

Automorphisms and twisted loop algebras of finite dimensional simple Lie superalgebras

Dimitar Grantcharov* and Arturo Pianzola†

Abstract

We describe the structure of the algebraic group of automorphisms of all simple finite dimensional Lie superalgebras. Using this and étale cohomology considerations, we list all different isomorphism classes of the corresponding twisted loop superalgebras.

2000 MSC: Primary 17B67. Secondary 17B40, 20G15.

1 Introduction

Twisted loop algebras were introduced by V. Kac to provide explicit realizations of all affine Kac-Moody Lie algebras (see Ch. 8 of [K1] for details). The basic ingredients of his construction are a finite dimensional simple complex Lie algebra \mathfrak{g} , and an automorphism σ of \mathfrak{g} of finite order. The loop algebras then appear as complex (infinite dimensional) Lie subalgebras $L(\mathfrak{g}, \sigma) \subset \mathfrak{g} \otimes \mathbb{C}[z^{\pm 1}]$. Deciding when two of these loop algebras are isomorphic is a delicate question. This can be done using the invariance of the underlying Cartan matrices (which requires conjugacy of Cartan subalgebras) or cohomological methods (see below). The final outcome is that \mathfrak{g} is an invariant of $L(\mathfrak{g}, \sigma)$, and that the σ 's may be chosen to come from the different symmetries of the Dynkin diagram (up to conjugation).

In [S] the author sets out to recreate this construction in the case when \mathfrak{g} is a finite dimensional simple complex Lie superalgebra. Therein one finds some rather intricate arguments (crucial to the present work, see (a) below) that determine the structure of the (abstract) quotient group $\text{Aut } \mathfrak{g}/G_0$, where G_0 is the group of “inner automorphisms” of \mathfrak{g} . (The group G_0 is related to the group of automorphisms of the even part of \mathfrak{g} , but it is not the connected component of $\text{Aut } \mathfrak{g}$.) The author shows that the elements of G_0 lead to (untwisted) loop algebras, and states that a given list of coset representatives of $\text{Aut } \mathfrak{g}/G_0$ accounts for the remaining different

*Supported by a Pacific Institute for the Mathematical Sciences postdoctoral fellowship

†Supported by NSERC operating grant A9343

isomorphism classes of twisted loop algebras. The issue of the invariance of \mathfrak{g} is not raised.¹

In [P] and [ABP] it is shown that the study of isomorphism classes of loop algebras of arbitrary algebras is a problem of Galois cohomology. More precisely, one views $L(\mathfrak{g}, \sigma)$ as an algebra over the ring $R = k[z^{\pm m}]$ (where m is the order of σ and k is our base field) and shows that if $S = k[z^{\pm 1}]$ then $L(\mathfrak{g}, \sigma) \otimes_R S \simeq (\mathfrak{g} \otimes R) \otimes_R S$ as S -algebras. Thus $L(\mathfrak{g}, \sigma)$ is an S/R form of $\mathfrak{g} \otimes R$. Since the extension S/R is étale (in fact finite Galois), we can attach to $L(\mathfrak{g}, \sigma)$ an element (isomorphism class of a torsor)

$$[L(\mathfrak{g}, \sigma)] \in H_{\text{ét}}^1(\text{Spec } R, \mathbf{Aut } \mathfrak{g}_R)$$

(see [P] for details). The study of this H^1 is enough to understand the isomorphism classes of loop algebras as *algebras over* R . The passage from R to the base field k is done by bringing the centroid of \mathfrak{g} into the picture ([ABP], §4). This is the approach that we will follow, and as can be seen from the above discussion then, there are three key ingredients that must be looked at:

- (a) The nature of the *algebraic group* $\mathbf{Aut } \mathfrak{g}$.
- (b) The computation of $H_{\text{ét}}^1(\text{Spec } R, \mathbf{Aut } \mathfrak{g}_R)$.
- (c) The supercentroid of \mathfrak{g} .

(a) is dealt with in §6. With (b) in mind we describe for each \mathfrak{g} (according to the classification due to Kac) an exact sequence of algebraic groups

$$1 \rightarrow \mathbf{G}^0 \rightarrow \mathbf{Aut } \mathfrak{g} \rightarrow \mathbf{F}_k \rightarrow 1$$

where \mathbf{G}^0 is the connected component of the identity of $\mathbf{Aut } \mathfrak{g}$ and \mathbf{F} is finite. The work of Serganova, which essentially tells us what $\mathbf{Aut } \mathfrak{g}(k)$ is, plays here a crucial role.²

(b) This is done in §8. With the aid of (a) and Grothendieck's work on the algebraic fundamental group, we obtain natural correspondences

$$H_{\text{ét}}^1(\text{Spec } R, \mathbf{Aut } \mathfrak{g}_R) \simeq H^1(\text{Spec } R, \mathbf{F}_R) \simeq \text{Conjugacy classes of } \mathbf{F}.$$

This crucial finiteness result is analogous to one of the deep theorems of classical Galois cohomology over fields ([CG], Ch. III, Théorème 4.4). Its validity rests ultimately on the very special nature of the Dedekind ring $k[t^{\pm 1}]$. The parameterization of R -isomorphism classes of loop algebras by conjugacy classes of \mathbf{F} will follow from this result.

- (c) This is the extent of §7. Once \mathfrak{g} is shown to be central, we reason as in [ABP].

Acknowledgements: We would like to thank the referee for suggesting the proof of Proposition 7.1 which is substantially shorter than the original one.

¹Serganova has informed us that she did have case-by-case computational proofs of some of these results in her unpublished Ph.D thesis. In any event, we believe that the cohomological point of view explains the result better and sheds considerably more insight into these questions.

²Care must be taken though, to avoid the use of the “analytic” part of her arguments using the exponential function over \mathbb{C} . This is done by introducing suitable Cartan subgroups.

2 Conventions and notation

Henceforth k will denote an algebraically closed field of characteristic zero. The category of associative commutative unital k -algebras will be denoted by $k\text{-alg}$. By \mathbb{Z}_n we denote the group $\mathbb{Z}/n\mathbb{Z}$. We adhere to the convention of writing algebra filtrations $A = A^l \supset A^{l+1} \supset \dots$ with upper indexes and algebra gradings $A = \oplus_i A_i$ with lower indexes.

Let $R \in k\text{-alg}$. Recall that a *Lie superalgebra over R* is a \mathbb{Z}_2 -graded R -module $L = L_{\bar{0}} \oplus L_{\bar{1}}$ together with a multiplication $[\cdot, \cdot] : L \times L \rightarrow L$ satisfying the following three axioms:

- (i) $[\cdot, \cdot]$ is R -bilinear,
- (ii) $[x, y] = -(-1)^{\deg x \deg y} [y, x]$ ($\deg u = i$ if $u \in L_{\bar{i}}$),
- (iii) $[x, [y, z]] + (-1)^{\deg x (\deg y + \deg z)} [y, [z, x]] + (-1)^{\deg z (\deg x + \deg y)} [z, [x, y]] = 0$.

An *automorphism* of L is an R -module automorphism φ of L satisfying:

- (i) φ stabilizes $L_{\bar{0}}$ and $L_{\bar{1}}$ (i.e. φ is homogeneous of degree $\bar{0}$),
- (ii) $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in L$.

Endomorphisms are defined in a similar fashion. Note that in some works (see [K2] for example) automorphisms and endomorphisms are not supposed to preserve the \mathbb{Z}_2 -grading. For details on all simple finite dimensional Lie superalgebras over k we refer the reader to [K2] and [Sch]. In the present paper we will use the notations of [Pen].

Assume henceforth that $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a finite dimensional simple Lie superalgebra over k .

By base change $\mathfrak{g}(R) = \mathfrak{g}_{\bar{0}}(R) \oplus \mathfrak{g}_{\bar{1}}(R)$ becomes a Lie superalgebra over R . Consider the k -group $\mathbf{Aut} \mathfrak{g} : R \rightarrow \mathbf{Aut}_{R\text{-}Ls} \mathfrak{g}(R)$ where the right hand side denotes the group of automorphisms of $\mathfrak{g}(R)$ as a Lie superalgebra over R . It is easy to see that $\mathbf{Aut} \mathfrak{g}$ is a closed subgroup of $\mathbf{GL}(\mathfrak{g})$, hence affine (see §5 below).

3 Loop algebras

We fix once and for all a compatible set $\{\zeta_m\}$ of primitive m -th roots of unity in k . (Thus $\zeta_{lm}^l = \zeta_m$). Let σ be an automorphism of \mathfrak{g} of finite period m . We then get a \mathbb{Z}_m -grading $\oplus_{i=0}^{m-1} \mathfrak{g}_{\bar{i}}$ of \mathfrak{g} via the eigenspace decomposition $\mathfrak{g}_{\bar{i}} = \{x \in \mathfrak{g} \mid \sigma(x) = \zeta_m^i x\}$. Define

$$L_m(\mathfrak{g}, \sigma) := \oplus_{i \in \mathbb{Z}} \mathfrak{g}_{\bar{i}} \otimes z^i \subset \mathfrak{g} \otimes k[z^{\pm 1}].$$

Since by definition σ preserves the even and odd parts of \mathfrak{g} , we see that $L_m(\mathfrak{g}, \sigma)$ is in natural way an (infinite dimensional) Lie superalgebra over k . An easy calculation shows that up to isomorphism $L_m(\mathfrak{g}, \sigma)$ does not depend on the choice of the period m of σ , therefore by a harmless abuse in notation we will henceforth simply write $L(\mathfrak{g}, \sigma)$ instead. This is the *loop algebra* of the pair (\mathfrak{g}, σ) . (The reason for writing periods and not orders is that one can then deal with the isomorphism question $L(\mathfrak{g}, \sigma) \simeq L(\mathfrak{g}, \tau)$ in easier terms).

4 Statements of the main theorems

Theorem 4.1 *There exists an exact sequence of linear algebraic k -groups*

$$1 \rightarrow \mathbf{G}^0 \rightarrow \mathbf{Aut} \mathfrak{g} \rightarrow \mathbf{F}_k \rightarrow 1.$$

where \mathbf{G}^0 (the connected component of the identity of $\mathbf{Aut} \mathfrak{g}$) and \mathbf{F}_k (finite and constant) are given by the following table.³

Table: Automorphism groups of the simple Lie superalgebras.⁴

\mathfrak{g}	$\mathbf{G}^0 = \mathbf{Aut}^0 \mathfrak{g}$	\mathbf{F}_k	Split
$\mathfrak{sl}(m n)$	$(\mathbf{SL}_m \times \mathbf{SL}_n \times \mathbf{G}_m)/(\boldsymbol{\mu}_m \times \boldsymbol{\mu}_n)$	$\mathbb{Z}_{2,k}$	No
$\mathfrak{psl}(n n), n > 2$	$(\mathbf{SL}_n \times \mathbf{SL}_n \times \mathbf{G}_m)/(\boldsymbol{\mu}_n \times \boldsymbol{\mu}_n)$	$\mathbb{Z}_{2,k} \times \mathbb{Z}_{2,k}$	No
$\mathfrak{psl}(2 2)$	$(\mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2)/(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$	$\mathbb{Z}_{2,k}$	Yes
$\mathfrak{sp}(n)$	$(\mathbf{SL}_n \times \mathbf{G}_m)/\boldsymbol{\mu}_n$	1_k	Yes
$\mathfrak{psq}(n)$	\mathbf{PGL}_n	$\mathbb{Z}_{4,k}$	Yes
$\mathfrak{osp}(2l 2n)$	$(\mathbf{SO}_{2l} \times \mathbf{Sp}_{2n})/\boldsymbol{\mu}_2$	$\mathbb{Z}_{2,k}$	Yes
$\mathfrak{osp}(2l+1 2n)$	$\mathbf{SO}_{2l+1} \times \mathbf{Sp}_{2n}$	1_k	Yes
$F(4)$	$(\mathbf{Spin}_7 \times \mathbf{SL}_2)/\boldsymbol{\mu}_2$	1_k	Yes
$G(3)$	$\mathbf{G}_2 \times \mathbf{SL}_2$	1_k	Yes
$D(\alpha), \alpha^3 \neq 1, \alpha \neq -(2)^{\pm 1}$	$(\mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2)/(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$	1_k	Yes
$D(\alpha), \alpha \in \{1, -2, -1/2\}$	$(\mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2)/(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$	$\mathbb{Z}_{2,k}$	Yes
$D(\alpha), \alpha^3 = 1, \alpha \neq 1$	$(\mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2)/(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$	$\mathbb{Z}_{3,k}$	Yes
$W(n)$	$\mathbf{N}_{W(n)} \rtimes \mathbf{GL}_n$	1_k	Yes
$S(n)$	$\mathbf{N}_{S(n)} \rtimes \mathbf{GL}_n$	1_k	Yes
$S'(2l)$	$\mathbf{N}_{S'(2l)} \rtimes \mathbf{SL}_{2l}$	1_k	Yes
$H(2l)$	$\mathbf{N}_{H(2l)} \rtimes ((\mathbf{SO}_{2l} \times \mathbf{G}_m)/\boldsymbol{\mu}_2)$	$\mathbb{Z}_{2,k}$	Yes
$H(2l+1)$	$\mathbf{N}_{H(2l+1)} \rtimes (\mathbf{SO}_{2l+1} \times \mathbf{G}_m)$	1_k	Yes

Note. It is natural to consider the supergroup (superspace) of automorphisms of \mathfrak{g} in the spirit of Bernstein, Leites, Manin, etc. (see [DM] and [Man]). More precisely, one defines a functor $\mathbf{Aut}^s \mathfrak{g} : R \rightarrow \mathbf{Aut}_{R\text{-}Ls} \mathfrak{g}(R)$ from the category $k\text{-salg}$ of associative unital k -superalgebras to the category of sets. We can then relate the purely even superspace $(\mathbf{Aut}^s \mathfrak{g})_{\text{rd}}$ to the (purely even) affine algebraic group $\mathbf{Aut} \mathfrak{g}$ above (recall that for a superspace (M, \mathcal{O}_M) , by M_{rd} we denote $(M, \mathcal{O}_M/\mathcal{J}_M)$, where $\mathcal{J}_M = \mathcal{O}_{M,1} + \mathcal{O}_{M,1}^2$). The explicit structure of $\mathbf{Aut}^s \mathfrak{g}$ is not needed in the present paper and will be left for future work.

