Integration of rational functions (Section 8.4)

Rational function: ratio (quotient) of two polynomials

\[ f(x) = \frac{P_m(x)}{Q_n(x)}, \]

\[ P_m(x) = b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0 \]

\[ Q_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \]

If \( m < n \) — **proper** rational function.

We need to find

\[ \int \frac{P_m(x)}{Q_n(x)} \, dx \]

Idea — represent a **proper** rational function as a sum of simple terms, ”partial fractions”, for example

\[ \frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{1}{2} \left( \frac{1}{1-x} + \frac{1}{1+x} \right). \]

For each term the integral can be evaluated by known methods.

**Step I. If \( m \geq n \), transform \( f \) into polynomial + proper rational function.**

Rule #1: Never integrate improper rational function. Transform.

Either divide \( P_m \) by \( Q_n \) and find the remainder \( P_{m-1} \): \( P_m(x) = g(x)Q_n(x) + P_{m-1}(x) \) or sequentially eliminate from \( P \) all terms with powers from \( m \) to \( n \).

**Example:** improper rational function

\[ f(x) = \frac{x^4 - x^2}{x^2 + x + 1} \]

\[ P_4(x) = x^4 - x^2, \quad Q_2(x) = x^2 + x + 1 \]

\[
\begin{array}{cccc}
  x^4 & -x^2 \\
  x^2 & x^4 & +x^3 & +x^2 \\
  & -x^3 & -2x^2 & (= P_4(x) - x^2Q_2(x)) \\
  & -x & -x^3 & -x^2 & -x \\
  & & -x^2 & +x & (= P_4(x) - (x^2Q_2(x) - xQ_2(x))) \\
  & & & -1 & -x^2 & -x & -1 \\
  & & & 2x & +1 & (= P_4(x) - (x^2Q_2(x) - xQ_2(x) - 1Q_2(x))) \\
\end{array}
\]
Therefore, we can conclude that

\[ P_4(x) = (x^2 - x - 1) Q_2 + 2x + 1. \]

Check it:

\[
(x^2 - x - 1) (x^2 + x + 1) + 2x - 1 = x^4 - (x + 1)^2 + 2x + 1 = x^4 - x^2 - 2x - 1 + 2x + 1 = x^4 - x^2.
\]

Now

\[ f(x) = (x^2 - x - 1) + \frac{2x + 1}{x^2 + x + 1}. \]

**Step 2.** Represent \( Q_n(x) \) as a product of standard factors \((ax - b)^r\) and \((ax^2 + bx + c)^r\)

**Example:** quadratic equation \( Q_2 = ax^2 + bx + c \).

a) \( b^2 - 4ac > 0 \), \( Q_2 = a(x - \alpha_1)(x - \alpha_2) = (ax - a\alpha_1)(x - \alpha_2) \). \( \alpha_1 \) and \( \alpha_2 \) are the roots of the quadratic equation \( ax^2 + bx + c = 0 \),

\[
\alpha_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \alpha_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.
\]

b) \( b^2 - 4ac = 0 \), \( Q_2 = a(x - \alpha_1)^2 \)

c) \( b^2 - 4ac < 0 \), \( Q_2 = a \left( x^2 + \frac{b}{a}x + \frac{c}{a} \right) \)

**General case (supplementary)**

(1) if \( x \) can take complex values \( z = \alpha + i\beta \), \( i^2 = -1 \), then there is a theorem: each polynomial \( Q_n(x) \) of degree \( n \) always has \( n \) roots (some of the roots may coincide). The polynomial can be rewritten in the form

\[ Q_n(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = a_n (x - z_1)(x - z_2)...(x - z_n) \]

So, every simple root contributes a factor \( (x - z_k) \), each repeated \( r \) times root \( -(x - z_k)^r \)

(2) All coefficients \( a_k \) are real numbers, then complex roots always appear in pairs: if \( \alpha + i\beta \) is a root, then \( \alpha - i\beta \) is a root too. Two such roots contribute a factor

\[
(x - \alpha - i\beta)(x - \alpha + i\beta) = (x - \alpha)^2 - (i\beta)^2 = x^2 - 2\alpha x + \alpha^2 + \beta^2
\]

This is a quadratic polynomial with the discriminant \( 4\alpha^2 - 4(\alpha^2 + \beta^2) = -4\beta^2 < 0 \).

