Twisted loop algebras and Galois cohomology

Arturo Pianzola

Dedicated to César Polcino Milies on his sixtieth birthday

A vida amigo, é arte do encontro
Embora haja tanto desencontro pela vida.
Vinicius de Moraes.

1 Loop Algebras

Throughout this note k will denote an algebraically closed field of characteristic 0[1]. The unadorned symbol $\otimes$ will always stand for tensor product over our base field k.

We fix once for all a choice of compatible primitive n-th roots of unity (ζ_n) in $k^\times$. Thus, $\zeta_{en}^e = \zeta_n$ for all $n, e \geq 1$. Our primary objects of study, loop algebras, are constructed out of a pair (A, σ) consisting of a k-algebra A together with an automorphism σ of A which is of finite order (henceforth denoted by m). Before defining loop algebras, let us introduce three rings $R \subset S_m \subset \widehat{S}$ that will play an important role in their study. They are

$$R = k[t^{\pm 1}], S_m = k[t^{\pm 1/m}], \text{ and } \widehat{S} = \varinjlim S_m.$$

This last ring has a very simple interpretation. As a space, $\widehat{S}$ can (and will) be naturally identified with $\oplus_{q \in \mathbb{Q}} kt^q$. The multiplication is then given by bilinear extension of $t^p t^q = t^{p+q}$.

Let then A be a k-algebra. At this point we do not put any assumption on the nature of A (In fact A may end up being something more general than an algebra as we shall see in the superconformal case later). Let $\sigma \in \operatorname{Aut}_k A$ be of finite order m. We have the eigenspace decomposition

$$A = \oplus_{i=0}^{m-1} A_{\bar{i}}, \ A_{\bar{i}} := \{a \in A \mid \sigma(a) = \zeta_m^i a\}$$

where $^- : \mathbb{Z} \to Z_m := \mathbb{Z}/m\mathbb{Z}$ is the canonical map. We then define the *loop algebra* of (A, σ) by

$$L(A, \sigma) := \oplus_{i \in \mathbb{Z}} A_{\bar{i}} \otimes t^{i/m} \subset A \otimes S_m \subset A \otimes \widehat{S}. \tag{1.1}$$

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[1] This assumption, though not crucial, does facilitate the exposition considerably without detracting form the spirit of the topic. For the more general point of view, the reader can turn to [ABP] and [P].

The loop algebra has a natural k-algebra structure (infinite dimensional whenever $A \neq 0$). Of course, if A is one of the "usual" type of algebras (e.g. associate, Lie, Jordan,...), then so is $L(A,\sigma)$.

Example. Clearly $L(A, \mathrm{Id}) = A \otimes R$. We say that $L(A,\sigma)$ is *trivial* if $L(A,\sigma) \simeq A \otimes R$ (isomorphism as k-algebras).

Remark. It follows from the definition that $L(A,\sigma)$ has a natural R-algebra structure. The contrast between k-isomorphisms and R-isomorphisms of loop algebras is very delicate, and it is usually handled using the centroid of $A \otimes R$ (see [ABP]). For the time being, let us point out that k-automorphism of $A \otimes \widehat{S}$ given by $a \otimes t^q \mapsto a \otimes t^{-q}$ maps $L(A,\sigma)$ onto $L(A,\sigma^{-1})$. Thus these two loop algebras are always isomorphic as k-algebras (but not necessarily so as R-algebras as we shall see).

2 The Loop Algebra principle

For motivational as well as historical reasons, let us first look at loop algebras of a finite dimensional simple Lie algebra $\mathfrak{g}$. We then have the exact sequence

$$1 \to G_{ad} \to \mathrm{Aut}\ \mathfrak{g} \overset{-}{\to} F \to 1. \tag{2.2}$$

Here G_{ad} is the (abstract) group of k-points of the (algebraic) group $\mathbf{G}_{ad}$ of "adjoint type" corresponding to $\mathfrak{g}$, and F is the finite group of automorphisms of the corresponding Coxeter-Dynkin diagram. This sequence splits: a section of $^-$ is easily seen to exist by the presentation of $\mathfrak{g}$ by generators and relations coming form the Cartan matrix.

The central result of the theory is the following.

Theorem 2.3 *(Kac) With the above notation we have*

(i) $L(\mathfrak{g},\sigma) \simeq L(\mathfrak{g},\tau) \Leftrightarrow \bar{\sigma} \sim \bar{\tau}^{\pm 1}$ (where $\sim$ denotes conjugacy on the finite group F);

(ii) $L(\mathfrak{g}_1,\sigma_1) \simeq L(\mathfrak{g}_2,\sigma_2) \Rightarrow \mathfrak{g}_1 \simeq \mathfrak{g}_2$;

(iii) Every affine Kac-Moody Lie algebra (derived modulo its center) is isomorphic to a loop algebra $L(\mathfrak{g},\pi)$ (with $\mathfrak{g}$ and $\pi \in F$ unique to the extent of (i) and (ii)).

