k(x)\text{–lin}}\mathfrak{g}(x)$ *is k-diagonalizable for all* $x \in \mathfrak{X}$.

(iii) *Assume that* $\mathtt{p}$ *is k-diagonalizable and* $f_{\mathrm{reg}}(\mathtt{p}) \neq 0$. *Then* $\mathtt{p}$ *is regular. In addition if* $x_0 \in \mathfrak{X}(k)$, *then* $\mathtt{p}(x)$ *and* $\mathtt{p}(x_0)$ *belong to the same orbit under the adjoint action of* $\mathfrak{G}(x)$ *on* $\mathfrak{g}(x)$.

Proof. The first two parts are easy. From the assumptions in (iii) one easily concludes that $f_{\mathrm{reg}}(\mathtt{p}) \in k^{\times}$, hence that $\mathtt{p}$ is regular. Given $x \in \mathfrak{X}$ then, there is no loss of generality in assuming that $\mathtt{p}(x)$ and $\mathtt{p}(x_0)$ belong to a split Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}(x)$. Now if these two elements were not to belong to the same $\mathfrak{G}(x)$-orbit, there would exist a polynomial function $f \in S(\mathfrak{h}^*)^W$ which would distinguish them. That this is not the case follows from the following two observations: Firstly that f is a linear combination of polynomial functions of the form $\mathfrak{h} \ni h \mapsto tr_V(\rho(h))^n$ where $n \in \mathbb{N}$ and $\rho : \mathfrak{g}(x) \to gl(V)$ is a finite dimensional representation of $\mathfrak{g}(x)$, and secondly that the eigenvalue assumption on $ad\,\mathtt{p}$ implies that the value of $tr_V(\rho(\mathtt{p}(x)))$ does not depend on $x \in \mathfrak{X}$. $\square$

Fix an element x_0 of $\mathfrak{X}(k)$. Let $\mathtt{p} \in \mathfrak{g}(R)$ be regular and k-diagonalizable, and set $\mathtt{p}_0 = \mathtt{p}(x_0)$. Proposition 1(iii) yields that $\mathtt{p}_0$ belongs to a unique split Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}$. Let $\mathfrak{T}_0$ be the split torus of $\mathfrak{G}$ corresponding to $\mathfrak{h}_0$. Once again we invoke Proposition 1(iii), this time to see that when $\mathtt{p}$ is viewed as an element of $\mathrm{Hom}_{k\text{–alg}}(S(\mathfrak{g}^*), R)$, it factors through the ideal defining the Zariski closed orbit $\mathfrak{G}(k) \cdot \mathtt{p}_0 \subset \mathfrak{g}$ (here and in what follows $\cdot$ denotes the appropriate adjoint action). As a consequence we deduce the existence of a compatible morphism of schemes $\Psi_{\mathtt{p}} : \mathfrak{X} \to \mathfrak{G}/\mathfrak{T}_0$. The next result is then clear.

Proposition 2. *The following are equivalent.*

(i) $\mathtt{p}$ *can be conjugated to* $\mathtt{p}_0$ *by* $\mathfrak{G}(R)$.

(ii) *There exists a morphism* $\widehat{\Psi_{\mathtt{p}}} : \mathfrak{X} \to \mathfrak{G}$ *rending the following diagram commutative*

$$\begin{array}{ccc} \widehat{\Psi_{\mathtt{p}}} & & \mathfrak{G} \\ & \nearrow & \downarrow^{q} \\ \mathfrak{X} & \underset{\Psi_{\mathtt{p}}}{\rightarrow} & \mathfrak{G}/\mathfrak{T}_0 \end{array}$$

(iii) *The pull-back* $\mathfrak{X} \times_{\mathfrak{G}/\mathfrak{T}_0} \mathfrak{G}$ *is a trivial principal* $\mathfrak{T}_0$*-bundle over* $\mathfrak{X}$. $\square$

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(3) This definition was suggested to me by J-P Serre. See also Expose XIII of SGA.

Proof of Theorem 1. Let $\mathfrak{h}$ be a maximal abelian and k-diagonalizable subalgebra of $\mathfrak{g}(R)$ containing a regular element $\mathfrak{p}$ of $\mathfrak{g}(R)$. The assumption on $\mathrm{Pic}(\mathfrak{X})$ ensures that the bundle of Proposition 2(iii) is trivial, hence that there exists $\mathtt{P} \in \mathfrak{G}(R)$ such that $\mathtt{P} \cdot \mathfrak{h} \subset \mathfrak{z}_{\mathfrak{g}(R)}\mathfrak{p}_0 = R \otimes \mathfrak{h}_0$. But since $\mathtt{P} \cdot \mathfrak{h}$ is k-diagonalizable we have that $\mathtt{P} \cdot \mathfrak{h} \subset k \otimes \mathfrak{h}_0$. Finally because $\mathfrak{h}$ is maximal, this last inclusion is in fact an equality. □

Remark 1. Let $\mathfrak{h}$ be an abelian and k-diagonalizable subalgebra of $\mathfrak{g}(R)$. If F is the field of quotients of R, then $F \otimes \mathfrak{h}$ is an abelian diagonalizable subalgebra of $\mathfrak{g}(F)$. As a consequence of this $\mathfrak{h} \subset \mathfrak{k}$ for some split Cartan subalgebra $\mathfrak{k}$ of $\mathfrak{g}(F)$ ([Slg] I §3 Theorem 2). It then follows from $\mathfrak{h}$ being k-diagonalizable that $\dim_k \mathfrak{h} \leq \mathrm{rank}\,(\mathfrak{g})$. If this last is an equality then $\mathfrak{h}$ is dense in $\mathfrak{k}$, hence regular.

Remark 2. The assumption on $\mathrm{Pic}(\mathfrak{X})$ is not superflous as can be easily seen from the case $\mathfrak{X} = \mathfrak{G}/\mathfrak{T}_0$.

Remark 3. Toroidal Lie algebras correspond to the case when $\mathfrak{X}$ is a split torus. Since R is then a noetherian factorial domain (Laurent polynomials in finitely many variables) $\mathrm{Pic}(\mathfrak{X})$ is trivial and the Theorem applies.

Remark 4. Since $\mathfrak{g}(R)$ is perfect, it admits a universal central extension $\mathfrak{e}$. The elements of $\mathfrak{G}(R)$ extend to automorphisms of $\mathfrak{e}$ in a natural way. Theorem 1 then holds if we replace $\mathfrak{g}(R)$ by $\mathfrak{e}$, and $\mathfrak{h}_0$ by $\mathfrak{h}_0 + \mathrm{centre}(\mathfrak{e})$.

Remark 5. Theorem 1 allows us to describe the structure of the group of automorphisms of $\mathfrak{g}(R)$. This as well the case of non regular maximal abelian and k-diagonalizable will be considered in future work.

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