Calculus and numerical methods (Section 8.7+)

Approximate evaluation of functions and Taylor polynomial

Sometimes it is necessary to approximate a complicated function $f(x)$ by a simpler one and to estimate the error of such approximation. Suppose we know $f(a)$. Sufficiently close to a $f(x) \approx P_1(x) = f(a) + f'(a)\,(x-a)$, that is we replace function by a tangent line, $P_1'(a) = f'(a)$. If we need better approximation, we can use a polynomial instead of a straight line, but this polynomial should be also "tangent" in a certain sense. Taylor polynomial solves this problem.
Assume that f has at least $k+1$ continuous derivatives near $x=a$. Then

$$f(x) = P_k(x) + E_{k+1}.$$

where

$$P_k(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2}f''(a) + \frac{(x-a)^3}{6}f''(a) + \ldots + \frac{(x-a)^k}{k!}f^{(k)}(a),$$
$$k! = 1\cdot 2\cdot 3\cdot \ldots \cdot k,$$

is called **Taylor polynomial**. Note that there is also **Taylor series** which has infinitely many terms, but we shall not talk about it — sum of an infinite series may not exist or be infinite for some x even if f has a finite values $f(x)$ and $f(a)$. Polynomials do not have this problem, they require only existence of necessary derivatives. Or more precisely, the problem reappears as a big approximation error.
There is an important property, at $x=a$

$$P_k(a) = f(a), \quad P_k'(a) = f'(a), \quad P_k''(a) = f''(a), \quad \ldots, \quad P_k^{(k)}(a) = f^{(k)}(a).$$

E_k is the error term, and

$$E_{k+1} = f(x) - P_k(x) = \frac{(x-a)^{k+1}}{(k+1)!}f^{(k+1)}(\xi), \qquad \xi \in [a,x].$$

In other words, if we replace $f(x)$ by $P_k(x)$, absolute value of error does not exceed $|E_{k+1}|$. The value of ξ is unknown, but instead of $f^{(k+1)}(\xi)$ we can take maximum value of $f^{(k+1)}(z)$ on some interval $[a,b]$. Then for any $x\in[a,b]$ the approximation error cannot exceed

$$|E_{k+1}| \le \frac{|x-a|^{k+1}}{(k+1)!}M_{k+1}, \qquad M_{k+1} = \max_{z\in[a,b]}\left|f^{(k+1)}(z)\right|.$$

Taylor polynomial with this technique of error estimate is a **major tool** of numerical applications of calculus.
The most famous polynomials, here $a=0$:

$$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots \frac{1}{k!}x^k + \ldots$$

$f^{(k)}(0) = e^0 = 1$,

$$\sin x \approx x - \frac{1}{6}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$$

$\sin 0 = 0$, $\cos 0 = 1$.

$$\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots$$

$$\ln(1+x) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots$$

Example 1. Evaluate $\sqrt{3}$ using 3 first terms of the Taylor polynomial and estimate the error.
$3 = 4 - 1 = 4(1-0.25)$, $\sqrt{3} = 2\sqrt{1-0.25}$, now we can use the function $f(x) = \sqrt{1+x} = (1+x)^{1/2}$ with $x = -0.25$.

$$f(x) = (1+x)^{1/2}, \qquad f(0) = 1$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}, \qquad f'(0) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-3/2}, \qquad f''(0) = -\frac{1}{4}$$

Therefore

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + E, \quad x = -\frac{1}{4},$$

$$\sqrt{3} \approx 2\left(1 - \frac{1}{8} - \frac{1}{128}\right) = \frac{111}{64} = 1.734375.$$

The "true" value is 1.73205..., and actual error is ≈ 0.0024. Let us find the error estimate $2E$, we need maximum value of $|f'''|$ on $\left[-\frac{1}{4}, 0\right]$,

$$M = \max_{-\frac{1}{4}\le x\le 0} |f'''(x)| = \max_{-\frac{1}{4}\le x\le 0} \frac{3}{8}(1+x)^{-5/2} = \frac{3}{8}\left(1 - \frac{1}{4}\right)^{-5/2} =$$

$$= \frac{3}{8}\left(\frac{4}{3}\right)^{5/2} = \frac{3\cdot 32}{8\cdot 9\sqrt{3}} < \frac{4}{3\cdot 1.7} \approx 0.7843 < 0.8,$$

$$|\text{error}| = 2|E| \le 2\frac{1}{3!}\left(\frac{1}{4}\right)^3 M = 2\frac{1}{6}\frac{1}{64}0.8 = \frac{1}{240} \approx 0.0042.$$