To put this into perspective, we turn to the classical theory. We can construct the Lie algebra $\mathfrak{g}$ by starting with the free Lie algebra on generators $e_i, f_i, h_i : 1 \leq i \leq \mathrm{rank}(\mathfrak{g})$, and imposing the relations coming from the corresponding Cartan matrix. Though this abstract presentation of $\mathfrak{g}$ is quite useful (in fact we just finish using it for constructing a section $F \hookrightarrow \mathrm{Aut}\ \mathfrak{g}$) above, it is also important to know that $\mathfrak{g}$ can be identified with a very concrete object: For example, if we start with the matrix of type B_2, then $\mathfrak{g}$ is nothing but $\mathfrak{so}_5$.

Kac-Moody algebras are *defined* by generators and relations as above, but using "generalized" Cartan matrices. . Part (iii) of Kac's Theorem says that, at least in the affine case (which is hands down the most interesting one besides the finite case), the algebras have very concrete realizations in terms of loop algebras.

In this paper we will be interested in the philosophy captured by parts (i) and (ii)of the Theorem (something that we will refer to as the "Loop algebra principle" below). Here are the constituents of these results.

(P1) $\bar{\sigma_1} \sim \bar{\sigma_2}^{\pm 1} \Rightarrow L(\mathfrak{g}, \sigma_1) \simeq L(\mathfrak{g}, \sigma_2)$.

(P2) $L(\mathfrak{g}, \sigma_1) \simeq L(\mathfrak{g}, \sigma_2) \Rightarrow \bar{\sigma}_1 \sim \bar{\sigma}_2^{\pm 1}$.

(P3) $L(\mathfrak{g}_1, \sigma_1) \simeq L(\mathfrak{g}_2, \sigma_2) \Rightarrow \mathfrak{g}_1 \simeq \mathfrak{g}_2$.

The implication in (P1) is readily available in the literature [K1]. The proof is based on the so called Gantmacher decomposition of the automorphism σ. The trick is that with respect to a clever choice of Cartan subalgebra, one can write $\sigma = \lambda\pi$ with λ "diagonal" and π in F. One then argues that λ can be "erased", namely $L(\mathfrak{g}, \lambda\pi) \simeq L(\mathfrak{g}, \pi)$.

On the other hand, (P2) and (P3) amount to the 16 types of affine Kac-Moody Lie algebras (Tables I, II, III as they are known) are indeed all non-isomorphic. Some awkward case by case specialization arguments can be put together to justify this claim. One can of course use conjugacy of Cartan subalgebras to prove this fact (but conjugacy was not available until much later on the Kac-Moody game ([PK]).

The main spirit of this paper is to explain how these results (and their analogous for arbitrary algebras) can be explained using Galois cohomology. Motivating by the foregoing we formulate the following.

Loop Algebra Principle: *The isomorphism class of $L(A, \sigma)$ depends only on the outer part $\bar{\sigma}$ of σ.*

The reader will immediately be struck by the unprecise nature of this statement, but this is intentionally so. First of all, our A is completely arbitrary, so the principle presumes some notion of "inner" and "outer part" of an automorphism. There may be no natural way of doing this in which case the principle can be turned around to indicate what the "correct" definition of inner automorphism is (namely one which makes the principle true!)[2]

The principle can be thought also in different degrees of strictness, like a *weak* version encompassed in spirit by (P1), and a *hard* one encompassing all three conditions.

Here is an example.

Theorem 2.4 *([ABP]) The hard Loop Algebra Principle holds for symmetrizable indecomposable Kac-Moody Lie algebras.*

Here the group of inner automorphisms is generated by e^{adx} with x in a real root space and the diagonal automorphisms corresponding to $\mathrm{Hom}(Q, k^{\times})$. The "outer" group F is generated by the symmetries of the Coxeter-Dynkin diagram and also ω (the Cartan involution) if the underlying Cartan matrix is not of finite type.

The ingredients of the proof are a generalized Gantmacher decomposition and Galois cohomology.

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[2] One can incorporate into the existence of the group of inner automorphisms into the Principle itself. See [P2].

3 Some geometry

Though it will seem at first as if we are going away from loop algebras, we are simply starting on a new path that will lead us back to them. Let us begin by looking at the following classical assertion.

A (topological) line bundle $\mathcal{L}$ over X looks locally like the line $\mathbb{A}^1_X$ over X.

