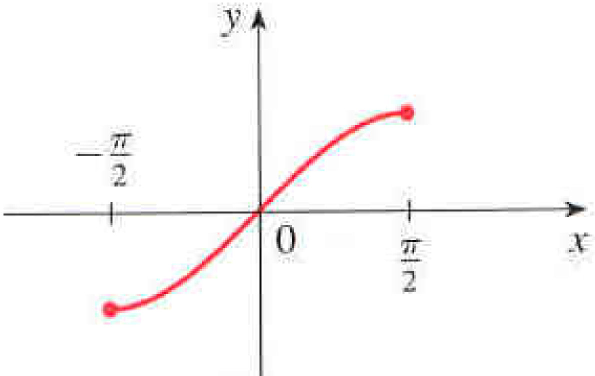
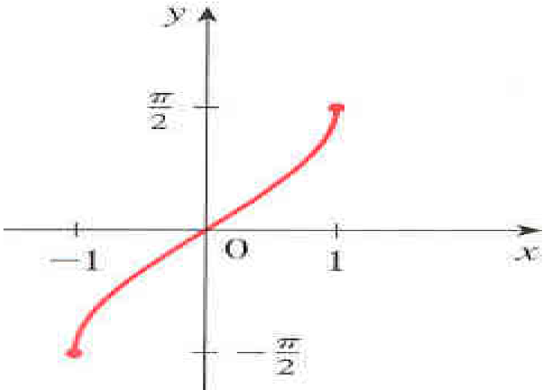
$\sin^{-1} x$ **or** $\arcsin x$





$$\sin x\text{: } D(\sin) = \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right],\ R(\sin) = [-1, 1]$$

$$\sin^{-1} x\text{: } D\left(\sin^{-1}\right) = [-1, 1],\ R\left(\sin^{-1}\right) = \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

Cancellation equalities

$$\sin\left(\sin^{-1} x\right) = x, \quad -1 \le x \le 1,$$

$$\sin^{-1}(\sin x) = x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

How to express trigonometric functions through each other

$$\cos x \ge 0, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \qquad \cos x = +\sqrt{1 - \sin^2 x}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\sin x}{\sqrt{1 - \sin^2 x}}$$

$$\tan^2 x + 1 = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}, \qquad \cos x = \frac{1}{\sqrt{1 + \tan^2 x}}$$

$$\sin x = \tan x \cos x = \frac{\tan x}{\sqrt{1 + \tan^2 x}}$$

combinations of direct and inverse functions

$$\cos\left(\sin^{-1} x\right) = \sqrt{1 - \sin^2\left(\sin^{-1} x\right)} = \sqrt{1 - x^2}$$

$$\tan\left(\sin^{-1} x\right) = \frac{\sin\left(\sin^{-1} x\right)}{\cos\left(\sin^{-1} x\right)} = \frac{x}{\sqrt{1 - x^2}}$$

Where do we need this? Substitution $x = \sin t$, $dx = \cos t\, dt$,

$$\int \sqrt{1-x^2}dx = \int \cos^2 t\, dt = \frac{1}{2}\int (1+\cos 2t)\, dt = \frac{t}{2} + \frac{\sin 2t}{4} + C =$$

$$= \frac{1}{2}t + \frac{1}{2}\sin t \cos t + C = \frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2} + C.$$