So we estimate $\sqrt{3} = 1.7343 \pm 0.0042$.
If we need better accuracy, we may take more terms in the Taylor polynomial.
Example 2. What is the accuracy of the approximate formula

$$\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120}, \qquad -1 \le x \le 1.$$

$$f = \sin x, \qquad f(0) = 0$$

$$f' = \cos x, \qquad f'(0) = 1$$

$$f'' = -\sin x, \qquad f''(0) = 0$$

$$f''' = -\cos x, \qquad f'''(0) = -1$$

$$f^{(4)} = \sin x, \qquad f^{(4)}(0) = 0$$

$$f^{(5)} = \cos x, \qquad f^{(5)}(0) = 1$$

$$f^{(6)} = -\sin x, \qquad f^{(6)}(0) = 0$$

$$f^{(7)} = -\cos x, \qquad M = \max\left|f^{(7)}\right| = 1$$

So in spite only x^5, we in fact have the approximation $P_6(x)$, and its error is described by $f^{(7)}$:

$$|E| \le \frac{1}{7!}1^7 M = \frac{1}{5040} < \frac{1}{5000} = 0.0002.$$

Actual error usually is smaller. Note also that if we use approximation $\sin x \approx x$, then the maximum true error is $1 - \sin 1 \approx 0.16$. So two addition terms increase accuracy more than 800 times.

Example 3. The integral

$$F(x) = \int_0^x \frac{\sin t}{t} dt$$

cannot be evaluated in elementary functions. If we need it for small x, we can use approximate formula

$$\sin t \approx t - \frac{t^3}{6} + \frac{t^5}{120}, \quad \frac{\sin t}{t} \approx 1 - \frac{t^2}{6} + \frac{t^4}{120},$$

$$F(x) \approx x - \frac{x^3}{18} + \frac{x^5}{600}.$$

It is possible to find the accuracy of this approximation by constructing Taylor polynomial for F ($F' = \sin x/x$ and so on) and estimate the error term for it.

Numerical integration (Section 8.7).

The midpoint rule.

Idea: use the Riemann sum to approximate the integral $S = \int_a^b f(x)dx$. Split the interval $[a, b]$ into n equal intervals $[x_0 = a, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, ..., $[x_{n-1}, x_n = b]$, where

$$x_0 = a, \quad x_1 = a + h, \quad x_2 = a + 2h, \ldots, \quad x_n = b = a + nh, \qquad h = \frac{b-a}{n} = x_i - x_{i-1}.$$

Note that there are n intervals and $n+1$ points x_i. Then we take $z_i \in [x_{i-1}, x_i]$ and

$$S \approx \sum_{i=1}^{n} f(z_i)h.$$

The **midpoint rule** corresponds to the choice $z_i = (x_{i-1} + x_i)/2$, or

$$\int_a^b f(x)dx \approx h\left(f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + \ldots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right).$$

The interesting point is to estimate the error of this approximation. Denote

$$F(x) = \int f(x)dx, \quad F'(x) = f, \quad F'' = f', \qquad F''' = f'',$$

Then

$$\int_{x_{i-1}}^{x_i} f(x)dx = F(x_i) - F(x_{i-1}).$$

Using the Taylor formula,

$$F(x_i) = F(z_i + \frac{h}{2}) = F(z_i) + \frac{h}{2}F'(z_i) + \frac{1}{2}\left(\frac{h}{2}\right)^2 F''(z_i) + \frac{1}{6}\left(\frac{h}{2}\right)^3 F'''(\xi_{1i}),$$

$$F(x_{i-1}) = F(z_i - \frac{h}{2}) = F(z_i) - \frac{h}{2}F'(z_i) + \frac{1}{2}\left(\frac{h}{2}\right)^2 F''(z_i) - \frac{1}{6}\left(\frac{h}{2}\right)^3 F'''(\xi_{2i}),$$