A good example is the case when $X = \mathbb{S}^1$ is the circle, and $\mathcal{L}$ the Möbius strip. Of course $\mathcal{L}$ looks nothing like a cylinder (which is the line over X) but locally they agree. So X can be covered by open subsets $U_i \simeq \mathbb{R}$ and over each U_i our $\mathcal{L}$ looks like $\mathbb{A}^1_{U_i} \cong \mathbb{R}^2$.

For each point $x \in U_i \cap U_j$ the transition function $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ yields an automorphism of the fiber $\mathbb{R}$. Thus $\varphi_{ij}(x) \in \mathbb{R}^\times$ and we have a continuous map $\varphi_{ij} : U_i \cap U_j \to \mathbb{R}^\times$. Thus if $\mathcal{O}_X$ is the structure sheaf of regular (= continuous in this case) functions on X and G_{m_X} (or $\mathcal{O}_X^\times$) its subsheaf of units, then we can think of φ_{ij} as an element of $G_{m_X}(U_i \cap U_j)$. This allows us to attach to $\mathcal{L}$ an element $[\mathcal{L}]$ of the first Cěch cohomology $H^1(X, G_{m_X})$ of X with values on the sheaf G_{m_X}. In fact, since compatible local data can be glued together (the continuous incarnation of descent theory), we have a natural correspondence.

(3.1) *Isomorphism classes of line bundles over $X \leftrightarrow H^1(X, G_{m_X})$*

With this in mind, we now go back to loop algebras. We now have the following (admittedly rather more outlandish looking) fact.

(3.2) *$L(A, \sigma)$ looks locally over $X = \operatorname{Spec} R$ like $A \otimes R$.*

Now $X = \operatorname{Spec} R$ is the geometric object that corresponds to $R = k[t^{\pm 1}]$. Recall the ring extension $R \subset S_m = k[t^{\pm 1/m}]$. This yields a scheme morphism $U := \operatorname{Spec} S_m \xrightarrow{\varphi} \operatorname{Spec} R = X$. Of course φ is not an open inversion (unless $m = 1$), but it is still a rather nice map (étale and finite, in fact Galois) that can be thought as an open cover of X (by one open set!) on a "topology" on X, called the étale topology, which is finer than the Zariski topology.

We saw observed that line bundle becomes trivial when restricted to some open subsets of X. The idea here is the same: $L(A, \sigma)$ as an object over $\operatorname{Spec} R$ becomes trivial, namely $A \otimes R$, when "restricted" to $\operatorname{Spec} S$. This is the content of the following straightforward lemma.

Lemma 3.3 $L(A, \sigma) \otimes_R S_m \simeq_{S_m-alg} A \otimes S_m \simeq_{S_m-alg} (A \otimes R) \otimes_R S_m$.

Just as in the classical situation as illustrated in (3.1) "objects that look locally like the fiber" are classified by the "cohomology of the sheaf of automorphisms of the fiber". Accordingly we have that

Isomorphisms classes of R-sheaf of algebras which
are locally isomorphic to $A \otimes R$ $\leftrightarrow$ $H^1(X, \mathbf{Aut}\, A_R)$

This is part of the theory of principal homogeneous spaces (torsors for short in the case of schemes) developed by A. Grothendieck (after a crucial first step by J.-P. Serre in the case of varieties). It is not our intention to go into any details about this theory (see [SGA3], [Mil] and [DG]). The important point is that in spirit, this theory allows for algebras over R that look locally (in the étale or more generally the flat topology on $\operatorname{Spec} R$) like $A \otimes R$ to be classified by some Cĕch-like non-abelian H^1. This whole philosophy will for simplicity be refered as the“Zen of torsors”[3]. We have

$$L(A,\sigma) \rightsquigarrow [\mathcal{L}(A,\sigma)] \in H^1(X, \mathbf{Aut}\, A_R),$$

$$L(A,\sigma) \sim_{R-alg} L(A,\tau) \Leftrightarrow [\mathcal{L}(A,\sigma)] = [\mathcal{L}(A,\tau)].$$

In this generality $\mathbf{Aut}\, A_R$ is just the sheaf of groups over R given by $\mathbf{Aut}\, A_R(R') = \mathrm{Aut}_{R'-alg}\, A \otimes_k R'$ for any (associative unital commutative) R-algebra R'. The torsor $\mathcal{L}$ corresponding to $L = L(A,\sigma)$ is the R-sheaf of sets

$$\mathcal{L} \,:\, R' \mapsto \mathrm{Isom}_{R'-alg}(L \otimes_R R', A \otimes R').$$

It is clear what the (right) action of $\mathbf{Aut}\, A_R$ is on the sheaf $\mathcal{L}$. Of course $\mathcal{L}$ need not be isomorphic to $\mathbf{Aut}\, A_R$. (For example $\mathcal{L}(R) = \varnothing$ for any of the twisted affine Kac-Moody algebras). But $\mathcal{L}$ “becomes” $\mathbf{Aut}\, A_R$ when we restrict to S_m since $\mathcal{L}(S_m) \neq \varnothing$ by Lemma 3.3.

