Trigonometric integrals

Previous lecture

$$\int \sin x \cos x = \frac{1}{2}\int \sin 2x \, dx = -\frac{1}{4}\cos 2x + C,$$

$$\int \sin^2 x \, dx = \frac{1}{2}\int (1-\cos 2x)\, dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$$

$$\int \cos^2 x \, dx = \frac{1}{2}\int (1+\cos 2x)\, dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$$

Integrals of the form

$$\int \sin ax \cos bx \, dx, \quad \int \sin ax \sin bx \, dx, \quad \int \cos ax \cos bx \, dx, \qquad a \neq b$$

Two basic identities:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

(for small x, y sin is increasing, "+", cos is decreasing, "$-$")

Change y to $-y$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$2\sin x \cos y = \sin(x+y) + \sin(x-y)$$

$$2\cos x \cos y = \cos(x+y) + \cos(x-y)$$

$$2\sin x \sin y = \cos(x-y) - \cos(x+y)$$

Application to integrals.

Example

$$\int \sin 2x \cos 3x \, dx = \frac{1}{2}\int (\sin(2x+3x) + \sin(2x-3x))\, dx =$$

$$= \frac{1}{2}\int \sin 5x \, dx - \frac{1}{2}\int \sin x \, dx =$$

$$= -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C.$$

Example

$$\int \cos 3x \cos 5x \, dx = \frac{1}{2}\int \left(\cos(3x+5x) + \cos(3x-5x)\right) dx =$$

$$= \frac{1}{2}\int \cos 8x \, dx + \frac{1}{2}\int \cos 2x \, dx =$$

$$= \frac{1}{16}\sin 8x + \frac{1}{4}\sin 2x + C.$$

Example

$$\int \sin 3x \sin 5x \, dx = \frac{1}{2}\int \left(\cos(3x-5x) - \cos(3x+5x)\right) dx =$$

$$= \frac{1}{2}\int \cos 2x \, dx - \frac{1}{2}\int \cos 8x \, dx =$$

$$= \frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C.$$

Integrals of the form

$$\int \sin^m x \cos^n x \, dx, \qquad \int \tan^m x \sec^n x \, dx, \qquad \int \cot^m x \csc^n x \, dx.$$

Idea — use substitution to transform to integral of polynomial

$$\int P_k(u) du \qquad \text{or} \qquad \int \frac{P_k(u)}{u^s} ds.$$

Actual substitution depends on m, n, and the type of the integral.

Typical substitutions: $u = \sin x$, $\cos x$, $\tan x$, $\sec x$, $\cot x$, $\csc x$.

Task: find, which substitution integral can be transformed $\int P(u)du$ without $\sqrt{1 \pm u^2}$ or similar.

Even or odd powers in $\int \cos^m x \sin^n x \, dx$

$$\cos^m x \sin^n x \, dx = \cos^{m-1} x \sin^n x \, \cos x dx$$

$\cos x dx = d(\sin x)$, $u = \sin x$, but have to express $\cos^{m-1} x$ through $\sin x$.

If $m = 2k+1$ (odd) — we can: $\cos^{2k} x = \left(\cos^2 x\right)^k = \left(1 - \sin^2 x\right)^k = \left(1-u^2\right)^k$

If $m = 2k$ (even) — we can not without $\sqrt{1-\sin^2 x}$.

Similarly

$$\cos^m x \sin^n x \, dx = \cos^m x \sin^{n-1} x \, \sin x dx$$

If $n = 2k + 1$ (odd) we can use $u = \cos x$, if $n = 2k$ (even) — cannot.

sin–cos rule: if one has odd power, use other for substitution.