(3) if the root \( \alpha + i\beta \) is repeated \( r \) times, then \( \alpha - i\beta \) is repeated \( r \) times too. Then \( Q_n(x) \) has a factor

\[
(x^2 - 2\alpha x + \alpha^2 + \beta^2)^r
\]
Conclusion:

\[ Q_n(x) = (a_1x - b_1)^{r_1} (a_2x - b_2)^{r_2} \ldots (a_kx - b_k)^{r_k} \left( a_{k+1}x^2 + b_{k+1}x + c_{k+1} \right)^{r_{k+1}} \ldots \left( a_{t}x^2 + b_{t}x + c_{t} \right)^{r_t}, \]

where \( b_1/a_1, b_2/a_2, \ldots, b_k/a_k \) — real roots of the equation \( Q(x) = 0 \),

\[ r_1 + r_2 + \ldots + r_k + 2r_{k+1} + 2r_{k+2} + \ldots + 2r_t = n. \]

Step 3. express proper fraction \( P/Q \) as a sum of partial fractions

\[
\frac{P_m(x)}{(a_1x - b_1)^{r_1} (a_2x - b_2)^{r_2} \ldots (a_kx - b_k)^{r_k} \left( a_{k+1}x^2 + b_{k+1}x + c_{k+1} \right)^{r_{k+1}} \ldots \left( a_{t}x^2 + b_{t}x + c_{t} \right)^{r_t}} = \sum \text{(Partial Fractions)}
\]

Each factor in \( Q_n(x) \) generates \( r_i \) terms in the sum of partial fractions:

<table>
<thead>
<tr>
<th>Factor</th>
<th>Terms</th>
<th>comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>((ax - b))</td>
<td>( \frac{A}{ax - b} )</td>
<td>1 term</td>
</tr>
<tr>
<td>((ax - b)^r)</td>
<td>( \frac{A_1}{ax - b} + \frac{A_2}{(ax - b)^2} + \ldots + \frac{A_r}{(ax - b)^r} )</td>
<td>( r ) terms</td>
</tr>
<tr>
<td>((ax^2 + bx + c))</td>
<td>( \frac{Ax + B}{ax^2 + bx + c} )</td>
<td>1 term</td>
</tr>
<tr>
<td>((ax^2 + bx + c)^r)</td>
<td>( \frac{A_1x + B_1}{ax^2 + bx + c} + \ldots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r} )</td>
<td>( r ) terms</td>
</tr>
</tbody>
</table>

Unknown coefficients \( A, B, C \) to be determined at the next step

Example:

\[ f(x) = \frac{x^7 - 3x^4}{2(x + 1)(x - 1)^3(x^2 + x + 1)^2} = \frac{1}{2} \left( \frac{A}{x + 1} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2} + \frac{D}{(x - 1)^3} + \frac{Ex + F}{x^2 + x + 1} + \frac{Gx + H}{(x^2 + x + 1)^2} \right) \]

where \( A \ldots H \) are unknown coefficients. Note: the number of unknown coefficients equals to the power of the polynomial in denominator — here 8.

Step 4. Find the coefficients for each partial fraction

- Bring the sum of partial fractions to common denominator
- Find the polynomial in the numerator,
- Coefficients at each power of \( x \) of this polynomial should be equal to that of \( P_m(x) \). This gives the system of equations for the coefficients at partial fractions
• Solve the system and find the coefficients.

**Example.**

\[
\frac{2x^2 + 1}{(x - 1)(x^2 + x + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1} = \\
= \frac{A(x^2 + x + 1) + (Bx + C)(x - 1)}{(x - 1)(x^2 + x + 1)} = \\
= \frac{Ax^2 + Ax + A + Bx^2 + Cx - Bx - C}{(x - 1)(x^2 + x + 1)} = \\
= \frac{(A + B)x^2 + (A - B + C)x + A - C}{(x - 1)(x^2 + x + 1)}.
\]

We need that for any \(x\)

\[
(A + B)x^2 + (A - B + C)x + A - C = 2x^2 + 1
\]

therefore coefficients at the same powers of \(x\) must coincide:

(a) \(A + B = 2\),  
(b) \(A - B + C = 0\),  
(c) \(A - C = 1\).