Let us finish our discussion of torsors by pointing out that because $\mathbf{Aut}\, A_R$ need not be abelian, our $H^1(X, \mathbf{Aut}\, A_R)$ are *not* groups, but simply *a sets with a distinguished element* (namely the isomorphism class of the trivial torsor corresponding to $A \otimes R$).

To reassure the reader that all this level of abstraction can be put to good use, let us begin by “explaining” why Kac’s Theorem is true.

The starting point is the sequence of k-algebraic groups

$$1 \to \mathbf{G}_{ad} \to \mathbf{Aut}\, \mathfrak{g} \xrightarrow{-} F_k \to 1, \tag{3.4}$$

whose k-points yield (2.2) (so F_k is the constant k-group corresponding to F). Passing to cohomology, we obtain

$$H^1(R, \mathbf{G}_R) \to H^1(R, \mathbf{Aut}\, \mathfrak{g}_R) \xrightarrow{\psi} H^1(R, F_R) \tag{3.5}$$

where for convenience we have written $\mathbf{G}$ instead of $\mathbf{G}_{ad}$. Throughout, the subindex R reminds us of the fact that we are now working on the category of affine group schemes over R (or affine R-groups for short).

Let us first look at $H^1(R, F_R)$. Grothendieck’s work (see [SGA1]) tells us that this H^1 measures Galois extensions of R with Galois group F, and can be computed by

$$H^1(R, F_R) \simeq H^1_{ct}(\pi_1(R), F).$$

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[3] Variations on a theme of Grothendieck, namely the “yoga of descent”.

This last is the "usual" continuous cohomology of the (profinite) algebraic fundamental group $\pi_1(R)$ of R (based at the geometric point $t = 1$). The Comparison Theorem (or Hurwitz formula), tells us that $\pi_1(R)$ has the same finite quotients than the (usual) fundamental group of $\mathbb{C}[t^{\pm 1}]$. In fact $\pi_1(R) = \widehat{\mathbb{Z}}$. This is a cyclic group (in the profinite sense). Because F is a constant group, the action of $\pi_1(R)$ on F_R is trivial, so an 1-cocycle in our H^1_{ct} is nothing but a (continuous) group homomorphism $\varphi : \widehat{\mathbb{Z}} \to F$. Since φ is then determined by $\varphi(1)$ we may identify φ with an element of F. Finally, two elements of F (namely two 1-cocycles) are cohomologous if they are conjugate. Thus

$$H^1(R, F_R) \simeq \text{Conjugacy classes of elements of } F. \tag{3.6}$$

This is a very satisfactory position to be in, as (3.6) is precisely the right hand side of Kac's Theorem (when one takes into account that $\pi \sim \pi^{-1}$ in F since $F \simeq \mathbb{Z}_2, \mathbb{Z}_3$, or $\mathfrak{S}_3$).

Next we have to look more closely to the torsor $\mathcal{L}(\sigma)$ attached to $L(\sigma) = L(\mathfrak{g}, \sigma)$. Our extension $S = S_m$ of R is Galois with Galois group $\Gamma = \mathbb{Z}_m$ given by $\bar{1}t^{1/m} = \zeta_m t^{1/m}$. This group Γ acts naturally on $\mathbf{Aut}\,\mathfrak{g}_R(S) = \mathrm{Aut}_{S-alg}\,\mathfrak{g} \otimes S$ by conjugation

$$\bar{i}_f = (1 \otimes \bar{i}) \circ f \circ (1 \otimes (-\bar{i})). \tag{3.7}$$

Because $\mathcal{L}(\sigma)$ is trivialized by S, we have

$$[\mathcal{L}(\sigma)] \in H^1(S/R, \mathbf{Aut}\,\mathfrak{g}_R) \simeq H^1(\Gamma, \mathrm{Aut}_S\,\mathfrak{g} \otimes S)$$

where this last is the usual (non-abelian) cohomology corresponding to the action (3.7) above. The standard 1-cocycle of $[\mathcal{L}(\sigma)]$ is given by $\bar{1} \mapsto 1 \otimes \sigma^{-1}$. It then follows that in (3.5) we have $\psi : [\mathcal{L}(\sigma)] \mapsto \bar{\sigma}^{-1}$.