The Lie algebra of the algebraic group $\mathbf{Aut} \mathfrak{g}$ is comprised of the (parity preserving) derivations of the algebra \mathfrak{g} . These are precisely the even derivations of \mathfrak{g}

³The \mathbf{N} 's appearing in the Cartan type series are unipotent groups. Their specific description is given in §6.9

⁴Although the Lie superalgebras $D(1)$ and $\mathfrak{osp}(4|2)$ are isomorphic we will treat them separately given that the two different viewpoints yield different information.

considered as a Lie superalgebra. In this way we recover from Theorem 4.1 the list of the even derivations $(\text{Der } \mathfrak{g})_0$ of all simple finite-dimensional Lie superalgebras \mathfrak{g} (for the full list see Appendix A in [Pen]).

Theorem 4.2 *Let $R = k[t^{\pm 1}]$. The canonical map $H_{\text{ét}}^1(R, \mathbf{Aut } \mathfrak{g}_R) \rightarrow H_{\text{ét}}^1(R, \mathbf{F}_R)$ is bijective.*

Theorem 4.3 *For $i = 1, 2$ let \mathfrak{g}_i be a finite dimensional simple Lie superalgebra over k , and $\sigma_i \in \mathbf{Aut } \mathfrak{g}_i$ an automorphism of finite order. If $L(\mathfrak{g}_1, \sigma_1) \simeq L(\mathfrak{g}_2, \sigma_2)$ then $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ (all isomorphisms as k -Lie superalgebras).*

Theorem 4.4 *The following two conditions are equivalent:*

- (i) $L(\mathfrak{g}, \sigma_1)$ and $L(\mathfrak{g}, \sigma_2)$ are isomorphic as k -Lie superalgebras,
- (ii) $\bar{\sigma}_1$ is conjugate to either $\bar{\sigma}_2$ or $\bar{\sigma}_2^{-1}$ in $\mathbf{F} := \mathbf{F}_k(k)$, where $\bar{\cdot} : \mathbf{Aut } \mathfrak{g}(k) \rightarrow \mathbf{F}$ is the surjective (abstract) group isomorphism arising from Theorem 4.1.

Theorem 4.5 *Let $R = k[t^{\pm 1}]$ and consider the R -Lie superalgebras $\mathfrak{g} \otimes_k R$. Let \mathfrak{f} be an R -form of $\mathfrak{g} \otimes_k R$; namely \mathfrak{f} is an R -Lie superalgebra such that $\mathfrak{f} \otimes_R S \simeq (\mathfrak{g} \otimes_k R) \otimes_R S \simeq \mathfrak{g} \otimes_k S$ as Lie superalgebras for some faithfully flat and finitely presented (commutative associative unital) R -algebra S . Then $\mathfrak{f} \simeq L(\mathfrak{g}, \sigma)$ for some σ and conversely. Furthermore, S can be taken to be finite étale (in fact Galois).*

5 Generalities about Lie superalgebras and algebraic groups

Because of our present needs, we will use the “functorial approach” to algebraic groups. This facilitates immensely the explicit definition of morphisms and quotients. We briefly recall how this goes (see [DG], [W], and specially Chapter 6 of [KMRT]).

A k -group is a functor from $k\text{-alg}$ into the category of groups. These form a category where morphisms are natural transformations. Thus a morphism $f : \mathbf{G} \rightarrow \mathbf{H}$ of k -groups is nothing but a functorial collection of abstract group homomorphisms $f_R : \mathbf{G}(R) \rightarrow \mathbf{H}(R)$, for $R \in k\text{-alg}$. In particular if $\varphi : R \rightarrow R'$ is a k -algebra homomorphism then the diagram

$$\begin{array}{ccc} \mathbf{G}(R) & \xrightarrow{f_R} & \mathbf{H}(R) \\ \mathbf{G}(\varphi) \downarrow & & \downarrow \mathbf{H}(\varphi) \\ \mathbf{G}(R') & \xrightarrow{f_{R'}} & \mathbf{H}(R') \end{array}$$

commutes.

Every finite (abstract) group \mathbf{F} gives raise to a finite constant k -group that will be denoted by \mathbf{F}_k . We have $\mathbf{F}_k(R) = \mathbf{F}$ whenever $\text{Spec } R$ is connected. Finally, if \mathbf{G} is a k -group and $R \in k\text{-alg}$, we let \mathbf{G}_R denote the R -group obtained by the base change $k \rightarrow R$.

An *affine k -group* is a k -group \mathbf{G} which is representable: There exists $k[\mathbf{G}] \in k\text{-alg}$ such that $\mathbf{G}(R) = \text{Hom}(k[\mathbf{G}], R)$ for all R in $k\text{-alg}$. By Yoneda's Lemma $k[\mathbf{G}]$ is unique up to isomorphism and has a natural commutative Hopf algebra structure. Conversely, every commutative Hopf algebra A over k gives rise to an affine k -group \mathbf{G}^A whose R -points are $\text{Hom}(A, R)$.

We will make repeated use of the following affine k -groups: $\mathbf{Hom}(U, V) : R \rightarrow \text{Hom}_{R\text{-mod}}(U \otimes R, V \otimes R)$ where U and V are finite dimensional k -spaces, $\mathbf{End}(V) = \mathbf{Hom}(V, V)$, $\mathbf{G}_a : R \rightarrow (R, +)$, and $\mathbf{G}_m : R \rightarrow R^\times$ (the units of R). In addition we will make repeated use of many of the classical groups \mathbf{GL} , \mathbf{SL} , etc; as well as some of the exceptional ones.

Though in the category of affine k -groups the concepts of kernel of a morphism, normal subgroups, etc. are obvious ones (“*définition ensembliste*”), those of image of a morphism and quotient are not. We illustrate this with an example that will appear later on.

Consider the k -group $\mathbf{Aut} \mathfrak{sl}_n : R \rightarrow \text{Aut}_{R\text{-Lie}}(\mathfrak{sl}_n(k) \otimes R)$ (this last being the group of automorphisms in the category of Lie algebras over R). This is an affine k -group and there is a natural homomorphism $\text{Ad} : \mathbf{SL}_n \rightarrow \mathbf{Aut} \mathfrak{sl}_n$ where $\text{Ad}_R X : A \rightarrow XAX^{-1}$ for all $A \in \mathfrak{sl}_n(k) \otimes R$ and $X \in \mathbf{SL}_n(R)$. If $n = 2$, this is a *surjective* homomorphism, i.e. the image of Ad equals $\mathbf{Aut} \mathfrak{sl}_2$. This *does not mean* that a typical element of $\theta \in \mathbf{Aut} \mathfrak{sl}_2(R)$ is of the form $\text{Ad} X$ for some $X \in \mathbf{SL}_2(R)$ (a false statement even for most fields), but rather that there exist a faithfully flat base change $R \rightarrow S$ and an element X of $\mathbf{SL}_2(S)$ such that $\theta_S = \text{Ad}_S X$, where $\theta_S = \theta \otimes 1 \in \text{Aut}_{S\text{-Lie}}(\mathfrak{sl}_2(k) \otimes R \otimes_R S) = \mathbf{Aut} \mathfrak{sl}_2(S)$.

The kernel of Ad is a closed subgroup isomorphic to μ_n , where $\mu_n(R) := \{r \in R \mid r^n = 1\}$. In particular, we have the exact sequence $1 \rightarrow \mu_n \rightarrow \mathbf{SL}_n \rightarrow \mathbf{Aut} \mathfrak{sl}_n$. The quotient (in the category of affine k -groups) \mathbf{SL}_n/μ_n is also denoted by \mathbf{PGL}_n .

We shall make repeated use of the following two standard results

Proposition 5.1 *Let $f : \mathbf{N} \rightarrow \mathbf{G}$ and $g : \mathbf{G} \rightarrow \mathbf{H}$ be morphisms of affine k -groups. The following are equivalent.*

(i) *The sequence $1 \rightarrow \mathbf{N} \xrightarrow{f} \mathbf{G} \xrightarrow{g} \mathbf{H} \rightarrow 1$ is exact (i.e. f is injective, $f(\mathbf{N}) = \ker g$, and g is surjective).*

(ii) *f is injective and $\mathbf{G}/f(\mathbf{N}) \simeq \mathbf{H}$.*

(iii) *$1 \rightarrow \mathbf{N}(R) \xrightarrow{f_R} \mathbf{G}(R) \xrightarrow{g_R} \mathbf{H}(R)$ is an exact sequence of abstract groups and the map $g_k : \mathbf{G}(k) \rightarrow \mathbf{H}(k)$ is surjective.*

(iv) *$1 \rightarrow \mathbf{N}(R) \xrightarrow{f_R} \mathbf{G}(R) \xrightarrow{g_R} \mathbf{H}(R)$ is an exact sequence of abstract groups and for all $h \in \mathbf{H}(R)$ there exist a faithfully flat base change $R \rightarrow S$ and $g \in \mathbf{G}(S)$ such that $g_S(g) = h_S$ (where h_S is the image of h under the map $\mathbf{H}(R) \rightarrow \mathbf{H}(S)$).*

Proof. The crucial point is that if $k[\mathbf{H}] \rightarrow k[\mathbf{G}]$ is injective then this map is faithfully flat. Furthermore, all our k -groups are smooth since we are in characteristic zero (Cartier's Theorem). See [W] and [KMRT]. \square

Proposition 5.2 *Let \mathbf{G} be a k -group and \mathbf{H} a closed subgroup of \mathbf{G} . The following are equivalent:*

- (i) \mathbf{H} is normal in \mathbf{G} (i.e. $\mathbf{H}(R)$ is normal in $\mathbf{G}(R)$ for all $R \in k\text{-alg}$).
- (ii) $\mathbf{H}(k)$ is a normal subgroup of $\mathbf{G}(k)$.

Proof. Follows from [DG], Ch. II, §5.4.1. Indeed, both \mathbf{G} and \mathbf{H} are smooth by Cartier's Theorem. \square

To illustrate some of these ideas we finish this section by giving the functorial version of a well known result that we shall need later on.

Lemma 5.3 (*Schur's Lemma*) *Let \mathfrak{a} be a Lie algebra over k , and let $\rho : \mathfrak{a} \rightarrow \mathfrak{gl}(V)$ be an irreducible finite dimensional representation of \mathfrak{a} . Consider the k -group \mathbf{C}_ρ given by*

$$\mathbf{C}_\rho(R) := \{\varphi \in \mathbf{End} V(R) \mid \varphi \text{ commutes with } \rho_R(\mathfrak{a}(R))\}.$$

The canonical map $\mathbf{G}_\mathfrak{a} \rightarrow \mathbf{C}_\rho$ is an isomorphism.