This is a system of equations for \(A, B, C\).

**Standard way of solving:**

From equation (a) \(A = 2 - B\), substitute this into two other equations:

(b) \(2 - B - B + C = 2 - 2B + C = 0\),  
(c) \(2 - B - C = 1\).

or

(b) \(2B - C = 2\),  
(c) \(B + C = 1\).

From (c) \(B = 1 - C\), substitute into (b):

(b) \(2 - 2C - C = 2\),  
\(3C = 0\),  
\(C = 0\).

Knowing \(C\) we can find \(B = 1 - C = 1\), and \(A = 2 - B = 1\).

There may be faster ways. For example, add all 3 equations, then

\[A + B + A - B + C + A - C = 3, \quad 3A = 3, \quad A = 1,\]

then from (a) \(B = 2 - A = 1\), (c) \(C = A - 1 = 0\).

Finally,

\[
\frac{2x^2 + 1}{(x - 1)(x^2 + x + 1)} = \frac{1}{x - 1} + \frac{x}{x^2 + x + 1}
\]
**The cross-out method**

Some coefficients, corresponding to the highest degree of \((ax - b)^r\) can be determined without solving a system of equations.

**Example.** Partial fractions give the relation with 3 unknown constants

\[
\frac{2x^2 + 1}{(x - 1)(x^2 + x + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}.
\]

Multiply this equality by \((x - 1)\)

\[
\frac{2x^2 + 1}{x^2 + x + 1} = A + (x - 1)\frac{Bx + C}{x^2 + x + 1}.
\]

There is no singularity at \(x = 1\) any longer. Let us set \(x = 1\),

\[
\frac{2 + 1}{1 + 1 + 1} = 1 = A.
\]

At \(x = 1\) terms containing \(B\) and \(C\) vanished, and we obtain the value of \(A\).

Unfortunately, \(B\) and \(C\) cannot be determined this way — you need complex numbers to do this.

**Example.** Let us consider a general case, \(P_m(x)/Q_n(x)\). Suppose that \(Q_n(x) = (ax - b)^r S_{n-r}(x)\), where \(S_{n-r}(x)\) does not contain factors \((ax - b)\). Then

\[
\frac{P_m(x)}{Q_n(x)} = \frac{P_m(x)}{(ax - b)^r S_{n-r}(x)} = \frac{A_r}{(ax - b)^r} + \frac{A_{r-1}}{(ax - b)^{r-1}} + \ldots + \frac{A_1}{(ax - b)} + \text{(other terms)}.
\]

Multiply by \((ax - b)^r\),

\[
\frac{P_m(x)}{S_{n-r}(x)} = A_r + A_{r-1} (ax - b) + \ldots + A_1 (ax - b)^{r-1} + (ax - b)^r \text{ (other terms)}.
\]

Now set \(x = b/a\), all \((ax - b) = 0\), what remains is

\[
A_r = \frac{P_m(b/a)}{S_{n-r}(b/a)}.
\]

Other coefficients \(A_{r-1}\) to \(A_1\) cannot be determined this way. Coefficients corresponding to quadratic terms \((ax^2 + bx + c)^r\) cannot be determined either.

This method can be **briefly formulated** as follows: to determine \(A_r\)

1) "cross out" the term \((ax - b)^r\) from the denominator in \(P(x)/Q(x)\)
2) set \(x = b/a\).
Step 5. Integrating