Theorem 3.8 *The map $\psi : H^1(R, \mathbf{Aut}\,\mathfrak{g}_R) \to H^1(R, F_R)$ is bijective.*

Before turning to the proof, let us observe that this result not only proves but also explains why Kac's result is true. Indeed, the "Zen of torsors" above and (3.6) yield that the bijectivity of ψ amounts to

$$L(\sigma) \sim_{R-alg} L(\tau) \Leftrightarrow \bar{\sigma} \sim \bar{\tau}.$$

All that is left is to contrast this with $L(\sigma) \sim_{k-alg} L(\tau)$. Using the fact that the centroid of $L(\sigma)$ is R, and that any k-isomorphism of $L(\sigma)$ and $L(\tau)$ induces an isomorphism of the respective centroids, it follows (see [ABP] for details) that

$$L(\sigma) \simeq_k L(\tau) \Rightarrow L(\sigma) \simeq_R L(\tau) \text{ or } L(\sigma) \simeq_R L(\tau^{-1}). \tag{3.9}$$

Since in F every element is conjugate to its inverse Kac's Theorem follows.

It would seem that the injectivity of ψ amounts to the kernel $H^1(R, \mathbf{G}_R)$ of ψ being trivial. But this need not be true. The problem lies in the fact that our H^1

are not groups but only sets with distinguished elements. Now there is an old trick of Serre, called "twisting", that makes any element of $H^1(R, F_R)$ the distinguished element . The preimage of this element is now measured by means of the twisted group ${}^{\pi}G_R$ (one twists by cocycles, or torsors in general. As we have seen however, in our case we may think of cocycles in $Z^1(R, F_R) \hookrightarrow H^1(R, \mathbf{Aut}\,\mathfrak{g}_R)$ as elements of F). These twisted groups are themselves R-groups which are forms of $\mathbf{G}_R$ (they have no analogue over k). The injectivity of ψ amounts to all twisted cohomologies $H^1(R, {}^{\pi}\mathbf{G}_R)$ vanishing.

4 The case of Lie superalgebras

Again we abide by convention and denote our finite dimensional simple Lie superalgebra by $\mathfrak{g}$ rather that A. The first person to look at the loop (super)algebras corresponding to these algebras was V. Serganova ([S]). The results therein, and much more, can be obtained by cohomological methods.

Theorem 4.1 *([GP]) The hard Loop Algebra Principle holds for finite dimensional Lie superalgebras.*

By definition, automorphisms of $\mathfrak{g}$ preserve parity. This is an algebraic condition, so we can look at the algebraic group $\mathbf{Aut}\,\mathfrak{g}$ of automorphisms of $\mathfrak{g}$. The connected component $\mathbf{G}^0$ of $\mathbf{Aut}\,\mathfrak{g}$ (and not the simply connected group of the even part of $\mathfrak{g}$) is the correct group of inner automorphisms that makes Loop Algebra Principle hold). We have the exact sequence

$$1 \to \mathbf{G}^0 \to \mathbf{Aut}\,\mathfrak{g} \to F \to 1$$

where F is a finite constant group. The groups $\mathbf{G}^0$ and F are as follows:

$\mathfrak{g}$	$\mathbf{G}^0 = \mathbf{Aut}^0\,\mathfrak{g}$	F_k	**Split**
$\mathfrak{sl}(m\mid n)$	$(\mathbf{SL}_m \times \mathbf{SL}_n \times \mathbf{G}_m)/(\boldsymbol{\mu}_m \times \boldsymbol{\mu}_n)$	$\mathbb{Z}_{2,k}$	No
$\mathfrak{psl}(n\mid n), n > 2$	$(\mathbf{SL}_n \times \mathbf{SL}_n \times \mathbf{G}_m)/(\boldsymbol{\mu}_n \times \boldsymbol{\mu}_n)$	$\mathbb{Z}_{2,k} \times \mathbb{Z}_{2,k}$	No
$\mathfrak{psl}(2\mid 2)$	$(\mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2)/(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$	$\mathbb{Z}_{2,k}$	Yes
$\mathfrak{sp}(n)$	$(\mathbf{SL}_n \times \mathbf{G}_m)/\boldsymbol{\mu}_n$	$\mathbf{1}_k$	Yes
$\mathfrak{psq}(n)$	$\mathbf{PGL}_n$	$\mathbb{Z}_{4,k}$	Yes
$\mathfrak{osp}(2l\mid 2n)$	$(\mathbf{SO}_{2l} \times \mathbf{Sp}_{2n})/\boldsymbol{\mu}_2$	$\mathbb{Z}_{2,k}$	Yes
$\mathfrak{osp}(2l+1\mid 2n)$	$\mathbf{SO}_{2l+1} \times \mathbf{Sp}_{2n}$	$\mathbf{1}_k$	Yes
$F(4)$	$(\mathbf{Spin}_7 \times \mathbf{SL}_2)/\boldsymbol{\mu}_2$	$\mathbf{1}_k$	Yes
$G(3)$	$\mathbf{G}_2 \times \mathbf{SL}_2$	$\mathbf{1}_k$	Yes
$D(\alpha), \alpha^3 \neq 1, \alpha \neq -(2)^{\pm 1}$	$(\mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2)/(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$	$\mathbf{1}_k$	Yes
$D(\alpha), \alpha \in \{1, -2, -1/2\}$	$(\mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2)/(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$	$\mathbb{Z}_{2,k}$	Yes
$D(\alpha), \alpha^3 = 1, \alpha \neq 1$	$(\mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2)/(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$	$\mathbb{Z}_{3,k}$	Yes
$W(n)$	$\mathbf{N}_{W(n)} \rtimes \mathbf{GL}_n$	$\mathbf{1}_k$	Yes
$S(n)$	$\mathbf{N}_{S(n)} \rtimes \mathbf{GL}_n$	$\mathbf{1}_k$	Yes
$S'(2l)$	$\mathbf{N}_{S'(2l)} \rtimes \mathbf{SL}_{2l}$	$\mathbf{1}_k$	Yes
$H(2l)$	$\mathbf{N}_{H(2l)} \rtimes ((\mathbf{SO}_{2l} \times \mathbf{G}_m)/\boldsymbol{\mu}_2)$	$\mathbb{Z}_{2,k}$	Yes
$H(2l+1)$	$\mathbf{N}_{H(2l+1)} \rtimes (\mathbf{SO}_{2l+1} \times \mathbf{G}_m)$	$\mathbf{1}_k$	Yes