Proof. It is clear that \mathbf{C}_ρ is a closed subgroup of $\mathbf{End} V$ and hence affine. If $r \in R$ then $r \text{Id}_V \in \mathbf{C}_\rho(R)$, which leads to a natural injection $\mathbf{G}_\mathfrak{a} \rightarrow \mathbf{C}_\rho$. By the usual Schur's Lemma $\mathbf{G}_\mathfrak{a}(k)$ surjects onto $\mathbf{C}_\rho(k)$ and therefore our map is an isomorphism of affine k -groups. \square

6 The proof of Theorem 4.1

6.1 Generalities on $\mathfrak{gl}(m|n)$ and $\mathfrak{sl}(m|n)$

Recall that

$$\mathfrak{gl}(m|n) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A \in \mathcal{M}_m, B \in \mathcal{M}_{m,n}, C \in \mathcal{M}_{n,m}, D \in \mathcal{M}_n \right\},$$

where by $\mathcal{M}_{p,q} = \mathcal{M}_{p,q}(k)$ we denote the space of $p \times q$ matrices with entries in k and $\mathcal{M}_p := \mathcal{M}_{p,p}$. For $(X, Y) \in (\mathbf{SL}_m \times \mathbf{SL}_n)(R)$ we define $\text{Ad}_R(X, Y) \in \mathbf{Aut} \mathfrak{gl}(m|n)(R)$ by

$$\text{Ad}_R(X, Y) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} XAX^{-1} & XBY^{-1} \\ YCX^{-1} & YDY^{-1} \end{pmatrix}.$$

This yields a group homomorphism $\text{Ad} : \mathbf{SL}_m \times \mathbf{SL}_n \rightarrow \mathbf{Aut} \mathfrak{gl}(m|n)$.

The *supertranspose* τ is the automorphism of $\mathfrak{gl}(m|n)$ given by

$$\tau : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} -A^t & C^t \\ -B^t & -D^t \end{pmatrix}.$$

The automorphism τ is of order 4 and leads to a constant subgroup $\langle \tau \rangle_k$ of $\mathbf{Aut} \mathfrak{gl}(m|n)$.

If $m = n$ we have an automorphism π of $\mathfrak{gl}(n|n)$ given by

$$\pi : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} D & C \\ B & A \end{pmatrix}$$

which leads to a constant group $\langle \pi \rangle_k$ of $\mathbf{Aut} \mathfrak{gl}(n|n)$.

We also have an injective group homomorphism $j : \mathbf{G}_m \rightarrow \mathbf{Aut} \mathfrak{gl}(m|n)$ given by $j_R(\lambda) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} A & \lambda B \\ \lambda^{-1}C & D \end{pmatrix}$ for all $\lambda \in \mathbf{G}_m(R) = R^\times$.

One easily checks that:

$$(6.1) \quad \begin{aligned} \tau_R \text{Ad}_R(X, Y) \tau_R^{-1} &= \text{Ad}_R((X^t)^{-1}, (Y^t)^{-1}), \pi_R \text{Ad}_R(X, Y) \pi_R^{-1} = \text{Ad}_R(Y, X), \\ \pi_R \tau_R \pi_R^{-1} &= \tau_R^3, \tau_R^2 = j_R(-1), \tau_R j_R(\lambda) \tau_R^{-1} = j_R(\lambda^{-1}). \end{aligned}$$

6.2 Automorphisms of $\mathfrak{sl}(m|n)$, $m > n \geq 1$

Recall that $\mathfrak{sl}(m|n)$ consists of those elements $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $\mathfrak{gl}(m|n)$ for which $\text{tr}(A) - \text{tr}(D) = 0$. As above we have a group morphism $\text{Ad} \times j : \mathbf{SL}_m \times \mathbf{SL}_n \times \mathbf{G}_m \rightarrow \mathbf{Aut} \mathfrak{sl}(m|n)$. The image of $\text{Ad} \times j$ is denoted by $\mathbf{Aut}^0 \mathfrak{sl}(m|n)$. Clearly $(X, Y, \lambda) \in \ker(\text{Ad} \times j)_R$ iff there exist $\alpha, \beta \in R^\times$ such that $X = \alpha I_m, Y = \beta I_n, \alpha^m = \beta^n = 1$, and $\lambda = \alpha^{-1} \beta$. Thus $\ker(\text{Ad} \times j) = \boldsymbol{\mu}_m \times \boldsymbol{\mu}_n$.

Serganova's reasoning shows that $\mathbf{Aut} \mathfrak{sl}(m|n)(k)$ is generated by $\mathbf{Aut}^0 \mathfrak{sl}(m|n)(k)$ and τ . From Proposition 5.1 and 5.2 it follows that $\mathbf{Aut}^0 \mathfrak{sl}(m|n)$ is normal in $\mathbf{Aut} \mathfrak{sl}(m|n)$ and that the canonical morphism $\langle \tau \rangle_k \rightarrow \mathbf{Aut} \mathfrak{sl}(m|n) / \mathbf{Aut}^0 \mathfrak{sl}(m|n)$ is surjective. With the aid of (6.1) and Proposition 5.1 the kernel of this last homomorphism is easily seen to be $\langle \tau^2 \rangle_k$.

The above definition yields a non-split exact sequence

$$1 \rightarrow (\mathbf{SL}_m \times \mathbf{SL}_n \times \mathbf{G}_m) / (\boldsymbol{\mu}_m \times \boldsymbol{\mu}_n) \rightarrow \mathbf{Aut} \mathfrak{sl}(m|n) \rightarrow \mathbb{Z}_{2,k} \rightarrow 1,$$

where $\langle \tau \rangle_k$ surjects onto $\mathbb{Z}_{2,k}$.

6.3 Automorphisms of $\mathfrak{psl}(n|n)$, $n > 2$

Recall that $\mathfrak{psl}(n|n) := \mathfrak{sl}(n|n) / \mathfrak{z}$ where $\mathfrak{z} := k \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}$. One reasons as in the last case to obtain a surjective group homomorphism $\langle \tau \rangle_k \rtimes \langle \pi \rangle_k \rightarrow \mathbf{Aut} \mathfrak{psl}(n|n) / \mathbf{Aut}^0 \mathfrak{psl}(n|n)$ (see Proposition 5.1). An easy reasoning again shows that the kernel of this map is $\langle \tau^2 \rangle_k$. Since the corresponding quotient is the Klein 4-group we have

$$1 \rightarrow (\mathbf{SL}_n \times \mathbf{SL}_n \times \mathbf{G}_m) / (\boldsymbol{\mu}_n \times \boldsymbol{\mu}_n) \rightarrow \mathbf{Aut} \mathfrak{psl}(n|n) \rightarrow \mathbb{Z}_{2,k} \times \mathbb{Z}_{2,k} \rightarrow 1.$$

The sequence is not split, but the second copy of $\mathbb{Z}_{2,k}$ (which corresponds to $\langle \pi \rangle_k$) does admit a section.

6.4 Automorphisms of $\mathfrak{psl}(2|2)$

In this case $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$, $\mathfrak{g}_{-1} \simeq V_2^* \otimes V_2$, and $\mathfrak{g}_1 \simeq V_2 \otimes V_2^*$, where V_2 is the standard \mathfrak{gl}_2 -module. In fact, \mathfrak{g}_{-1} and \mathfrak{g}_1 are isomorphic \mathfrak{g}_0 -modules. We explicitly describe how this is done to better understand Serganova's argument. Let $\{e_1, e_2\}$ be the standard basis of V_2 , and $\{e_1^*, e_2^*\}$ the corresponding dual basis of V_2^* . Then $\varphi : V_2 \xrightarrow{\cong} V_2^*$ as \mathfrak{sl}_2 -modules via $\varphi(e_1) = -e_2^*$, $\varphi(e_2) = e_1^*$. This yields the isomorphism $\phi := \varphi \times \varphi^{-1} : V_2 \otimes V_2^* \rightarrow V_2^* \otimes V_2$. The matricial description of ϕ is as follows. Identify $V_2 \otimes V_2^* \xrightarrow{\eta} \mathcal{M}_2(k)$ via $e_i \otimes e_j^* \mapsto E_{ij}$, and $V_2^* \otimes V_2 \xrightarrow{\eta^*} \mathcal{M}_2(k)$ via $e_i^* \otimes e_j \mapsto E_{ij}$. We then have the following commutative diagram:

$$\begin{array}{ccc} V_2 \otimes V_2^* & \xrightarrow{\phi} & V_2^* \otimes V_2 \\ \eta \downarrow & & \downarrow \eta^* \\ \mathcal{M}_2(k) & \xrightarrow{\psi} & \mathcal{M}_2(k) \end{array}$$

where $\psi(E) = -JE^tJ^{-1}$ with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Furthermore, $\psi([B, C]) = [\psi(B), \psi(C)]$ for all B and C in $\mathcal{M}_2(k)$. Using the canonical R -module isomorphism $(V_2 \otimes R)^* \simeq V_2^* \otimes R$, one easily sees that the above construction is ‘‘functorial on R ’’. More precisely, we have the commutative diagram

$$(6.2) \quad \begin{array}{ccccccc} \phi_R : & V_2(R) \otimes V_2(R)^* & \simeq & (V_2 \otimes V_2^*)(R) & \xrightarrow{\phi \otimes 1} & (V_2^* \otimes V_2)(R) & \simeq & V_2(R)^* \otimes V_2(R) \\ & \eta_R \downarrow & & \eta \otimes 1 \downarrow & & \downarrow \eta^* \otimes 1 & & \downarrow \eta_R^* \\ \psi_R : & \mathcal{M}_2(R) & \simeq & \mathcal{M}_2(k) \otimes R & \xrightarrow{\psi \otimes 1} & \mathcal{M}_2(k) \otimes R & \simeq & \mathcal{M}_2(R) \end{array}$$

where the compositions ϕ_R and ψ_R of all horizontal isomorphisms are $(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2)(R)$ -module isomorphisms.

For $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbf{SL}_2(R)$ consider the R -module endomorphism $\rho_R \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of $\mathfrak{psl}(2|2)$ given by

$$\rho_R \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & \alpha B + \beta \psi_R(C) \\ \gamma \psi_R(B) + \delta C & D \end{pmatrix}.$$

A direct calculation using (6.2) shows that this is in fact an element of $\mathbf{Aut} \mathfrak{psl}(2|2)$, and that this construction yields a k -group homomorphism $\rho : \mathbf{SL}_2 \rightarrow \mathbf{Aut} \mathfrak{psl}(2|2)$ with trivial kernel. A direct computation shows that $\rho(\mathbf{SL}_2)$ commutes with the image of Ad . This leads to a group homomorphism

$$\text{Ad} \times \rho : \mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2 \rightarrow \mathbf{Aut} \mathfrak{psl}(2|2)$$

with kernel $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ corresponding to $(-I_2, I_2, -I_2)$ and $(I_2, -I_2, -I_2)$. Furthermore, $\pi_R(\text{Ad} \times \rho)_R(X, Y, Z)\pi_R^{-1} = (\text{Ad} \times \rho)_R(Y, X, (Z^t)^{-1})$ and it easily follows from Proposition 5.2 that the image of $\text{Ad} \times \rho$ is normal in $\mathbf{Aut} \mathfrak{psl}(2|2)$. Serganova's argument

shows that $\mathbf{Aut} \mathfrak{psl}(2|2)(k)$ is generated by the image of $(\mathrm{Ad} \times \rho)_k$ and π . By Proposition 5.1, we obtain the split exact sequence

$$1 \rightarrow (\mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2)/(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \rightarrow \mathbf{Aut} \mathfrak{psl}(2|2) \rightarrow \mathbb{Z}_{2,k} \rightarrow 1.$$

6.5 Automorphisms of $\mathfrak{g} = \mathfrak{sp}(n)$

The Lie subsuperalgebra $\mathfrak{p}(n)$ of $\mathfrak{gl}(n|n)$ in matrix form is defined by

$$\mathfrak{p}(n) := \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A \in \mathfrak{gl}_n, B = B^t, C = -C^t \right\}.$$

By $\mathfrak{sp}(n)$ we denote the quotient $\mathfrak{p}(n)/\mathfrak{z}$ where $\mathfrak{z} := \left\{ \begin{pmatrix} \lambda I_n & 0 \\ 0 & -\lambda I_n \end{pmatrix} \mid \lambda \in k \right\}$. We have that $\mathfrak{sp}(n)_{\bar{0}} \simeq \mathfrak{sl}_n$, $\mathfrak{sp}(n)_{\bar{1}} = \mathfrak{sp}(n)_1 \oplus \mathfrak{sp}(n)_{-1}$, $\mathfrak{sp}(n)_1 \simeq S^2 V_n$, and $\mathfrak{sp}(n)_{-1} \simeq \Lambda^2 V_n$.