For each of the partial fractions evaluate the integral and obtain the final answer. For each partial fraction the integral can be evaluated by known methods:

\[
\text{(a) } \int \frac{dx}{ax - b} = \frac{1}{a} \int \frac{d(ax - b)}{ax - b} = \frac{1}{a} \ln |ax - b| + C,
\]

\[
\text{(b) } \int \frac{dx}{(ax - b)^r} = \frac{1}{a} \int (ax - b)^{-r} d(ax - b) = -\frac{1}{a(r - 1)} (ax - b)^{-(r - 1)} + C, \quad r > 1,
\]

\[
\text{(c) } \int \frac{Ax + B}{ax^2 + bx + c} dx = \frac{A}{a} \int \frac{u}{u^2 + \beta^2} du + \frac{B}{a} \int \frac{1}{u^2 + \beta^2} du.
\]

here denominator has to be transformed to a standard form without linear term,

\[
u = x + \frac{b}{2a}, \quad ax^2 + bx + c = au^2 + a\beta^2, \quad \beta^2 = \frac{4ac - b^2}{4a^2} > 0,
\]

\[
= \frac{1}{a} \int \frac{Au + (B - \frac{Ab}{2a})}{u^2 + \beta^2} du = \frac{A}{a} \int \frac{u}{u^2 + \beta^2} du + \frac{B}{a} \int \frac{1}{u^2 + \beta^2} du.
\]

Evaluate the integrals separately

\[
\int \frac{u}{u^2 + \beta^2} du = \left[ v = u^2 + \beta^2 \right] = \frac{1}{2} \int \frac{1}{v} dv = \frac{1}{2} \ln (u^2 + \beta^2) + C = \frac{1}{2} \ln \left( \frac{ax^2 + bx + c}{a} \right) + C,
\]

\[
\int \frac{1}{u^2 + \beta^2} du = \left[ u = \beta v \right] = \frac{\beta}{\beta^2} \int \frac{1}{1 + v^2} dv = \frac{1}{\beta} \tan^{-1} v + C = \frac{1}{\beta} \tan^{-1} \left( \frac{2ax + b}{\sqrt{4ac - b^2}} \right) + C
\]

The last one we consider only schematically:

\[
\text{(d) } \int \frac{Ax + B}{(ax^2 + bx + c)^r} dx = \frac{1}{a^r} \int \frac{A_1u + B_1}{(u^2 + \beta^2)^r} du = \frac{A_1}{a^r} \int \frac{u}{(u^2 + \beta^2)^r} du + \frac{B_1}{a^r} \int \frac{1}{(u^2 + \beta^2)^r} du
\]

In the first integral we use substitution \(v = u^2 + \beta^2\), in the second \(u = \beta \tan \theta\), so

\[
\int \frac{1}{(u^2 + \beta^2)^r} du = \beta^{1-2r} \int \cos^{2r} \theta \sec^{2} \theta d\theta = \beta^{1-2r} \int \cos^{2r-2} \theta d\theta.
\]

This integral can be evaluated by successive application of double angle identity.
Examples

1. \[
\int \frac{1}{1-x^2} \, dx
\]
\[
\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}
\]
Here both coefficients can be obtained by cross-out method:
\[
A = \frac{1}{1+x} \bigg|_{x=1} = \frac{1}{2}, \quad B = \frac{1}{1-x} \bigg|_{x=-1} = \frac{1}{2}
\]
\[
\int \frac{1}{1-x^2} \, dx = \frac{1}{2} \int \frac{1}{1-x} \, dx + \int \frac{1}{1+x} \, dx = \frac{1}{2} \ln |1-x| + \frac{1}{2} \ln |1+x| + C = \frac{1}{2} \ln \frac{|1+x|}{|1-x|} + C.
\]