The canonical map

$$H^1(R, \mathbf{Aut}\,\mathfrak{g}) \to H^1(R, F_R) \tag{4.2}$$

is bijective and the loop algebras are therefore parameterized as follows

$$L(\mathfrak{g}, \sigma) \simeq_R L(\mathfrak{g}, \tau) \Leftrightarrow \bar{\sigma} \sim \bar{\tau}$$

while

$$L(\mathfrak{g}, \sigma) \simeq_k L(\mathfrak{g}, \tau) \Leftrightarrow \bar{\sigma} \sim \bar{\tau}^{\pm 1}.$$

Furthermore, the base algebra is an invariant $L(\mathfrak{g}_1, \sigma_1) \sim_k L(\mathfrak{g}_2, \sigma_2) \Rightarrow \mathfrak{g}_1 \simeq \mathfrak{g}_2$.

As explained in §2, to establish (4.2) one must show that $H^1(R, {}^{\pi}\mathbf{G}) = 1$ for all twisted versions of $\mathbf{G}$. Unlike §2, the groups now in question are not in general reductive, and even their semisimple parts may be of neither adjoint nor simply connected type. The vanishing of H^1 is established in [GP] by choosing, for each type of superalgebra, a suitable exact sequences that reduces the semisimple case to the results established in [P]. To deal with twisted tori, a different approach is needed. We illustrate how this goes by looking at the case of the twisted multiplicative group ${}^{\pi}G_{m_R}$. The Weil restriction of G_{m_S} by S/R. This is the R-group whose functor of points is given by

$$\mathcal{R}_{S/R}\mathrm{G}_{m_S}(R') = \mathrm{G}_{m_S}(S \otimes_R R') = (S \otimes_R R')^{\times}.$$

The norm map $N : S \to R$ leads to the exact sequence of R-groups

$$1 \to {}^{\pi}\mathrm{G}_{m_R} \to \mathcal{R}_{S/R}\,\mathrm{G}_{m_S} \to \mathrm{G}_{m_R} \to 1.$$

Passing to cohomology yields

$$S^\times \xrightarrow{N} R^\times \to H^1_{\acute{e}t}(R, {}^\pi\mathbf{G}_{m_R}) \to H^1_{\acute{e}t}(R, \mathcal{R}_{S/R}\mathbf{G}_{m_S}).$$

This last H^1 vanishes by Shapiro's Lemma, while the norm map is surjective (since k is algebraically closed). Thus H^1 is trivial.

As we shall presently see, the vanishing of $H^1_{\acute{e}t}(R, -)$ for the above twisted groups is part of a much more general and deeper fact.

5 Serre's Conjecture I

If K is a field then $\operatorname{Spec} K$ is reduced to one element. Thus, the only Zariski cover of $\operatorname{Spec} K$ is the trivial one. However, there are many beautiful covers of $\operatorname{Spec} K$ in the étale topology. In fact, if F is a Galois extension of K, then $\operatorname{Spec} F \to \operatorname{Spec} K$ is such a covering. Usual Galois cohomology fits thus nicely within the theory we have looked at. Serre's Conjecture I is one of the most striking and deepest results in classical Galois cohomology.

Theorem 5.3 *Let* $\mathbf{G}$ *be a* connected *linear algebraic group over a field* K *of dimension* 1. *If* $\operatorname{char} \mathrm{K} \neq 0$, *assume further that* $\mathbf{G}$ *is reductive. Then* $H^1(K, \mathbf{G}) = 1$.