There is a natural homomorphism $\mathrm{Ad} : \mathbf{SL}_n \rightarrow \mathbf{Aut} \mathfrak{g}$ given by $X \mapsto \mathrm{Ad}(X, X^*)$, where $X^* := (X^t)^{-1}$. Explicitly

$$\mathrm{Ad}_R(X) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} XAX^{-1} & XBX^t \\ X^*CX^{-1} & X^*DX^t \end{pmatrix}$$

for all $X \in \mathbf{SL}_n(R)$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{sp}(n)(R)$. Combined with the map j as above this yields a group homomorphism

$$\mathrm{Ad} \times j : \mathbf{SL}_n \times \mathbf{G}_m \rightarrow \mathbf{Aut} \mathfrak{sp}(n)$$

that can be seen to be surjective on k -points by reasoning as Serganova's. Clearly (X, λ) belongs to $\ker(\mathrm{Ad} \times j)_R$ iff $X = \alpha I_n$ with $\alpha^n = 1$ and $\alpha^2 \lambda = 1$. Thus:

$$\mathbf{Aut} \mathfrak{sp}(n) \simeq (\mathbf{SL}_n \times \mathbf{G}_m)/\boldsymbol{\mu}_n.$$

6.6 Automorphisms of $\mathfrak{psq}(n)$

The Lie subsuperalgebra $\mathfrak{q}(n)$ of $\mathfrak{gl}(n|n)$ is defined in matrix form by

$$\mathfrak{q}(n) := \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A, B \in \mathfrak{gl}_n \right\}.$$

We denote by $\mathfrak{sq}(n)$ the subsuperalgebra of odd-traceless matrices $\left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid \mathrm{tr}(B) = 0 \right\}$,

and by $\mathfrak{psq}(n)$ the quotient $\mathfrak{sq}(n)/\mathfrak{z}$, where $\mathfrak{z} := \left\{ \begin{pmatrix} \lambda I_n & 0 \\ 0 & \lambda I_n \end{pmatrix} \mid \lambda \in k \right\}$. We have also that $\mathfrak{psq}(n)_{\bar{0}} \simeq \mathfrak{sl}_n$ and $\mathfrak{psq}(n)_{\bar{1}} \simeq \mathfrak{sl}_n$ is isomorphic to the adjoint representation of \mathfrak{sl}_n . Note that $\mathfrak{q}(n)$ is not invariant with respect to the supertransposition, however

it is so with respect to the q -supertransposition $\sigma_q : \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto \begin{pmatrix} A^t & \zeta B^t \\ \zeta B^t & A^t \end{pmatrix}$, where $\zeta = \zeta_4 \in k$ is our fixed primitive 4th root of unity. There is a natural homomorphism $\mathbf{SL}_n \rightarrow \mathbf{Aut} \mathfrak{psq}(n)$ arising from $X \mapsto \text{Ad}_R(X, X)$. The kernel of this map is $\boldsymbol{\mu}_n$ showing that $\mathbf{Aut} \mathfrak{psq}(n)$ contains a copy of \mathbf{PGL}_n . We also have a constant subgroup $\langle \sigma_q \rangle_k \simeq \mathbb{Z}_{4,k}$ of $\mathbf{Aut} \mathfrak{psq}(n)$. The considerations of §5 and Serganova's original argument show that we have a split exact sequence

$$1 \rightarrow \mathbf{PGL}_n \rightarrow \mathbf{Aut} \mathfrak{psq}(n) \rightarrow \mathbb{Z}_{4,k} \rightarrow 1.$$

6.7 Automorphisms of $\mathfrak{osp}(m|2n)$

The orthosymplectic Lie superalgebra is defined by

$$\mathfrak{osp}(m|2n) := \{Z \in \mathfrak{gl}(m|2n) \mid Z^{\text{st}} B_{m,n} + B_{m,n} Z = 0\},$$

where $B_{m,n} := \begin{pmatrix} I_m & 0 \\ 0 & J_n \end{pmatrix}$ and $J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Equivalently, we have

$$\mathfrak{osp}(m|2n) = \left\{ \begin{pmatrix} A & B \\ J_n B^t & D \end{pmatrix} \mid A \in \mathfrak{so}_m, B \in \mathcal{M}_{m,2n}, D \in \mathfrak{sp}_{2n} \right\}.$$

We have that $\mathfrak{osp}(m|2n)_{\bar{0}} = \mathfrak{so}_m \oplus \mathfrak{sp}_{2n}$ and $\mathfrak{osp}(m|2n)_{\bar{1}}$ is the $(\mathfrak{so}_m \oplus \mathfrak{sp}_{2n})$ -module $V_m \otimes V_{2n}$. Note that $\mathfrak{osp}(2|2n)_{\bar{1}} \simeq (k \otimes V_{2n}) \oplus (k \otimes V_{2n})$.

We have a natural homomorphism

$$\text{Ad} : \mathbf{O}_m \times \mathbf{Sp}_{2n} \rightarrow \mathbf{Aut} \mathfrak{osp}(m|2n)$$

which is surjective on k -points as shown by Serganova. The kernel of Ad is isomorphic to $\boldsymbol{\mu}_2$ coming from $\text{Ad}(-I_m, -I_{2n})$. Thus

$$\mathbf{Aut} \mathfrak{osp}(m|2n) \simeq (\mathbf{O}_m \times \mathbf{Sp}_{2n}) / \boldsymbol{\mu}_2.$$

Consider the canonical map $\tau : \mathbf{SO}_m \times \mathbf{Sp}_{2n} \rightarrow (\mathbf{O}_m \times \mathbf{Sp}_{2n}) / \boldsymbol{\mu}_2$. If m is odd then τ is injective and clearly surjective on k -points because $(X, Y) \cong ((-I_m)X, (-I_{2n})Y) \pmod{\boldsymbol{\mu}_2(k)}$. Thus

$$\mathbf{Aut} \mathfrak{osp}(m|2n) \simeq \mathbf{SO}_m \times \mathbf{Sp}_{2n} \text{ if } m \text{ is odd.}$$

Assume now that m is even. Let $r_m \in \mathbf{O}_m(k)$ be an element satisfying $\det r_m = -1$ and $r_m^2 = I_m$. Let $r := \text{Ad}(r_m, I_{2n}) \in \mathbf{Aut} \mathfrak{osp}(m|2n)$. Then r leads to a constant subgroup $\langle r \rangle_k \simeq \mathbb{Z}_{2,k}$. It is immediate that $r_R \text{Ad}_R(X, Y) r_R^{-1} = \text{Ad}_R(r_{m,R} X r_{m,R}^{-1}, Y)$ where $r_{m,R}$ is the image of r_m under the canonical map $\mathbf{O}_m(k) \rightarrow \mathbf{O}_m(R)$. This yields a group homomorphism

$$(\mathbf{SO}_m \times \mathbf{Sp}_{2n}) \rtimes \langle r \rangle_k \rightarrow \mathbf{O}_m \times \mathbf{Sp}_{2n} \rightarrow (\mathbf{O}_m \times \mathbf{Sp}_{2n}) / \boldsymbol{\mu}_2 \simeq \mathbf{Aut} \mathfrak{osp}(m|2n)$$

which is surjective on k -points by Serganova's original reasoning. Since the kernel is precisely the group $\boldsymbol{\mu}_2 \times I$ described above we have the split exact sequence

$$1 \rightarrow (\mathbf{SO}_m \times \mathbf{Sp}_{2n}) / \boldsymbol{\mu}_2 \rightarrow \mathbf{Aut} \mathfrak{osp}(m|2n) \rightarrow \mathbb{Z}_{2,k} \rightarrow 1$$

whenever m is even.

6.8 The exceptional Lie superalgebras

In this subsection we focus on the three exceptional Lie superalgebras $\mathfrak{g} = G(3)$, $\mathfrak{g} = F(4)$, and $\mathfrak{g} = D(\alpha)$. First we recall some basic facts about the multiplication structure of each \mathfrak{g} . For more details we refer the reader to [K2] and [S]. To describe the Lie bracket on \mathfrak{g} we introduce some notations. Denote by p the bilinear form on V_2 with matrix J , i.e. $p(u, v) := u^t J v$ for $u, v \in V_2$. For $c \in k^\times$ we define a map $\Psi^c : V_2 \otimes V_2 \rightarrow \mathfrak{sl}_2$ by the formula $\Psi^c(u, v)w := c(p(v, w)u - p(w, u)v)$ and write simply Ψ for Ψ^1 . One checks immediately that if $A \in \mathbf{GL}_2(k)$ then $A\Psi^c(u, v)A^{-1} = \Psi^c(Au, Av)$ for all u, v iff $p(Au, Av) = p(u, v)$ for all u, v iff $\det A = 1$.

6.8.1 The case $\mathfrak{g} = G(3)$. We have $\mathfrak{g}_0 \simeq G_2 \oplus \mathfrak{sl}_2$ and \mathfrak{g}_1 is isomorphic to the simple \mathfrak{g}_0 -module $\text{stan}_{G_2} \otimes V_2$, where G_2 is the exceptional Lie algebra $\text{Der}(\mathbb{O})$ of the derivations of the octonions \mathbb{O} , and stan_{G_2} is the standard 8-dimensional G_2 -module \mathbb{O} . Let $\pi_1 : G_2 \oplus \mathfrak{sl}_2 \rightarrow G_2$ and $\pi_2 : G_2 \oplus \mathfrak{sl}_2 \rightarrow \mathfrak{sl}_2$ be the natural projections. The projection π_2 of the restriction $[\cdot, \cdot]_{|\mathfrak{g}_1 \times \mathfrak{g}_1} : \mathfrak{g}_1 \times \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$ of the Lie bracket on \mathfrak{g}_1 is described by the formula: $\pi_2[x_1 \otimes y_1, x_2 \otimes y_2] = (x_1, x_2)_\mathbb{O} \Psi(y_1, y_2)$, where $x_i \otimes y_i \in \text{stan}_{G_2} \otimes V_2 \simeq \mathfrak{g}_1$ and $(x_1, x_2)_\mathbb{O} := x_1 x_2^* + x_2 x_1^*$ is the non-degenerate G_2 -invariant form on \mathbb{O} (for more details see for example [F]). The formula of π_1 , on the other hand, comes from the adjoint representation.

It is clear then that we have an injective k -group homomorphism $\text{Ad} : \mathbf{G}_2 \times \mathbf{SL}_2 \rightarrow \mathbf{Aut} G(3)$ given by:

$$\text{Ad}_R(X, Y) : ((G, A), u \otimes v) \mapsto ((XGX^{-1}, YAY^{-1}), Xu \otimes Yv).$$

Since by Serganova's original work this map is surjective on k -points we obtain:

$$\mathbf{Aut} G(3) \simeq \mathbf{G}_2 \times \mathbf{SL}_2.$$

6.8.2 The case $\mathfrak{g} = F(4)$. We have $\mathfrak{g}_0 \simeq \mathfrak{so}_7 \oplus \mathfrak{sl}_2$ and \mathfrak{g}_1 is isomorphic to the simple \mathfrak{g}_0 -module $\text{spin}_7 \otimes V_2$, where spin_7 is the Spin-representation of \mathfrak{so}_7 whose space is the Clifford algebra Cliff_6 . If π_1 and π_2 are the canonical projections of \mathfrak{g}_0 to its first and second summands, respectively, then we have again that $\pi_2[x_1 \otimes y_1, x_2 \otimes y_2] = (x_1, x_2) \Psi(y_1, y_2)$ where $x_i \otimes y_i \in \text{spin}_7 \otimes V_2 \simeq \mathfrak{g}_1$ and (\cdot, \cdot) is an \mathfrak{so}_7 -invariant form on spin_7 . Similarly to the case $\mathfrak{g} = G(3)$ we conclude that

$$\mathbf{Aut} F(4) \simeq (\mathbf{Spin}_7 \times \mathbf{SL}_2) / \boldsymbol{\mu}_2.$$

Here \mathbf{Spin}_7 is the simply connected k -group of type B_3 and $\boldsymbol{\mu}_2$ is the diagonal subgroup of the center $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ of $\mathbf{Spin}_7 \times \mathbf{SL}_2$.

6.8.3 The case $\mathfrak{g} = D(\alpha)$ (often denoted by $D(2, 1, \alpha)$). Now we have $\mathfrak{g}_0 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\mathfrak{g}_1 \simeq V_2 \otimes V_2 \otimes V_2$. Denote by $(\mathfrak{sl}_2)_i$ and $(V_2)_i$ the i -th components of \mathfrak{g}_0 and \mathfrak{g}_1 respectively and by $\pi_i : \mathfrak{g}_0 \rightarrow (\mathfrak{sl}_2)_i$ the natural projections. Then for $\alpha_1 := \alpha, \alpha_2 := 1, \alpha_3 := -1 - \alpha$ and $u_i \in (V_2)_i$ we have

$$[u_1 \otimes u_2 \otimes u_3, v_1 \otimes v_2 \otimes v_3] = \sum_{\sigma \in \mathfrak{S}_3} p_{\sigma(1)}(u_{\sigma(1)}, v_{\sigma(1)}) p_{\sigma(2)}(u_{\sigma(2)}, v_{\sigma(2)}) \Psi_{\sigma(3)}^{\alpha_{\sigma(3)}}(u_{\sigma(3)}, v_{\sigma(3)}),$$

where p_j and Ψ_j^c are the maps p and Ψ^c on $(\mathfrak{sl}_2)_j$. Note that $D(\alpha) \simeq D(\alpha')$ if and only if $\alpha' = -(\alpha + 1)^{\pm 1}$, $\alpha' = \alpha^{-1}$, or $\alpha' = -(\alpha/(\alpha + 1))^{\pm 1}$. In future we consider $\alpha \neq 0, -1$ since $D(0)$ and $D(-1)$ are not simple.

