Trigonometric substitutions

$$f(t) = \sqrt{At^2 + Bt + C}$$

Purpose: eliminate $\sqrt{}$ or $(...)^{m/2}$ from the expression.

Technique: reduce to a standard form and then to trigonometric integral

general quadratic polynomial $f(t) \longrightarrow$ one of the three standard forms

$$f(x) = a^2x^2 + b^2, \quad f(x) = a^2x^2 - b^2, \quad f(x) = b^2 - a^2x^2,$$

$\longrightarrow$ trigonometric integral $f(\theta)$

$$At^2 + Bt + C = A\left(t^2 + 2\frac{B}{2A}t + \frac{B^2}{4A^2}\right) + C - \frac{B^2}{4A} = Ax^2 + D$$

$$x = t + \frac{B}{2A}, \qquad D = C - \frac{B^2}{4A} \neq 0$$

Trigonometric substitutions

are based upon the identities

$$\sqrt{\tan^2\theta + 1} = \sec\theta, \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\sqrt{\sec^2\theta - 1} = \tan\theta, , \quad \theta \in [0, \frac{\pi}{2})$$

$$\sqrt{1 - \sin^2\theta} = \cos\theta, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

A	D	$f(x)$	substitution	$\theta \in$	$f(\theta)$	$\theta(x)$
a^2	b^2	$\sqrt{a^2x^2+b^2}$	$ax = b\tan\theta$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$b\sec\theta$	$\theta = \tan^{-1}\left(\frac{ax}{b}\right)$
a^2	$-b^2$	$\sqrt{a^2x^2-b^2}$	$ax = b\sec\theta$	$[0, \frac{\pi}{2})$	$b\tan\theta$	$\theta = \sec^{-1}\left(\frac{ax}{b}\right)$ $\theta = \cos^{-1}\left(\frac{b}{ax}\right)$ $\theta = \frac{\pi}{2} - \sin^{-1}\left(\frac{b}{ax}\right)$
$-a^2$	b^2	$\sqrt{b^2-a^2x^2}$	$ax = b\sin\theta$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$b\cos\theta$	$\theta = \sin^{-1}\left(\frac{ax}{b}\right)$

Hyperbolic substitutions

based upon

$$\sqrt{\sinh^2\theta + 1} = \cosh\theta, \quad \theta \in (-\infty, \infty)$$

$$\sqrt{\cosh^2\theta - 1} = \sinh\theta, \quad \theta \in [0, \infty)$$

$$\sqrt{1 - \frac{1}{\cosh^2\theta}} = \tanh\theta, \quad [0, \infty)$$

A	D	$f(x)$	substitution	$\theta \in$	$f(\theta)$	$\theta(x)$
a^2	b^2	$\sqrt{a^2x^2+b^2}$	$ax = b\sinh\theta$	$(-\infty, \infty)$	$b\cosh\theta$	$\theta = \sinh^{-1}\left(\frac{ax}{b}\right)$
a^2	$-b^2$	$\sqrt{a^2x^2-b^2}$	$ax = b\cosh\theta$	$[0, \infty)$	$b\sinh\theta$	$\theta = \cosh^{-1}\left(\frac{ax}{b}\right)$
$-a^2$	b^2	$\sqrt{b^2-a^2x^2}$	$ax = \dfrac{b}{\cosh\theta}$	$[0, \infty)$	$b\tanh\theta$	$\theta = \cosh^{-1}\left(\frac{b}{ax}\right)$