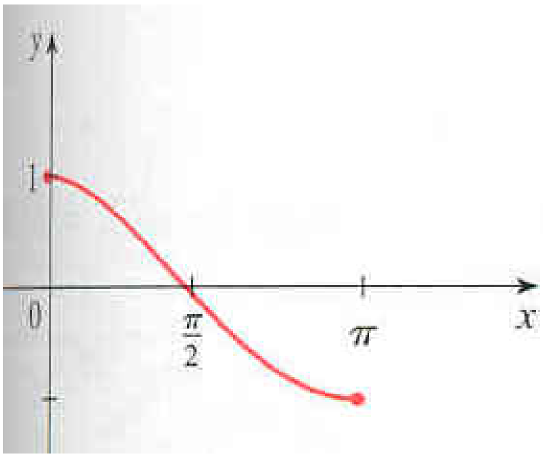
Derivative

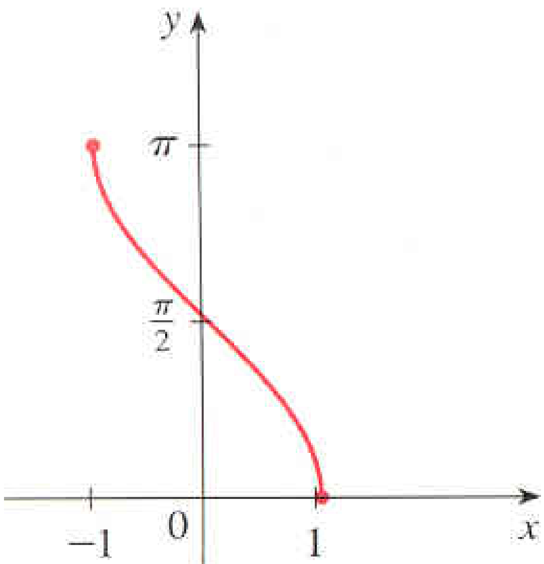
$$y = \sin^{-1} x, \qquad x = \sin y$$

$$(x)' = 1 = (\sin y)' = \cos y\, y' = \sqrt{1-\sin^2 y}\, y' = \sqrt{1-x^2}\, y'$$

$$\left(\sin^{-1} x\right)' = \frac{1}{\sqrt{1-x^2}}, \qquad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$\cos^{-1} x$ or $\arccos x$





$\cos x$: $D(\cos) = [0,\pi]$, $R(\cos) = [-1,1]$

$\cos^{-1} x$: $D(\cos^{-1}) = [-1,1]$, $R(\cos^{-1}) = [0,\pi]$

Cancellation equalities

$$\cos\left(\cos^{-1} x\right) = x, \quad -1 \le x \le 1,$$

$$\cos^{-1}(\cos x) = x, \quad 0 \le x \le \pi$$

How to express trigonometric functions through each other

$$\sin x \geq 0, \quad x \in [0, \pi], \qquad \sin x = +\sqrt{1 - \cos^2 x}$$

$$\cot x = \frac{\cos x}{\sin x} = \frac{\cos x}{\sqrt{1 - \cos^2 x}}$$

$$\cot^2 x + 1 = \frac{\cos^2 x + \sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}, \qquad \sin x = \frac{1}{\sqrt{1 + \cot^2 x}}$$

$$\cos x = \cot x \sin x = \frac{\cot x}{\sqrt{1 + \cot^2 x}}$$

combinations of direct and inverse functions

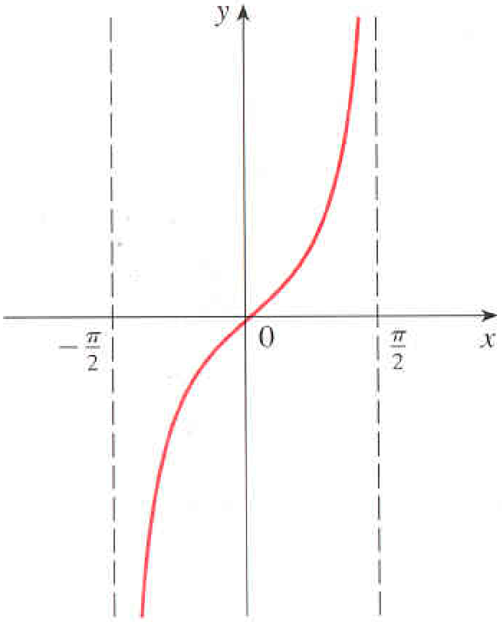
$$\sin\left(\cos^{-1} x\right) = \sqrt{1 - x^2}, \qquad \cot\left(\cos^{-1} x\right) = \frac{x}{\sqrt{1 - x^2}}$$