The natural homomorphism $\text{Ad} : \mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2 \rightarrow \mathbf{Aut} D(\alpha)$ has kernel $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ coming from $(-I_2, I_2, -I_2)$ and $(I_2, -I_2, -I_2)$. As usual we let $\mathbf{Aut}^0 D(\alpha)$ denote the image of Ad . Because $\mathbf{Aut}(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2) \simeq (\mathbf{PGL}_2)^3 \rtimes \mathfrak{S}_{3,k}$ the outer group can be found inside $\mathfrak{S}_{3,k}$. The answer depends on the value of α . Fix $\sigma \in \mathfrak{S}_3$ and $\lambda \in k^\times$. Define $\theta(\sigma, \lambda) \in \mathbf{GL}(D(\alpha))$ by

$$\theta(\sigma, \lambda) : ((A_1, A_2, A_3), u_1 \otimes u_2 \otimes u_3) \rightarrow ((A_{\sigma(1)}, A_{\sigma(2)}, A_{\sigma(3)}), \lambda u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes u_{\sigma(3)}).$$

Following Serganova we verify that:

- (i) If $\alpha = 1$ then $\theta((1, 2), 1) \in \mathbf{Aut} D(\alpha)$,
 - (ii) If $\alpha^3 = 1, \alpha \neq 1$ then $\theta((1, 2, 3), \lambda) \in \mathbf{Aut} D(\alpha)$, where $\lambda^2 = \frac{1}{\alpha}$.
- A straightforward calculation shows that

$$\theta(\sigma, \lambda)_R \text{Ad}(X_1, X_2, X_3) \theta(\sigma, \lambda)_R^{-1} = \text{Ad}(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}).$$

We thus have the following injective R -group homomorphisms:

$$\begin{aligned} \mathbf{Aut}^0 D(\alpha) &\rightarrow \mathbf{Aut} D(\alpha) && \text{for } \alpha \notin \{1, -1/2, -2\} \text{ and } \alpha^3 \neq 1, \\ \mathbf{Aut}^0 D(1) \rtimes \langle \theta((1, 2), 1) \rangle_k &\rightarrow \mathbf{Aut} D(1) \\ \mathbf{Aut}^0 D(\alpha) \rtimes \langle \theta((1, 2, 3), \lambda) \rangle_k &\rightarrow \mathbf{Aut} D(\alpha) && \text{for } \alpha \neq 1 \text{ and } \alpha^3 = 1. \end{aligned}$$

Serganova's original argument shows that these three homomorphisms are surjective on k -points. Since $D(\alpha) \simeq D(\beta)$ iff $\alpha, \beta \in \{1, -1/2, -2\}$, or $\alpha^3 = \beta^3 = 1$ and $\alpha \neq 1, \beta \neq 1$, we obtain:

- (a) If $\alpha \notin \{1, -1/2, -2\}$ and $\alpha^3 \neq 1$ then

$$\mathbf{Aut} D(\alpha) \simeq (\mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2) / (\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2).$$

- (b) If $\alpha \in \{1, -1/2, -2\}$ there is a split exact sequence

$$1 \rightarrow (\mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2) / (\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \rightarrow \mathbf{Aut} D(\alpha) \rightarrow \mathbb{Z}_{2,k} \rightarrow 1.$$

- (c) If $\alpha^3 = 1$ and $\alpha \neq 1$ there is a split exact sequence

$$1 \rightarrow (\mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2) / (\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \rightarrow \mathbf{Aut} D(\alpha) \rightarrow \mathbb{Z}_{3,k} \rightarrow 1.$$

6.9 The Cartan type Lie superalgebras

In this subsection we consider the simple Cartan type Lie superalgebras $W(n), S(n), S'(n)$ (for $n = 2k$), and $H(n)$. A brief summary for these Lie superalgebras is as follows (see [K2] for more details). First recall that every \mathbb{Z} -graded associative algebra $A = \bigoplus_{i \in \mathbb{Z}} A_i$ has a natural \mathbb{Z}_2 -grading $A = A_{\bar{0}} \oplus A_{\bar{1}}$ where $A_{\bar{0}} := \bigoplus_{i \in \mathbb{Z}} A_{2i}$ and $A_{\bar{1}} := \bigoplus_{i \in \mathbb{Z}} A_{2i+1}$. A becomes a Lie superalgebra defining a Lie superbracket on A by the equality $[a, b] := ab - (-1)^{\deg a \deg b} ba$, where $\deg x = \bar{i}$ whenever $x \in A_{\bar{i}}$. (We

will also write $\deg x = i$ for $x \in A_i$). By $W(n)$ we denote the (super)derivations of the Grassmann algebra $\Lambda(n) := \Lambda(\xi_1, \dots, \xi_n)$. Every element D of $W(n)$ has the form $D = \sum_{i=1}^n P_i(\xi_1, \dots, \xi_n) \frac{\partial}{\partial \xi_i}$ where by definition $\frac{\partial}{\partial \xi_i}(\xi_j) = \delta_{ij}$. Both $\Lambda(n)$ and $W(n)$ have natural gradings $\Lambda(n) = \bigoplus_{i=0}^n \Lambda(n)_i$ and $W(n) = \bigoplus_{j=-1}^{n-1} W(n)_j$, where $\Lambda(n)_i := \{P(\xi_1, \dots, \xi_n) \mid \deg P = i\}$ and $W(n)_j := \{\sum_{i=1}^n P_i \frac{\partial}{\partial \xi_i} \mid \deg P_i = j + 1\}$. Note that $[\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j}] = \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \xi_j} + \frac{\partial}{\partial \xi_j} \frac{\partial}{\partial \xi_i} = 0$ (the first equality holds because $\deg \frac{\partial}{\partial \xi_i} = \bar{1}$).

There are two associative superalgebras of differential forms defined over $\Lambda(n)$, namely $\Omega(n)$ and $\Theta(n)$. The superalgebra $\Omega(n)$ has generators $d\xi_1, \dots, d\xi_n$ and defining relations $d\xi_i \circ d\xi_j = d\xi_j \circ d\xi_i$, $\deg d\xi_i = \bar{0}$, while the superalgebra $\Theta(n)$ has generators $\theta\xi_1, \dots, \theta\xi_n$ and relations $\theta\xi_i \wedge \theta\xi_j = -\theta\xi_j \wedge \theta\xi_i$, $\deg \theta\xi_i = \bar{1}$. Note that the differentials d and θ are derivations of degrees $\bar{1}$ and $\bar{0}$ respectively. Let $\Delta_n := \theta\xi_1 \wedge \dots \wedge \theta\xi_n$ be the standard volume form in $\Omega(n)$, and let $\omega_n := \sum_{i=1}^n d\xi_i \circ d\xi_i$ be the standard Hamiltonian form in $\Theta(n)$. Put $\Delta'_n := (1 + \xi_1 \dots \xi_n) \Delta_n$.

Every derivation D of $W(n)$ and every automorphism Φ of $\Lambda(n)$ extend uniquely to a derivation \tilde{D}^d and an automorphism $\tilde{\Phi}^d$ (respectively, \tilde{D}^θ and $\tilde{\Phi}^\theta$) of $\Omega(n)$ (resp., $\Theta(n)$) so that $[\tilde{D}^d, d] = 0$ and $[\tilde{\Phi}^d, d] = 0$ (resp., $\tilde{D}^\theta \theta f - \theta \tilde{D}^\theta f = 0$ and $\tilde{\Phi}^\theta \theta f - \theta \tilde{\Phi}^\theta f = 0$ for every $f \in \Lambda(n)$). We denote by $S(n)$ the Lie superalgebra $\{D \in W(n) \mid \tilde{D}^\theta(\Delta_n) = 0\}$ and by $S'(n)$ (not to be confused with the derived superalgebra $S(n)'$ of $S(n)$) the Lie superalgebra $\{D \in W(n) \mid \tilde{D}^\theta(\Delta'_n) = 0\}$. Since $S'(2k+1) \simeq S(2k+1)$ we are interested in $S'(n)$ only for even numbers n . The Hamiltonian Lie superalgebras are defined by $\tilde{H}(n) := \{D \in W(n) \mid \tilde{D}^d \omega_n = 0\}$ and $H(n) := [\tilde{H}(n), \tilde{H}(n)]$. The following explicit description of the Cartan type series S , S' , H , and \tilde{H} is very helpful for our considerations:

$$S(n) = \text{Span}_k \left\{ \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial \xi_i} \mid f \in \Lambda(n), i, j = 1, \dots, n \right\},$$

$$S'(n) = \text{Span}_k \left\{ (1 - \xi_1 \dots \xi_n) \left(\frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial \xi_i} \right) \mid f \in \Lambda(n), i, j = 1, \dots, n \right\},$$

$$\tilde{H}(n) = \text{Span}_k \left\{ D_f := \sum_i \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial \xi_i} \mid f \in \Lambda(n), f(0) = 0, i, j = 1, \dots, n \right\},$$

$$\tilde{H}(n) = H(n) \oplus kD_{\xi_1 \dots \xi_n}.$$

We have also that $[D_f, D_g] = D_{\{f, g\}}$ where $\{f, g\} := (-1)^{\deg f} \sum_{i=1}^n \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_i}$.

For any Lie subsuperalgebra \mathfrak{k} of $W(n)$ define $\mathfrak{k}_i := \{D \in \mathfrak{k} \mid \deg D = i\}$ and $\mathfrak{k}^i := \bigoplus_{j \geq i} \mathfrak{k}_j$. If \mathfrak{k} is any of the following Lie superalgebras $W(n)$, $S(n)$, $S'(n)$, $\tilde{H}(n)$, $H(n)$ then $\mathfrak{k} = \mathfrak{k}^{-1} \supset \mathfrak{k}^0 \supset \dots \supset \mathfrak{k}^{l_{\mathfrak{k}}} \supset 0$ is a filtration of \mathfrak{k} , where $l_{\mathfrak{k}} = n - 1$ for $\mathfrak{k} = W(n)$, $l_{\mathfrak{k}} = n - 2$ for $\mathfrak{k} = S(n)$, $\mathfrak{k} = S'(n)$, and $\mathfrak{k} = \tilde{H}(n)$, and $l_{\mathfrak{k}} = n - 3$ for $\mathfrak{k} = H(n)$. We also have that $\mathfrak{k} = \bigoplus_{i=-1}^{l_{\mathfrak{k}}} \mathfrak{k}_i$ is a grading for $\mathfrak{k} = W(n)$, $S(n)$, $\tilde{H}(n)$ and $H(n)$, and that the graded superalgebra $\text{Gr}(S'(n))$ of $S'(n)$ is isomorphic to $S(n)$. The degree-zero component of \mathfrak{k} is described by the isomorphisms $W(n)_0 \simeq \mathfrak{gl}_n$, $S(n)_0 \simeq S'(n)_0 \simeq \mathfrak{sl}_n$, $\tilde{H}(n)_0 \simeq H(n)_0 \simeq \mathfrak{so}_n$.

There is a natural homomorphism $\text{Ad} : \mathbf{Aut} \Lambda(n) \rightarrow \mathbf{Aut} W(n)$ given by $\text{Ad}_R(\varphi_R)(D) := \varphi_R D \varphi_R^{-1}$ for $D \in W(n)(R)$ and $\varphi_R \in \mathbf{Aut} \Lambda(n)(R)$. Note that in this case Ad is injective. Indeed, let $\varphi_R \xi_i = \sum_j c_{ij} \xi_j + f_i$ where $f_i \in \bigoplus_{j \geq 2} \Lambda(n)_i(R)$ (there is no constant term in the expression of $\varphi_R \xi_i$ because $\xi_i^2 = 0$). Then applying the identity $\varphi_R D f = D \varphi_R f$ for $D = \frac{\partial}{\partial \xi_j}$ and $f = \xi_i$ we find $\frac{\partial}{\partial \xi_j} f_i = 0$ and thus $f_i = 0$. Applying the same identity for $D \in W(n)_0(R) \simeq \mathfrak{gl}_n(R)$ we conclude by Schur's Lemma (Lemma 5.3) that $(\varphi_R)|_{\Lambda(n)_1} = \lambda \text{Id}$ for some $\lambda \in R^\times$, and finally we let again $D \in W(n)_{-1}(R)$ to find $\lambda = 1$. Since any element φ of $\mathbf{Aut} \Lambda(n)$ is completely determined by its restriction $\varphi|_{\Lambda(n)_1}$ the kernel of Ad is trivial as desired.