2. \[
\int \frac{d\theta}{\cos^5 \theta} = [x = \sin \theta] = \int \frac{dx}{(1-x^2)^3} = \int \frac{dx}{(1-x)^3 (1+x)^3} = \frac{A}{(1-x)^3} + \frac{B}{(1-x)^2} + \frac{C}{1-x} + \frac{D}{(1+x)^3} + \frac{E}{(1+x)^2} + \frac{F}{1+x}
\]
\[A \text{ and } D \text{ can be determined by cross-out:}
\]
\[
A = \frac{1}{(1+x)^3} \bigg|_{x=1} = \frac{1}{8}, \quad B = \frac{1}{(1-x)^3} \bigg|_{x=-1} = \frac{1}{8}.
\]
For other coefficients it is necessary to solve a system of equations.
\[
\frac{1}{8} (1+x)^3 + B (1-x) (1+x)^3 + C (1-x)^2 (1+x)^3 + \\
+ \frac{1}{8} (1-x)^3 + E (1+x) (1-x)^3 + F (1+x)^2 (1-x)^3 = 1
\]
\[
\frac{1}{4} (1+3x^2) + B (1-x^2) (1+x^2 + 2x) + C (1-x^2)^2 (1+x) + \\
+ E (1-x^2) (1+x^2 - 2x) + F (1-x^2)^2 (1-x) = 1
\]
\[
B (1-x^4+2x-2x^3) + C (1-2x^2+x^4+x-2x^3+x^5) + \\
+ E (1-x^4-2x+2x^3) + F (1-2x^2+x^4-x+2x^3-x^5) = \frac{3}{4} - \frac{3}{4} x^2
\]
\[
\begin{align*}
x^0 & : \quad B + C + E + F = \frac{3}{4} \\
x^1 & : \quad 2B + C - 2E - F = 0 \\
x^2 & : \quad -2C - 2F = -\frac{3}{4} \\
x^3 & : \quad -2B - 2C + 2E + 2F = 0
\end{align*}
\]
\[ x^4: \quad -B + C - E + F = 0 \]
\[ x^5: \quad C - F = 0 \]

So \( C = F \),

\[ 2C + 2F = 4C = \frac{3}{4}, \quad C = F = \frac{3}{16}, \]

\[ B + C = E + F, \quad B = E \]

\[ B + E = 2B = \frac{3}{4} - C - F = \frac{3}{4} - \frac{3}{8} = \frac{3}{8}, \quad B = E = \frac{3}{16}. \]

\[
\int \frac{dx}{(1 - x^2)^3} = \frac{1}{8} \int \frac{dx}{(1 - x)^3} + \frac{3}{16} \int \frac{dx}{(1 - x)^2} + \frac{3}{16} \int \frac{dx}{1 - x} + \\
+ \frac{1}{8} \int \frac{dx}{(1 + x)^3} + \frac{3}{16} \int \frac{dx}{(1 + x)^2} + \frac{3}{16} \int \frac{dx}{1 + x} = \\
= \frac{1}{16} \left( \frac{1}{1 - x} \right)^2 + \frac{3}{16} \ln |1 - x| - \frac{3}{16} \ln |1 + x| + C = \\
= \frac{1}{16} \left( \frac{1}{1 - x^2} \right)^2 - \frac{3}{16} \ln |1 + x| + C = \\
= \frac{x}{4(1 - x^2)^2} + \frac{3x}{8(1 - x^2)} + \frac{3}{16} \ln \left| \frac{1 + x}{1 - x} \right| + C = \ldots (x = \sin \theta) \]

3.

\[ J = \int \frac{3x^2 + 2}{(x - 1)(x^2 + 2x + 2)} dx \]

\[
\frac{3x^2 + 2}{(x - 1)(x^2 + 2x + 2)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 2x + 2},
\]

\[ A = \left. \frac{3x^2 + 2}{(x^2 + 2x + 2)} \right|_{x=1} = 1, \]

Bring to common denominator,

\[ 3x^2 + 2 = x^2 + 2x + 2 + (Bx + C)(x - 1), \]

\[ Bx^2 + Cx - Bx - C = 2x^2 - 2x, \]

\[ B = 2, \quad C - B = -2, \quad -C = 0. \]

\[
\int \frac{3x^2 + 2}{(x - 1)(x^2 + 2x + 2)} dx = \int \frac{1}{x - 1} dx + 2 \int \frac{x}{x^2 + 2x + 2} dx = \\
\int \frac{1}{x - 1} dx = \ln |x - 1| + C,
\]

\[ \int \frac{x}{x^2 + 2x + 2} dx = \int \frac{x + 1 - 1}{(x + 1)^2 + 1} dx = [u = x + 1] \]

\[ = \int \frac{u - 1}{u^2 + 1} du = \int \frac{du}{u^2 + 1} - \int \frac{du}{u^2 + 1} = \frac{1}{2} \ln (u^2 + 1) - \tan^{-1} u + C. \]

\[ J = \ln |x - 1| + \ln (x^2 + 2x + 2) - 2 \tan^{-1} (x + 1) + C \]