1) Odd power at $\cos x$, $u = \sin x$, $du = \cos x\, dx$, $\cos^2 x = 1 - u^2$

$$\int \cos^{2k+1} x \sin^n x\, dx = \int \cos^{2k} x \sin^n x\, \cos x dx = \int \left(1 - u^2\right)^k u^n du$$

Example: no $\sin x$, but still $u = \sin x$

$$\int \cos^5 x\, dx = \int \left(\cos^2 x\right)^2 \cos x\, dx = \int \left(1 - u^2\right)^2 du =$$

$$= \int \left(1 - 2u^2 + u^4\right) du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C =$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

2) Odd power at $\sin x$, $u = \cos x$, $du = -\sin x\, dx$, $\sin^2 x = 1 - u^2$

$$\int \cos^m x \sin^{2k+1} x\, dx = \int \cos^m x \sin^{2k} x\, \sin x dx = -\int u^m \left(1 - u^2\right)^k du$$

3) Odd powers at both: use for u one with bigger power:

Example: a) $u = \cos x$

$$\int \cos^3 x \sin^9 x\, dx = \int \cos^3 x \sin^8 x\, \sin x dx = -\int u^3 \left(1 - u^2\right)^4 du =$$

$$= -\int u^3 \left(1 - 4u^2 + 6u^4 - 4u^6 + u^8\right) du =$$

$$= -\frac{1}{4}\cos^4 x + \frac{4}{6}\cos^6 x - \frac{6}{8}\cos^8 x + \frac{4}{10}\cos^{10} x - \frac{1}{12}\cos^{12} x + C$$

b) $u = \sin x$

$$\int \cos^3 x \sin^9 x\, dx = \int \cos^2 x \sin^9 x\, \cos x dx = \int \left(1 - u^2\right) u^9 du =$$

$$= \int \left(u^9 - u^{11}\right) du = \frac{1}{10}u^{10} - \frac{1}{12}u^{12} + C = \frac{1}{10}\sin^{10} x - \frac{1}{12}\sin^{12} x + C$$

which is **simpler**?

4) Even powers at both. No good substitution. Transform with double angle formulas ($x \to 2x$) to smaller powers:

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin x \cos x = \frac{\sin 2x}{2}$$

Example:

$$
\begin{aligned}
\int \sin^4 x \cos^6 x dx &= \int (\sin x \cos x)^4 \cos^2 x = \\
&= \int \frac{1}{16} \sin^4 2x \frac{1+\cos 2x}{2} \frac{1}{2} d(2x) = [z = 2x] \\
&= \frac{1}{64} \int \sin^4 z dz \ \ [\text{double angle}] + \frac{1}{64} \int \sin^4 z \cos z dz \ \ [u = \sin z] = \\
&= \frac{1}{64} \int \left(\frac{1-\cos 2z}{2} \right)^2 dz + \frac{1}{64} \int u^4 du = \\
&= \frac{1}{256} \int \left(1 - 2\cos 2z + \cos^2 2z \right) dz + \frac{1}{320} u^5 + C = \\
&= \frac{z}{256} - \frac{1}{256} \sin z + \frac{1}{512} \int (1 + \cos 4z) \, dz + \frac{1}{320} \sin^5 z + C = \\
&= \frac{x}{128} - \frac{1}{256} \sin 2x + \frac{x}{256} + \frac{1}{2048} \sin 4z + \frac{1}{320} \sin^5 2x + C = \\
&= \frac{3}{256} x - \frac{1}{256} \sin 2x + \frac{1}{2048} \sin 8x + \frac{1}{320} \sin^5 2x + C.
\end{aligned}
$$

Even or odd powers in $\int \tan^m x \sec^n x \, dx$

- If $m = 2k+1$ is **odd**, you can rewrite and use the substitution $u = \cos x$

$$
\int \tan^{2k+1} x \sec^n x \, dx = \int \frac{\sin^{2k+1} x}{\cos^{2k+1+n} x} dx = -\int \frac{\left(1-u^2\right)^k}{u^{2k+1+n}} du.
$$

Example:

$$
\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx \ \ [u = \cos x] \ \ = -\int \frac{du}{u} = -\ln|\cos x| + C
$$

Example:

$$
\begin{aligned}
\int \tan^3 x \, dx &= \int \frac{\sin^3 x}{\cos^3 x} dx = \ \ [u = \cos x] \ \ = -\int \frac{1-u^2}{u^3} du = \\
= -\int u^{-3} du + \int u^{-1} du &= \frac{1}{2} u^{-2} + \ln|u| + C = \frac{1}{2} \sec^2 x + \ln|\cos x| + C
\end{aligned}
$$