It is clear that each element θ_R of $\mathbf{GL}_n(R) = \mathbf{Aut}_{R\text{-mod}}(R\xi_1 \oplus \dots \oplus R\xi_n)$ extends to an automorphism of the R -algebra $\Lambda(n) \otimes_k R$ (by the universal nature of Λ). This yields an embedding $\mathbf{GL}_n \subset \mathbf{Aut} \Lambda(n)$. Let $\mathbf{N}_{W(n)}$ be the subgroup of $\mathbf{Aut} \Lambda(n)$ defined as follows:

$$\mathbf{N}_{W(n)}(R) := \{\nu \in \mathbf{Aut} \Lambda(n)(R) \mid (\nu - \text{Id})(\Lambda(n)_i(R)) \subset \bigoplus_{j > i} \Lambda(n)_j(R), \text{ for all } 0 \leq i \leq n\}.$$

It is clear that $\mathbf{N}_{W(n)}$ is a closed (hence affine) subgroup of $\mathbf{Aut} \Lambda(n)$. Set for simplicity $\mathbf{N} := \mathbf{N}_{W(n)}$, $V := \Lambda(n)_1 = \bigoplus_{i=1}^n k\xi_i$ and $U := \bigoplus_{i \geq 1} \Lambda(n)_{2i+1} = \bigoplus_{i \geq 1} \Lambda^{2i+1} V$.

Lemma 6.3 $\mathbf{N} \simeq \mathbf{Hom}(V, U)$ as k -functors.

Proof. If $\nu \in \mathbf{N}(R)$ then clearly $\nu - \text{Id} : V(R) \rightarrow U(R)$ (remember that ν preserves $\Lambda(n)_{\bar{1}}$). Since ν is completely determined by its values on $V(k) = V$ the above yields an injective natural transformation $\mathbf{N} \rightarrow \mathbf{Hom}(V, U)$. Assume now that an R -linear map $\varphi : V(R) \rightarrow U(R)$ is given. We show that φ is in the image of \mathbf{N} . Let $f_i^{(1)} := \varphi(\xi_i) \in U(R)$. Then $\delta_1 := \sum_{i=1}^n f_i^{(1)} \frac{\partial}{\partial \xi_i}$ is an n -step nilpotent derivation of $\Lambda(n)(R)$. Thus $e^{\delta_1} := \sum_{i=0}^n \frac{\delta_1^i}{i!} \in \mathbf{Aut} \Lambda(n)(R)$ and $e^{\delta_1}(\xi_i) = \xi_i + f_i^{(1)} + f_i^{(2)}$ for some $f_i^{(2)} \in \bigoplus_{j \geq 2} \Lambda^{2j+1} V(R)$. We next proceed by induction setting $e^{\delta_{j-1}} \dots e^{\delta_1}(\xi_i) = \xi_i + f_i^{(1)} + f_i^{(j)}$ for some $f_i^{(j)} \in \bigoplus_{i \geq j} \Lambda^{2i+1} V(R)$ and $\delta_j := -\sum_{i=1}^n f_i^{(j)} \frac{\partial}{\partial \xi_i}$. It is clear that there exists l for which $f_i^{(l)} \neq 0$ for some i and $f_i^{(j)} = 0$ for all $j > l$ and all i . Then $e^{\delta_l} \dots e^{\delta_1} - \text{Id} = \varphi$, and the claim follows. \square

It follows from this that as an affine scheme $\mathbf{N} = \text{Spec } k[\mathbf{N}]$ where $k[\mathbf{N}] \simeq S_k(V^* \otimes U)$ (S_k stands for the symmetric algebra over k). In particular \mathbf{N} is connected. It is also clear by looking at the identical representation of \mathbf{N} on $\Lambda(n)$ that \mathbf{N} is unipotent. Furthermore, with the language of [DG], Ch. IV, §2, 4.5, we see that \mathbf{N} is the unipotent group that corresponds to the nilpotent subalgebra $(W(n)^2)_{\bar{0}}$ of $W(n)$. (In fact one can use this approach to show that \mathbf{N} is connected).

Using the definitions of the Cartan type series S , S' , and H we easily verify that the following homomorphisms defined by Ad take place:

$$(6.4) \quad \mathbf{GL}_n \rightarrow \mathbf{Aut} S(n), \mathbf{SL}_{2n} \rightarrow \mathbf{Aut} S'(2n), \mathbf{O}_n \times \mathbf{G}_m \rightarrow \mathbf{Aut} H(n)$$

where \mathbf{G}_m corresponds to the scalar matrices λI_n , $\lambda \in \mathbf{G}_m(R) \simeq R^\times$. All three homomorphisms come from linear maps $\Lambda(n)_1 \rightarrow \Lambda(n)_1$ and thus preserve the gradings

of $\Lambda(n)$ and $W(n)$. The kernel of the last homomorphism is isomorphic to μ_2 coming from $(-I_n, -I_n)$ and all other kernels are trivial.

Define the subgroups $\mathbf{N}_{S(n)}$, $\mathbf{N}_{S'(n)}$, and $\mathbf{N}_{H(n)}$ of $\mathbf{Aut} \Lambda(n)$ by the identities $\mathbf{N}_{S(n)}(R) := \{\nu \in \mathbf{N}(R) \mid \tilde{\nu}^\theta(\Delta_n) = \Delta_n\}$, $\mathbf{N}_{S'(n)}(R) := \{\nu \in \mathbf{N}(R) \mid \tilde{\nu}^\theta(\Delta'_n) = \Delta'_n\}$, $\mathbf{N}_{H(n)}(R) := \{\nu \in \mathbf{N}(R) \mid \tilde{\nu}^d(\omega_n) = \omega_n\}$. The following analog of Lemma 6.3 describes the nature of these three subgroups.

Lemma 6.5 *The following isomorphisms of k -functors take place:*

- (i) $\mathbf{N}_{S(n)} \simeq \mathbf{N}_{S'(n)} \simeq (\oplus_{i \geq 1} (V^* \otimes \Lambda^{2i+1}V) / \Lambda^{2i}V)_a$
- (ii) $\mathbf{N}_{H(n)} \simeq (\oplus_{i \geq 2} \Lambda^{2i}V)_a$

Proof. The isomorphisms follow from Propositions 3.3.1 and 3.3.6 in [K2] but for the sake of clearness we sketch the proof. There is a natural isomorphism $\mathcal{D} : \mathbf{Hom}(V(R), \Lambda^{2i+1}V(R)) \rightarrow W(n)_{2i}(R)$ defined by $\mathcal{D} : \psi \mapsto \sum_{i=1}^n \psi(\xi_j) \frac{\partial}{\partial \xi_i}$. With some abuse of notation we will write simply $\tilde{\psi}^\theta$ and $\tilde{\psi}^d$ instead of $\widetilde{\mathcal{D}(\psi)}^\theta$ and $\widetilde{\mathcal{D}(\psi)}^d$.

(i) Let $\varphi \in \mathbf{Hom}(V(R), \Lambda^{2i+1}V(R))$ be defined by $\varphi(\xi_j) = f_j \xi_j + g_j$, where $f_j, g_j \in \Lambda(n)(R)$ and no summand $c_{j_1, \dots, j_{2i+1}} \xi_{j_1} \dots \xi_{j_{2i+1}}$ of $g_j = \sum_{(j_1, \dots, j_{2i+1})} c_{j_1, \dots, j_{2i+1}} \xi_{j_1} \dots \xi_{j_{2i+1}}$ contains ξ_j . Then we present φ in the following way: $\varphi = \alpha + \beta$, where $\alpha(\xi_j) = \frac{1}{n}(\sum_l f_l) \xi_j$ and $\beta(\xi_j) = g_j + (f_j - \frac{1}{n}(\sum_l f_l)) \xi_j$. Note that $\tilde{\alpha}^\theta(\Delta_n) = (\sum_l f_l) \Delta_n$ and $\tilde{\beta}^\theta \Delta_n = 0$ (similar identities for Δ'_n). This presentation of φ leads to the following isomorphism:

$$\mathbf{Hom}(V(R), \Lambda^{2i+1}V(R)) \simeq \Lambda^{2i}V(R) \oplus \mathcal{D}^{-1}(S(n)_{2i}(R))$$

where the first summand corresponds to the subset of $\mathbf{Hom}(V(R), \Lambda^{2i+1}V(R))$ spanned by the inner derivations $i_f : V \rightarrow \Lambda^{2i+1}V$, $i_f(g) := fg, f \in \Lambda^{2i}V$. Now using arguments similar to those in the proof of Lemma 6.3 we show (i).

(ii) The homomorphism $\Lambda^{2i+2}V(R) \rightarrow \mathbf{Hom}(V(R), \Lambda^{2i+1}V(R))$ defined by $f \mapsto D_f$ is injective for $i \geq 1$. Its image is $\tilde{H}(n)_{2i}$ (by the definition of H) and we obtain the isomorphism $\Lambda^{2i+2}V(R) \simeq \{\nu \in \mathbf{Hom}(V(R), \Lambda^{2i+1}V(R)) \mid \tilde{\nu}^d \omega_n = 0\}$ from which the statement easily follows. \square

Note. Let \mathfrak{g} be one of the Lie superalgebras $W(n), S(n), S'(n)$ (if $n = 2l$), $H(n)$, or $\tilde{H}(n)$. Denote by $\mathfrak{n}_{\mathfrak{g}}$ the nilpotent Lie algebra $(\mathfrak{g}^2)_{\bar{0}}$. Note that $\mathfrak{n}_{\mathfrak{g}}$ is the radical of the Lie algebra $\mathfrak{g}_{\bar{0}}$ for $\mathfrak{g} = S(n), S'(2l), H(n), \tilde{H}(n)$ and it is the commutator of the radical of $\mathfrak{g}_{\bar{0}}$ for $\mathfrak{g} = W(n)$. We can easily show that $\mathbf{N}_{\mathfrak{g}}$ is the unipotent group that corresponds to $\mathfrak{n}_{\tilde{\mathfrak{g}}}$, where $\tilde{\mathfrak{g}} := \tilde{H}(2l)$ for $\mathfrak{g} = H(2l)$ and $\tilde{\mathfrak{g}} := \mathfrak{g}$ in all other cases (the arguments for $\mathfrak{g} = S, S', H$ are similar to those for $\mathfrak{g} = W$).

From the discussion above we see that $\text{Ad} : \mathbf{N}_{\mathfrak{g}} \rightarrow \mathbf{Aut} \mathfrak{g}$ is an injective homomorphism. Combining this homomorphism with the homomorphisms in (6.4) we find the following injective homomorphisms:

$$\begin{aligned} \mathbf{N}_{W(n)} \times \mathbf{GL}_n &\rightarrow \mathbf{Aut} W(n), \mathbf{N}_{S(n)} \times \mathbf{GL}_n \rightarrow \mathbf{Aut} S(n), \\ \mathbf{N}_{S'(2l)} \times \mathbf{SL}_{2l} &\rightarrow \mathbf{Aut} S'(2l), \mathbf{N}_{H(n)} \times ((\mathbf{O}_n \times \mathbf{G}_m) / \mu_2) \rightarrow \mathbf{Aut} H(n). \end{aligned}$$

We easily see that $(\mathbf{O}_n \times \mathbf{G}_m)/\mu_2 \simeq \mathbf{SO}_n \times \mathbf{G}_m$ for odd n . In the case when n is even one reasons as in §6.7 and shows that there is a split exact sequence:

$$1 \rightarrow \mathbf{N}_{H(n)} \rtimes ((\mathbf{SO}_n \times \mathbf{G}_m)/\mu_2) \rightarrow \mathbf{Aut} H(n) \rightarrow \mathbb{Z}_{2,k} \rightarrow 1$$

where $\mathbb{Z}_{2,k} = \langle r_m \rangle_k$ for $r_m \in \mathbf{O}_n$, $\det r_m = -1$, and $r_m^2 = 1$. As usual, using Serganova's results that the corresponding homomorphisms are surjective on k -points we show that all four homomorphisms above are isomorphisms.

7 The supercentroid of \mathfrak{g}

In this section we find the *supercentroid* $\text{Ctd}(\mathfrak{g})$ of \mathfrak{g} , where

$$\text{Ctd}_k(\mathfrak{g}) := \{\chi \in \text{End}_k(\mathfrak{g}) \mid \chi([x, y]) = [\chi(x), y] \text{ for all } x, y \in \mathfrak{g}\}.$$

Recall that by definition of $\text{End}_k(\mathfrak{g})$, we have $\varphi(\mathfrak{g}_i) \subseteq \mathfrak{g}_i$ for all $\varphi \in \text{End}_k(\mathfrak{g})$ and $\bar{i} \in \mathbb{Z}_2$. Let $\lambda_{\mathfrak{g}} : k \rightarrow \text{End}_k(\mathfrak{g})$ be defined by $\lambda_{\mathfrak{g}}(a)(x) := ax$ for $a \in k$ and $x \in \mathfrak{g}$. We call \mathfrak{g} *central* if $\text{Ctd}_k(\mathfrak{g}) = \lambda_{\mathfrak{g}}(k)$.