Note. If K is perfect the result is due to Steinberg. The general case was proved later by Borel and Springer. The concept of dimension 1 of K is rather technical (the Brauer group $\operatorname{Br}(K)$ is trivial for every finite separable field extension K of k). It suffices for us to know that, as a consequence of T'sen's theorem, the field $K = k(t)$ is of dimension 1.

It should be clear by now that the Loop Algebra Principle is intimately related to the vanishing of certain $H^1(R, \mathbf{G})$. Unfortunately, R is not a field and our $\mathbf{G}$, though reductive (as affine group schemes over $\operatorname{Spec} R$ in the sense of [SGA3]), do not in general come from linear algebraic groups (as exemplified by the twisted groups already encountered). It seems natural to ask for a ring analogue of Serre's conjecture I.

Which rings R then, have the property that $H^1(R, \mathbf{G}) = 1$ for all reductive (connected) R-group schemes? There is an immediate obstruction. If S/R is finite étale then $\mathcal{R}_{S/R}\mathrm{G}_{m_S}$ is a reductive R-group and

$$H^1(R, \mathcal{R}_{S/R}\mathrm{G}_{m_S}) = H^1(S, \mathrm{G}_{m_S}) = \operatorname{Pic}(S).$$

It is thus inevitable to make the following assumption on R.

TPEC *Every finite connected étale cover* S *of* R *has trivial Picard group.*

Nicely enough, in dimension one this condition is essentially sufficient.

Theorem 5.4 *([P2]) Let* $X = \operatorname{Spec} D$ *be a Dedekind scheme whose generic fiber is the spectrum of a field* K *of dimension* 1. *The following two conditions are equivalent*

(i) X *satisfies condition* **TPEC**,

(ii) $H^1(X, \mathbf{G}) = 1$ *for all reductive group schemes* $\mathbf{G}$ *over* X.

From this we get the following version of the Loop Algebra Principle.

Corollary 5.5 *Let* $\mathbf{Aut}\,A \subset \mathbf{GL}_k(A)$ *be the linear algebraic group of automorphisms of a finite dimensional* k*-algebra* A*. Let* $\mathrm{F}_k := \mathbf{Aut}\,A/\mathbf{Aut}^0 A$ *be the (finite constant) group of connected components of* $\mathbf{Aut}\,A$ *and* $^- : \mathbf{Aut}\,A(k) \to \mathrm{F} := \mathrm{F}_k(k)$ *the corresponding canonical (abstract) group homomorphism. Finally, let* σ *and* τ *be two elements of* $\mathbf{Aut}\,A(k)$ *of finite order. Then:*

(i) $L(A,\sigma) \simeq_{R\text{-}alg} L(A,\tau) \Leftrightarrow \bar{\sigma} \sim \bar{\tau}$

(ii) If the centroids of the k*-algebras* $L(A,\sigma)$ *and* $L(A,\tau)$ *coincide with* R *(with* R *acting naturally by scalar multiplication. For example, if* A *is central and perfect), then*

$$L(A,\sigma) \simeq_{k\text{-}alg} L(A,\tau) \Leftrightarrow \bar{\sigma} \sim \bar{\tau}^{\pm 1}.$$

6 Lie superconformal algebras

In this final section we illustrate the flexibility of the cohomological point of view we have taken. For basic concepts related to superconformal algebras see [K2], specially Chapter 2.

Let $A/\mathbb{C}[\delta]$ be a Lie superconformal algebra. We will assume that, as a $\mathbb{C}[\delta]$-module, A is free of finite rank (this covers all the N-superconformal cases, which is what interests us). With $\sigma \in \mathrm{Aut}\,A$ of order m, we construct the loop algebra as usual by using (1.1). The important thing is that the n-product of the affinization $A \otimes \widehat{S}$ are given by

$$a \otimes f_{(n)} b \otimes g = \sum_{j \geq 0} a_{(n+j)} b \otimes \delta^{(j)}(f) g \tag{6.6}$$

where $\delta = d/dt$ and $\delta^{(j)} = \frac{1}{j!}\delta^j$. Then $\widehat{A}/\mathbb{C}[\widehat{\delta}]$ is a Lie superconformal superalgebra where $\widehat{\delta} = \delta \otimes 1 + 1 \otimes \delta$, and $L(A,\sigma) \subset \widehat{A}$ is easily seen to be a subalgebra (in the conformal sense).

The quotient spaces

$$Lie(A,\sigma) := L(A,\sigma)/\widehat{\delta} L(A,\sigma) \subset \widehat{A}/\widehat{\delta} A =: Lie\widehat{A}$$

are Lie algebras with respect to the bracket induced by the 0-product.