- Other possibility — to use substitutions

$$
\text{a)} \qquad u = \tan x, \quad du = \sec^2 x \, dx, \quad \sec^2 x = 1 + u^2
$$

so the power at $\sec x$ must be **even**, power at $\tan x$ may be any;

$$
\text{b)} \qquad u = \sec x, \quad du = \tan x \sec x \, dx, \quad \tan^2 x = u^2 - 1
$$

So the power at $\tan x$ must be **odd**, power at $\sec x$ may be any;

1) Even power at $\sec x$, $n = 2k+2$, $u = \tan x$,

$$\int \tan^m x \sec^{2k+2} x\, dx = \int u^m \left(1+u^2\right)^k du.$$

Example:

$$\int \tan^4 x\ \sec^4 x\, dx = \int u^4 \left(1+u^2\right) du = \int \left(u^4+u^6\right) du =$$

$$= \frac{1}{5}u^5 + \frac{1}{7}u^7 + C = \frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C$$

Example: no $\tan x$, but still $u = \tan x$

$$\int \sec^4 x\, dx = \int \sec^2 x\ \sec^2 x\, dx = \int \left(1+u^2\right) du =$$

$$= u + \frac{1}{3}u^3 + C = \tan x + \frac{1}{3}\tan^3 x + C.$$

2) Odd power at $\tan x$, $m = 2k+1$, $u = \sec x$,

$$\int \tan^{2k+1} x \sec^n x\, dx = \int \left(\tan^2 x\right)^k \sec^{n-1} x\ \tan x \sec x dx = \int \left(u^2-1\right)^k u^{n-1} du.$$

Example:

$$\int \tan x dx = \int \tan^1 x \sec^0 x dx = \int (\sec x)^{-1} \tan x \sec x dx = \int u^{-1} du =$$

$$= \ln|u| + C = \ln|\sec x| + C = -\ln|\cos x| + C$$

Other types: $\int \tan^{2k+2} x\, dx$

Reduction formula

$$\int \tan^{2k+2} x\, dx = \int \tan^{2k} x \sin^2 x \sec^2 x\, dx =$$

$$= \int \tan^{2k} x \left(1-\cos^2 x\right) \sec^2 x\, dx = \int \tan^{2k} x \sec^2 x\, dx\ \ [u = \tan x] +$$

$$+ \int \tan^{2k} x\, dx = \int u^{2k} du + \int \tan^{2k-2} x \left(1-\cos^2 x\right) \sec^2 x\, dx = ...$$

Other types: $\int \sec^k x\,dx$, $k = 1$, 3, 5, ...

Denote for brevity $J_k = \int \sec^k x\,dx$

- $k = 1$

$$J_1 = \int \sec x dx = \int \frac{dx}{\cos x} = \int \frac{\cos x dx}{\cos^2 x} = [u = \sin x]$$

$$= \int \frac{du}{1-u^2} = \frac{1}{2}\int \left(\frac{1}{1+u} + \frac{1}{1-u}\right) du =$$

$$= \frac{1}{2}\left(\ln|1+u| - \ln|1-u|\right) + C = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C =$$

$$= \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}\right| + C = \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{(1-\sin^2 x)}\right| + C = \ln\left|\frac{(1+\sin x)}{\cos x}\right| + C =$$

$$= \ln|\sec x + \tan x| + C$$

- $k \geq 3$: reduction formula

$$J_k = \int \sec^k x\,dx = \int \sec^{k-2} x \sec^2 x dx = \int \sec^{k-2} x\,(\tan x)'\,dx =$$

$$= \sec^{k-2} x \tan x - \int \tan x\ (k-2)\sec^{k-3} x \tan x \sec x dx =$$

$$= \sec^{k-2} x \tan x - (k-2)\int \tan^2 x \sec^{k-2} x dx =$$

$$= \sec^{k-2} x \tan x - (k-2)\int \left(\sec^2 x - 1\right)\sec^{k-2} x dx =$$

$$= \sec^{k-2} x \tan x - (k-2)\left(J_k - J_{k-2}\right)$$

- Reduction formula

$$J_k = \frac{1}{k-1}\sec^{k-2} x \tan x + \frac{k-2}{k-1}J_{k-2}$$

- Application:

$$J_3 = \frac{1}{2}\sec x \tan x + \frac{1}{2}J_1 = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

- Integrals of $I_k = \int \csc^k x\,dx$ can be evaluated in a similar way..