Proposition 7.1 *All simple finite dimensional Lie superalgebras are central.*

Proof. It is clear that an endomorphism χ is in the centroid of \mathfrak{g} if and only if $\chi \circ \text{ad} = \text{ad} \circ \chi$. The proposition follows directly from the supersversion of the classical Schur's Lemma (see §1.1.6 in [K2]) since the adjoint representation of \mathfrak{g} is irreducible. \square

8 The Proofs of Theorems 4.2, 4.3, 4.4, and 4.5

8.1 Cohomological considerations

We refer the reader to [Mil], [DG], and [SGA] for general facts about étale nonabelian H^1 . Let $L(\mathfrak{g}, \sigma)$ be as in §3 with σ of period m . Recall that for $S = k[z^{\pm 1}]$ viewed as an $R = k[t^{\pm 1}]$ -algebra via $t \mapsto z^m$ we have

$$L(\mathfrak{g}, \sigma) \otimes_R S \simeq_S \mathfrak{g} \otimes S \simeq (\mathfrak{g} \otimes_k R) \otimes_R S.$$

Because of this, we can attach to $L(\mathfrak{g}, \sigma)$ an element

$$[L(\mathfrak{g}, \sigma)] \in H_{\text{ét}}^1(S/R, \mathbf{Aut} \mathfrak{g}_R) \subset H_{\text{ét}}^1(R, \mathbf{Aut} \mathfrak{g}_R).$$

Now S/R is a Galois extension with Galois group $\Gamma \simeq \mathbb{Z}_m$ given by $\Gamma = \langle \gamma \rangle$ where $\gamma(z) = \zeta_m z$. One knows that there is a natural correspondence (see [W] or [Mil], §III.2, Example 2.6) $H_{\text{ét}}^1(S/R, \mathbf{Aut} \mathfrak{g}_R) \xrightarrow{\sim} H^1(\Gamma, \mathbf{Aut} \mathfrak{g}_R(S))$ where this last is the “usual” nonabelian cohomology. Here Γ acts on $\mathbf{Aut} \mathfrak{g}_R(S) := \text{Aut}_S(\mathfrak{g} \otimes R) \otimes_S S = \text{Aut}_S \mathfrak{g} \otimes S$ by “acting on each of the coordinates” (the elements of $\mathbf{Aut} \mathfrak{g}_R(S)$ can, after fixing

a homogeneous basis of \mathfrak{g} , be thought as $n \times n$ matrices with entries in S where $n = \dim_k \mathfrak{g}$. In terms of maps, under this action $\gamma\theta : \mathfrak{g} \otimes S \rightarrow \mathfrak{g} \otimes S$ is given by $\gamma\theta := (1 \otimes \gamma) \circ \theta \circ (1 \otimes -\gamma)$. The action of Γ can of course also be described functorially as follows: If $\theta \in \mathbf{Aut} \mathfrak{g}_R(S) = \mathrm{Hom}_{R\text{-alg}}(R[\mathbf{Aut} \mathfrak{g}_R], S)$, then $\gamma\theta = \gamma \circ \theta$. An obvious calculation shows that the unique homomorphism $u : \Gamma \rightarrow \mathbf{Aut} \mathfrak{g}_R(S) := \mathrm{Aut}_S(\mathfrak{g} \otimes_k S)$ given by $\gamma \mapsto \sigma^{-1} \otimes 1$ is in fact a cocycle in $Z^1(\Gamma, \mathbf{Aut} \mathfrak{g}_R(S))$. Then $[L(\mathfrak{g}, \sigma)]$ is nothing but the cohomology class of this cocycle.

Via our choice of compatible roots of unity, the algebraic fundamental group of R based at the geometric point 1 can be identified with $\varprojlim \mathbb{Z}_n$, hence cyclic in the topological sense. From Grothendieck's work we obtain the correspondences

$$H_{\acute{e}t}^1(R, F_R) \simeq \text{Conjugacy classes of } F \simeq \text{Galois extensions of } R \text{ with group } F.$$

(see [SGA1], §5, Exp. 11 and [P]). As it seems inevitable, under the canonical map $H_{\acute{e}t}^1(R, \mathbf{Aut} \mathfrak{g}_R) \rightarrow H_{\acute{e}t}^1(R, F_R)$ we have $[L(\mathfrak{g}, \sigma)] \mapsto \{\text{conjugacy class of } \bar{\sigma}^{-1}\}$ where $\bar{\cdot} : \mathbf{Aut} \mathfrak{g}(k) \rightarrow F$ arises from Theorem 4.1.

8.2 Proof of Theorem 4.2

That the map $H_{\acute{e}t}^1(R, \mathbf{Aut} \mathfrak{g}_R) \rightarrow H_{\acute{e}t}^1(R, F_R)$ is surjective follows from the fact that in all cases (as made clear in the proof of Theorem 4.1) the group F_k lifts to a finite constant subgroup \tilde{F}_k of $\mathbf{Aut} \mathfrak{g}$, and one can now conclude by the considerations of §8.1.

Let $\pi \in F$, and lift π to an element $\tilde{\pi}$ of \tilde{F} . Then $\tilde{\pi}$ induces an automorphism $\tilde{\pi}_0$ of \mathbf{G}^0 and hence by base change also one of \mathbf{G}_R^0 . One can then consider the twisted group $\tilde{\pi} \mathbf{G}_R^0$ constructed as follows. Let l be the order of $\tilde{\pi}_0$. Consider $S = k[z^{\pm 1}]$ which we view as an R -algebra via $t \mapsto z^l$. As we have seen S/R is a Galois extension with Galois group $\Gamma \simeq \mathbb{Z}_l$ generated by γ where $\gamma(z) = \zeta z$ where $\zeta = \zeta_l$.

If $A \in R\text{-alg}$ and $x \in \mathbf{G}_R^0(S \otimes_R A) := \mathrm{Hom}_R(R[\mathbf{G}^0], S \otimes_R A)$ we define $\gamma x \in \mathbf{G}_R^0(S \otimes_R A)$ by $\gamma x = (\gamma \otimes 1) \circ x$. Then the twisted group $\tilde{\pi} \mathbf{G}_R^0$ is the R -group whose functor of points is given by $\tilde{\pi} \mathbf{G}_R^0(A) = \{x \in \mathbf{G}_R^0(S \otimes_R A) : \tilde{\pi}_{0_{S \otimes_R A}}(\gamma x) = x\}$.

Remark. That $\tilde{\pi} \mathbf{G}_R^0$ is a an R -group (i.e. affine) follows in general from descent. In the present case one can easily see what the coordinate ring is by viewing π as an automorphism of the Hopf algebra $R[\mathbf{G}^0] = R \otimes_k k[\mathbf{G}^0]$. Then $R[\tilde{\pi} \mathbf{G}_R^0] = \{f \in S \otimes_R R[\mathbf{G}^0] \mid \tilde{\pi}_{0_S}(\gamma \otimes 1)(f) = f\}$. (This amounts to the usual cocycle condition of descent because Γ is generated by γ and the action of Γ commutes with that of $\tilde{\pi}$). Up to an isomorphism the twisted group $\tilde{\pi} \mathbf{G}_R^0$ does not depend on the choice of the lift $\tilde{\pi}$ and will henceforth simply be denoted by $\pi \mathbf{G}_R^0$.

We now come to the most delicate point of the proof, namely showing that the map $H_{\acute{e}t}^1(R, \mathbf{Aut} \mathfrak{g}_R) \rightarrow H_{\acute{e}t}^1(R, F_R)$ is injective. The complication stems from the fact that these H^1 's are not groups, but rather sets with a distinguished element. As a consequence, to show injectivity it is not enough that $H_{\acute{e}t}^1(R, \mathbf{G}_R^0) = 0$, but rather that *all* twisted $H_{\acute{e}t}^1(R, \pi \mathbf{G}_R^0)$ also vanish. Unlike \mathbf{G}_R^0 , which is obtained from the k -group \mathbf{G}^0 by base change, the twisted $\pi \mathbf{G}_R^0$ (unless $\pi = 1$) *do not come from any*

algebraic k -group (there are no twisted forms of \mathbf{G}^0 over k). Thus $\pi\mathbf{G}_R^0$ is a reductive group over R in the sense of [SGA3]. Computing its cohomology is therefore not surprisingly much more delicate than in the untwisted case.

The general strategy to establish Theorem 4.2 will be as follows. We will consider an exact sequence of the form

$$(8.1) \quad 1 \rightarrow \mathbf{z} \rightarrow \mathbf{G} \rightarrow \mathbf{G}^0 \rightarrow 1.$$

where \mathbf{G} is reductive and \mathbf{z} is central and finite, i.e. an isogeny. We also assume that this sequence can be twisted by π to yield

$$1 \rightarrow \pi\mathbf{z}_R \rightarrow \pi\mathbf{G}_R \rightarrow \pi\mathbf{G}_R^0 \rightarrow 1.$$

Passing to cohomology we obtain

$$\rightarrow H_{\acute{e}t}^1(R, \pi\mathbf{G}_R) \rightarrow H_{\acute{e}t}^1(R, \pi\mathbf{G}_R^0) \rightarrow H_{\acute{e}t}^2(R, \pi\mathbf{z}_R).$$

By [SGA4], Exp. IX, 5.7, and Exp. X, 5.2 our ring R is of cohomological dimension 1. As a consequence $H_{\acute{e}t}^2(R, \pi\mathbf{z}_R) = 1$ and therefore Theorem 4.2 will follow once we establish that in each case for all $\pi \in \mathbb{F}$

$$(8.2) \quad H_{\acute{e}t}^1(R, \pi\mathbf{G}_R) = 1.$$

8.2.1 The case $\mathbb{F} = 1$. Let $\mathbf{z} = 1_k$ in (8.1). Then $\mathbf{Aut} \mathfrak{g} = \mathbf{G}^0$ is a connected linear algebraic k -group. By [P], Proposition 5, $H_{\acute{e}t}^1(R, \mathbf{G}_R^0) = 1$.

In view of this we may therefore assume that $\mathbb{F} \neq 1$, that $\pi \neq 1$, and verify (8.2) in each of the cases.

8.2.2 The case $\mathfrak{g} = \mathfrak{sl}(m|n)$. The generator of the outer \mathbb{Z}_2 lifts to the element r of $\mathbf{Aut} \mathfrak{sl}(m|n)(k)$, which acts on each of the factors of $\mathbf{SL}_m \times \mathbf{SL}_n \times \mathbf{G}_m$ via $\pi_R : X \mapsto (X^t)^{-1}$ in the \mathbf{SL} 's and $\gamma_R : X \mapsto X^{-1}$ in \mathbf{G}_m . This action stabilizes $\mu_m \times \mu_n$ where it acts via $(\alpha, \beta) \mapsto (\alpha^{-1}, \beta^{-1})$. The exact sequence (8.1) is given by

$$1 \rightarrow \mu_m \times \mu_n \rightarrow \mathbf{SL}_m \times \mathbf{SL}_n \times \mathbf{G}_m \rightarrow \mathbf{SL}_m \times \mathbf{SL}_n \times \mathbf{G}_m / (\mu_m \times \mu_n) \rightarrow 1.$$

To establish (8.2) we need

$$(8.3) \quad H_{\acute{e}t}^1(R, \gamma\mathbf{G}_{m_R}) = H_{\acute{e}t}^1(R, \pi\mathbf{SL}_{l_R}) = 1$$

Let $S = k[z^{\pm 1}]$ which we view as an R -algebra via $t \mapsto z^2$. This is a Galois extension of degree 2. The norm map $N : S^\times \rightarrow R^\times$ induces a surjective R -group homomorphism

$$N : \mathcal{R}_{S/R}\mathbf{G}_{m_S} \rightarrow \mathbf{G}_{m_R}$$

where \mathcal{R} denotes Weyl restriction. A direct calculation shows that our twisted multiplicative group is the kernel of N . Thus

$$1 \rightarrow \gamma\mathbf{G}_{m_R} \rightarrow \mathcal{R}_{S/R}\mathbf{G}_{m_S} \rightarrow \mathbf{G}_{m_R} \rightarrow 1.$$

By passing to cohomology we get

$$S^\times \xrightarrow{N} R^\times \rightarrow H_{\acute{e}t}^1(R, \gamma \mathbf{G}_{m_R}) \rightarrow H_{\acute{e}t}^1(R, \mathcal{R}_{S/R} \mathbf{G}_{m_S}).$$

The last H^1 vanishes by Shapiro's Lemma since

$$H_{\acute{e}t}^1(R, \mathcal{R}_{S/R} \mathbf{G}_{m_S}) = H_{\acute{e}t}^1(S, \gamma \mathbf{G}_{m_S}) = \text{Pic}(S) = 1.$$

The norm map is given by $\lambda z^l \mapsto (-1)^l \lambda^2 t^l$, hence it is surjective since k is algebraically closed. It follows that $H_{\acute{e}t}^1(R, \gamma \mathbf{G}_{m_R}) = 1$ as desired.