Both of these are naturally R-modules but not in general R-Lie algebras. This stems from the fact that the 0-product of $\widehat{A}$ *does not come* from that of A by R-linear extension.

The functoriality that has served us so well heretofore is now lost. One can nevertheless define a continuous action of $\widehat{\mathbb{Z}}$ on $\mathrm{Aut}_{\widehat{S}}\widehat{A}$ (automorphisms of $\widehat{A}$ which are $\widehat{S}$-linear), and still study the isomorphism classes of $Lie(A,\sigma)$ by means of the continuous non-abelian cohomology $H^1(\widehat{\mathbb{Z}},\ \mathrm{Aut}_{\widehat{S}}\widehat{A})$. The absence of functoriality brings also a mysterious possibility into the picture. Unlike the algebra case where

$\mathrm{Aut}_{\widehat{S}}\widehat{A} = \mathbf{Aut}\,A(\widehat{S})$, the groups $\mathrm{Aut}\,A$ and $\mathrm{Aut}_{\widehat{S}}\widehat{A}$ may a priori be completely unrelated. As we shall see below, this is likely the case.

We finish by looking at the development of the $N = 2$ and $N = 4$ superconformal algebras through the years. As we shall see, this is at the same time both highly amusing and instructive (see [SS] for details).

$N = 2$: Like in the classical case, the analogue of the Ramond and Nevue-Schwarz algebra are originally thought to be different. Later on, these two algebras are viewed as part of an infinite discrete family of algebras. There is also an "exotic" algebra that no one looks (or cares) about. Pictorially, this is the situation

$$\cdots \circ \;\; \circ \cdots \circ \;\; \circ \cdots \bullet$$

A direct calculation shows that $\mathrm{Aut}\ A = \mathbb{C}^\times \rtimes \mathbb{Z}_2$. If this was the algebra case, cohomology would at once dictate that only two distinct isomorphism classes of algebras can exist, hence that the above picture cannot be correct. But as we have cautioned above, functoriality may now be lost: There is no a priori way of telling the nature of $\mathrm{Aut}_{\widehat{S}}(\widehat{A})$ out of that of $\mathrm{Aut}\,A$. This however does not happen and

$$\mathrm{Aut}_{\widehat{S}}\widehat{A} = \widehat{S} \rtimes \mathbb{Z}_2$$

(so that $\mathrm{Aut}\,A$ seems to behave like the functor $G_m \rtimes \mathbb{Z}_{2,k}$). Since $H^1(\widehat{\mathbb{Z}}, \widehat{S}^\times \rtimes \mathbb{Z}_2) \simeq \mathbb{Z}_2$ the above picture *is* wrong. Indeed in [SS] the authors exhibit an explicit isomorphism between any of the two algebras of the series depicted by $\cdots \circ \cdots \circ \cdots$ above (in particular the Nevue-Schwarz and Ramond algebras are in this case isomorphic). So the correct picture, just as cohomology would have predicted, is in fact

$$\circ \;\; \bullet$$

$N = 4$: The tables are now reversed. The group $\mathrm{Aut}\ A = (SL_2(\mathbb{C}) \times SL_2(\mathbb{C}))/\pm 1$ is connected (so one will expect *all* loop algebras to be trivial), yet the physicists assert the existence of an infinite discrete family of non-isomorphic algebras!

Again, the correct answer would be given by computing a suitable $H^1(\widehat{\mathbb{Z}}, \mathrm{Aut}_{\widehat{S}}\widehat{A})$. Unfortunately, the author does not know[4] the structure of the group $\mathrm{Aut}_{\widehat{S}}\widehat{A}$. But there are two possible (or likely) answers. They are as follows.

Possibility 1. $\mathrm{Aut}_{\widehat{S}}\widehat{A} = (SL_2(\widehat{S}) \times SL_2(\widehat{S}))/\pm 1$.

If so, all loop algebras are isomorphic and (again like in the case $N = 2$) the discrete family is a mirage. This scenario is unlikely given the physicists believe[5]

Could cohomology account for an discrete infinite family of algebras? It does. Furthermore, the answer would be quite beautiful and in perfect agreement with the physicists picture.

Possibility 2. $\mathrm{Aut}_{\widehat{S}}\widehat{A} = (SL_2(\widehat{S}) \times SL_2(\mathbb{C}))/\pm 1$.

Now our H^1 is basically the set of conjugacy classes of elements of finite order of $SL_2(\mathbb{C})$. This infinite discrete set is exactly of the kind appearing in the parametrization given in [SS] (allowing passage from the compact to the complex case).

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[4] Due to his own limitations. A. Retakh assures him this can be done.

[5] Though admittedly, the $N = 2$ fiasco does not inspire much confidence.

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Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta T6G 2G1
CANADA

E-mail: a.pianzola@math.ualberta.ca