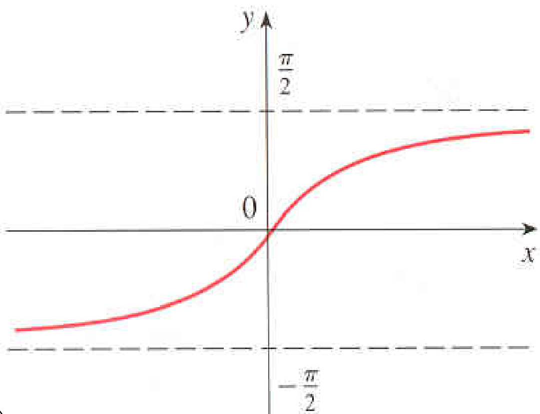
$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \qquad \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

Derivative

$$\left(\cos^{-1} x\right)' = -\left(\sin^{-1} x\right)' = -\frac{1}{\sqrt{1 - x^2}},$$

$\tan^{-1} x$ or $\arctan x$





$\tan x$: $D(\tan) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $R(\tan) = (-\infty, \infty)$

$\tan^{-1} x$: $D(\tan^{-1}) = (-\infty, \infty)$, $R(\tan^{-1}) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\lim_{x\to\frac{\pi}{2}^-} \tan x = \infty, \qquad \lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$$

$$\lim_{x\to-\frac{\pi}{2}^+} \tan x = -\infty, \qquad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Cancellation equalities

$$\tan\left(\tan^{-1} x\right) = x, \quad -1 < x < 1,$$

$$\tan^{-1}(\tan x) = x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

How to express trigonometric functions through each other

$$\cos x > 0, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad \cos x = +\sqrt{1 - \sin^2 x}$$

$$\sin x = \frac{\tan x}{\sqrt{1 + \tan^2 x}}, \qquad \cos x = \frac{1}{\sqrt{1 + \tan^2 x}}$$

combinations of direct and inverse functions

$$\sin\left(\tan^{-1} x\right) = \frac{x}{\sqrt{1+x^2}}, \qquad \cos\left(\tan^{-1} x\right) = \frac{1}{\sqrt{1+x^2}}.$$

Derivative

$$y = \tan^{-1} x, \qquad x = \tan y$$

$$(x)' = 1 = (\tan y)' = \frac{1}{\cos^2 y}\, y' = \frac{1}{\cos^2(\tan^{-1} x)}\, y' = \left(1 + x^2\right) y'$$

$$\left(\tan^{-1} x\right)' = \frac{1}{1+x^2}, \qquad \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

Example. Using implicit differentiation, find derivative of $y = \sec^{-1} x$, $x \in [1, \infty)$, $y \in [0, \frac{\pi}{2})$

$$x = \sec y = 1/\cos y, \;\; \cos y = 1/x,$$

$$\frac{d}{dx}x = 1 = \frac{d}{dx}\left(\frac{1}{\cos y}\right) = \frac{\sin y}{\cos^2 y}y' = \frac{\sqrt{1-\cos^2 y}}{\cos^2 y}y',$$

$$y' = \frac{\cos^2 y}{\sqrt{1-\cos^2 y}} = \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}} = \frac{1}{x\sqrt{x^2-1}}.$$

Example. Find

$$\lim_{x\to\infty} \tan^{-1}\left(e^x\right)$$

Denote $u = e^x$, $u \to \infty$ as $x \to \infty$, so

$$\lim_{x\to\infty} \tan^{-1}\left(e^x\right) = \lim_{u\to\infty} \tan^{-1} u = \frac{\pi}{2}.$$

Example. Evaluate integral

$$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx$$

Substitution $u = \sin^{-1} x$, $du = dx/\sqrt{1-x^2}$,

$$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx = \int u du = \frac{1}{2}u^2 + C = \frac{\left(\sin^{-1} x\right)^2}{2} + C.$$

Example. Evaluate integral

$$\int \frac{dx}{x^2+a^2}$$

Substitution $u = x/a$, $x = au$, $dx = adu$,

$$\int \frac{dx}{x^2+a^2} = \int \frac{adu}{a^2+a^2u^2} = \frac{1}{a}\int \frac{du}{1+u^2} =$$

$$= \frac{1}{a}\tan^{-1} u + C = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$