Finally ${}^\pi \mathbf{SL}_{l_R}$ is quasisplit of simply connected type so its H^1 is trivial by [P], Proposition 3. This finishes the proof of (8.3).

8.2.3 The case $\mathfrak{g} = \mathfrak{psl}(n|n), n > 2$. The reasoning is similar to that of $\mathfrak{sl}(m|n)$ except that we now have more twisted groups to content with. If σ denotes the element of $\mathbf{Aut}(\mathbf{SL}_n \times \mathbf{SL}_n)$ that comes from "switching" the two factors, we must show in addition to (8.3) that

$$(8.4) \quad H_{\acute{e}t}^1(R, {}^\sigma(\mathbf{SL}_n \times \mathbf{SL}_n)_R) = H_{\acute{e}t}^1(R, {}^{\sigma\pi}(\mathbf{SL}_n \times \mathbf{SL}_n)_R) = 1.$$

Again both ${}^\sigma(\mathbf{SL}_n \times \mathbf{SL}_n)_R$ and ${}^{\sigma\pi}(\mathbf{SL}_n \times \mathbf{SL}_n)_R$ are quasisplit of simply connected type so their $H_{\acute{e}t}^1(R, -)$ are trivial.

8.2.4 The case $\mathfrak{g} = \mathfrak{psl}(2|2)$. Take $\mathbf{z} = 1_k$ in (8.1). The cohomology that needs to vanish is $H_{\acute{e}t}^1(R, \gamma \mathbf{G}_m)$, and we have already seen that this is indeed the case.

8.2.5 The case $\mathfrak{g} = \mathfrak{psq}(n)$. Take $\mathbf{z} = 1_k$ in (8.1). All twisted forms of \mathbf{PGL}_{n_R} are R -groups of adjoint type and hence have trivial $H_{\acute{e}t}^1(R, -)$ by Proposition 3 of [P].

8.2.6 The case $\mathfrak{g} = \mathfrak{osp}(m|2n), m$ even. Again we take $\mathbf{z} = 1_k$ in (8.1). We need

$$H_{\acute{e}t}^1(R, {}^r(\mathbf{SO}_m \times \mathbf{Sp}_{2n})_R) = 1.$$

Now, ${}^r(\mathbf{SO}_m \times \mathbf{Sp}_{2n})_R \simeq {}^\pi \mathbf{SO}_{m_R} \times \mathbf{Sp}_{2n_R}$ where π is the automorphism of the "tail" of the Dynkin diagram of the group \mathbf{SO}_m . One knows that $H_{\acute{e}t}^1(R, {}^\pi \mathbf{SO}_m) = 0$. To see this consider the simply connected cover

$$1 \rightarrow \boldsymbol{\mu}_2 \rightarrow \mathbf{Spin}_m \rightarrow \mathbf{SO}_m \rightarrow 1$$

and conclude with the aid of Proposition 3 of [P] by passing to cohomology.

8.2.7 The case $\mathfrak{g} = D(1)$. For (8.1) we take

$$1 \rightarrow \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \rightarrow \mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2 \rightarrow \mathbf{Aut} D(1) \rightarrow 1,$$

which we need to twist by π where π switches the first two factors of $(\mathbf{SL}_2)^3$ and the two factors of $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. The twisted $(\mathbf{SL}_2)_R^3$ is of simply connected type and hence has trivial $H_{\acute{e}t}^1(R, -)$.

8.2.8 The case $\mathfrak{g} = D(\alpha), \alpha \neq 1, \alpha^3 = 1$. The reasoning is similar to that of $D(1)$.

8.2.9 The case $\mathfrak{g} = H(2l)$. The reason is similar to that of $\mathfrak{g} = \mathfrak{osp}(2l|2n)$. To deal with the twisted unipotent group, one reasons by dévissage on a suitable central composition series.

8.3 Proof of Theorem 4.3

The plan is to describe how the results we have obtained thus far allow us to apply the central ideas of [ABP] to obtain Theorem 4.3

First we observe that because \mathfrak{g} is perfect and central (see Proposition 7.1), $\text{Ctd}_k(L(\mathfrak{g}, \sigma)) \simeq \text{Ctd}_R(L(\mathfrak{g}, \sigma)) \simeq R$ (reason as in [ABP], Lemma 4.3). Now if $\varphi : L(\mathfrak{g}_1, \sigma_1) \rightarrow L(\mathfrak{g}_2, \sigma_2)$ is an isomorphism of k -Lie superalgebras, there exists a unique $\tilde{\varphi} \in \mathbf{Aut}_k(R)$ such that $\varphi(rx) = \tilde{\varphi}(r)\varphi(x)$ (ibid Lemma 4.4). To say that φ is an isomorphism of R -Lie superalgebras is equivalent to saying that $\tilde{\varphi} = \text{Id}$.

The trick now is to try to replace φ by another isomorphism φ' which is R -linear (there may be a price for doing this as we will see). We have $\mathbf{Aut}_k(R) \simeq k^\times \rtimes \mathbb{Z}_2$ where $\lambda \in k^\times$ accounts for $t \mapsto \lambda t$ and the generator of \mathbb{Z}_2 for the switch $t \rightarrow t^{-1}$. The up shot of this is that

$$(8.5) \quad L(\mathfrak{g}_1, \sigma_1) \simeq_k L(\mathfrak{g}_2, \sigma_2) \Leftrightarrow L(\mathfrak{g}_1, \sigma_1) \simeq_R L(\mathfrak{g}_2, \sigma_2) \text{ or } L(\mathfrak{g}_1, \sigma_1) \simeq_R L(\mathfrak{g}_2, \sigma_2^{-1}).$$

(ibid Theorem 4.6).

The reason behind this artifice is to bring us to the situation where the loop algebras are viewed as R -Lie superalgebras, and as such, their isomorphism classes are completely understood by cohomological methods. Assume $L(\mathfrak{g}_1, \sigma_1) \simeq_R L(\mathfrak{g}_2, \sigma_2)$ with σ_1 and σ_2 of common period m (the case of σ_2^{-1} is similar). Then as S -Lie superalgebras (see §8.1)

$$\mathfrak{g}_1 \otimes_k S \simeq L(\mathfrak{g}_1, \sigma_1) \otimes_R S \simeq L(\mathfrak{g}_2, \sigma_2) \otimes_R S \simeq \mathfrak{g}_2 \otimes_k S.$$

Now tensoring over S with the algebraic closure K of the quotient field of S we obtain $\mathfrak{g}_1 \otimes_k K \simeq \mathfrak{g}_2 \otimes_k K$ as k -Lie superalgebras and hence $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ by the classification of simple Lie superalgebras over algebraically closed fields of characteristic zero. \square

8.4 Proof of Theorem 4.4

Assume $L(\mathfrak{g}, \sigma_1) \simeq_R L(\mathfrak{g}, \sigma_2)$. Then $[L(\mathfrak{g}, \sigma_1)] = [L(\mathfrak{g}, \sigma_2)]$ and hence by §8.1 we obtain $\sigma_1^{-1} \sim \sigma_2^{-1}$ hence $\bar{\sigma}_1 \sim \bar{\sigma}_2$ in F . Similarly, if $L(\mathfrak{g}, \sigma_1) \simeq_R L(\mathfrak{g}, \sigma_2^{-1})$ then $\bar{\sigma}_1 \sim \bar{\sigma}_2^{-1}$. That (i) implies (ii) now follows from (8.5). For the converse observe that if $\bar{\sigma}_1 \sim \bar{\sigma}_2$ then $[L(\mathfrak{g}, \sigma_1)] = [L(\mathfrak{g}, \sigma_2)]$ by §8.1 in view of Theorem 4.2. But then $L(\mathfrak{g}, \sigma_1) \simeq_R L(\mathfrak{g}, \sigma_2)$ and a fortiori these two are isomorphic as k -Lie superalgebras. Similarly $\bar{\sigma}_1 \sim \bar{\sigma}_2^{-1}$ implies $L(\mathfrak{g}, \sigma_1) \simeq_k L(\mathfrak{g}, \sigma_2^{-1})$. But since $L(\mathfrak{g}, \sigma) \simeq_k L(\mathfrak{g}, \sigma^{-1})$ always ([ABP], Lemma 2.4(a)) the proof is now complete. \square

8.5 Proof of Theorem 4.5

The Zen of torsors shows that the isomorphism classes of forms of the R -Lie superalgebra $\mathfrak{g} \otimes_k R$ are classified by $H_{\text{ét}}^1(R, \mathbf{Aut} \mathfrak{g}_R)$ (we may replace the flat site by the étale site of $\text{Spec } R$ since our R -group $\mathbf{Aut} \mathfrak{g}_R$ is smooth). By applying now Theorem

4.2 one concludes that our forms are parameterized by $H_{\acute{e}t}^1(R, F_R)$. But the considerations of §8.1 show that all elements of $H_{\acute{e}t}^1(R, F_R)$ are of the form $[L(\mathfrak{g}, \tilde{\pi})]$ for some finite order automorphism $\tilde{\pi}$ of \mathfrak{g} . \square

References

- [ABP] B. Allison, S. Berman, A. Pianzola, Covering algebras II: Isomorphisms of Loop algebras, *J. Reine Angew. Math.* **571** (2004), 39-71.
- [CG] J.-P. Serre, Cohomologie Galoisienne (5^e Edt.), *Lecture Notes Math.* **5**, Springer, Berlin, 1994.
- [DG] M. Demazure, P. Gabriel, Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs. (French) Masson & Cie, diteur, Paris; North-Holland Publishing Co., Amsterdam, 1970.
- [DM] P. Deligne, J. Morgan, Notes on supersymmetry (following Joseph Bernstein), Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), 41–97, American Mathematical Society, Providence, RI, 1999.
- [F] H. Freudental, Oktaven, Ausnahmegruppen und Oktavengeometrie, [Octaves, exceptional groups and geometry of octaves], *Geom. Dedicata* **19** (1985), 7-63 (German).
- [K1] V. Kac, Simple irreducible graded Lie algebras of finite growth, *Math USSR-Izvestija* **2** (1968), 1271–1311.
- [K2] V. Kac, Lie superalgebras, *Adv. Math.* **26** (1977), 8-96.
- [KMRT] M.-A. Knus, A. Merkurjev, M. Rost, J.-P. Tignol, The book of involutions, *American Mathematical Society Colloquium Publications* **44**, American Mathematical Society, Providence, RI, 1998.
- [L] J. W. van de Leur, Contragredient Lie superalgebras of finite growth, Thesis Utrecht University, May 1986.
- [Man] Yu. Manin, Gauge field theory and complex geometry, *Grundlehren der Mathematischen Wissenschaften* **289**, Springer-Verlag, Berlin, 1988.
- [Mil] J. S. Milne, Étale cohomology, Princeton University Press, Princeton, 1980.
- [Pen] I. Penkov, Characters of strongly generic irreducible Lie superalgebra representations, *Internat. J. Math.* **9** (1998), 331–366.
- [P] A. Pianzola, Affine Kac-Moody Lie algebras as torsors over the punctured line, *Indag. Math.* **13** (2002), 249–257.

- [P2] A. Pianzola, Vanishing of H^1 for Dedekind rings and applications to loop algebras, preprint, 2004.
- [Sch] M. Scheunert, The theory of Lie superalgebras. An introduction, *Lecture Notes Math.* **716**, Springer, Berlin, 1979.
- [S] V. Serganova, Automorphisms of simple Lie superalgebras, *Math. USSR Izvestiya* **24** (1985), 539-551.
- [SGA1] Séminaire de Géométrie Algébrique du Bois Marie, 1960–1961, Revêtements étales et groupe fondamental, dirigé par A. Grothendieck, *Lecture Notes Math.* **224**, Springer-Verlag, Berlin-New York, 1971.
- [SGA3] Séminaire de Géométrie Algébrique du Bois Marie, 1963–1964, Schémas en groupes, dirigé par M. Demazure et A. Grothendieck, *Lecture Notes Math.* **151-153**, Springer-Verlag, Berlin-New York, 1970.
- [SGA4] Séminaire de Géométrie Algébrique du Bois Marie, 1963–1964, Théorie des topos et cohomologie étale des schémas, *Lecture Notes Math.* **269, 270, 305**, Springer-Verlag, Berlin-New York, 1972-1973.
- [W] W. Waterhouse, Introduction to affine group schemes, Springer-Verlag, New York, 1979.

Department of Mathematical and Statistical Sciences
 University of Alberta
 Edmonton, Alberta T6G 2G1
 CANADA

E-mails: grant@math.ualberta.ca, a.pianzola@math.ualberta.ca