subtracting the lower from the upper gives

$$\int_{x_{i-1}}^{x_i} f(x)dx = F(x_i) - F(x_{i-1}) = hF'(z_i) + \frac{h^3}{48}\left(F'''(\xi_{1i}) + F'''(\xi_{2i})\right) =$$

$$= hf(z_i) + \frac{h^3}{48}\left(f''(\xi_{1i}) + f''(\xi_{2i})\right).$$

Note that terms containing $h^2 f'$ **vanished**. This is because of the symmetry — we choose the middle of the interval. If we choose a single point within an interval any other way, these terms will remain, and accuracy of the method will be worse. So **the midpoint rule is the best choice** of z_i.
Now we can sum up both sides from 1 to n and

$$\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)dx = \int_a^b f(x)dx = \sum_{i=1}^{n} hf(z_i) + \sum_{i=1}^{n} \frac{h^3}{48}\left(f''(\xi_{1i}) + f''(\xi_{2i})\right),$$

therefore

$$\text{error} = E = \left|\int_a^b f(x)dx - \sum_{i=1}^{n} hf(z_i)\right| = \left|\sum_{i=1}^{n} \frac{h^3}{48}\left(f''(\xi_{1i}) + f''(\xi_{2i})\right)\right| \leq$$

$$\leq \frac{h^3}{48}\sum_{i=1}^{n}\left(|f''(\xi_{1i})| + |f''(\xi_{2i})|\right) \leq \frac{h^3}{48}2nK = \frac{h^2}{24}(b-a)K,$$

$$K = \max_{[a,b]}|f''(x)|, \qquad (b-a) = nh,$$

Or, substituting $h = (b-a)/n$

$$|E| \leq \delta = \frac{(b-a)^3 K}{24n^2}.$$

So, knowing an estimate for the second derivative we can solve two kinds of problems:

- For the given number of intervals n estimate the error E;
- For the given accuracy E find the required n.

Example 4. For the integral

$$\int_0^1 \sin\left(x^2\right)dx$$

1) write down the midpoint rule for $n = 5$ and estimate the accuracy,
2) find n for which the midpoint rule gives accuracy 10^{-4}.
Solution. 1) $h = 1/5 = 0.2$, so

$$\int_0^1 \sin\left(x^2\right)dx \approx 0.2\cdot\left(\sin\left(0.1^2\right) + \sin\left(0.3^2\right) + \sin\left(0.5^2\right) + \sin\left(0.7^2\right) + \sin\left(0.9^2\right)\right).$$

Now let's estimate the error bound δ,

$$f'(x) = 2x\cos\left(x^2\right), \qquad f''(x) = 2\cos\left(x^2\right) + 4x^2\sin\left(x^2\right),$$

$$K = \max_{[0,1]}|f''(x)| \leq 2\max_{[0,1]}\left|\cos\left(x^2\right)\right| + 4\max_{[0,1]}\left|x^2\sin\left(x^2\right)\right| \leq 2 + 4 = 6,$$

$$|E| \leq \delta = \frac{1^3\cdot 6}{24\cdot 5^2} = 0.01.$$

2) The given $\delta = 10^{-4}$, so the equation for n is

$$10^{-4} = \frac{1^3 \cdot 6}{24 \cdot n^2} = \frac{1}{4n^2}, \qquad n = \frac{100}{2} = 50.$$

So to increase the accuracy 100 times (from 10^{-2} to 10^{-4}) it is necessary to increase n only 10 times or decrease h 10 times — this is because of the expression for $\delta \sim h^2$ or $\delta \sim n^{-2}$. Because of the power 2 the midpoint rule is called **method of the second order**.

Trapezoidal rule

The midpoint rule can be interpreted as follows: at the interval $[x_{i-1}, x_i]$ we approximate $f(x)$ by a constant function, $f(x) \approx f(z_i)$, and then we integrate this approximate function instead of the original one. Note that this approximate function is discontinuous at $x = x_i$, but we split the integral into sum of the integral over intervals $[x_{i-1}, x_i]$, where the function can be made continuous.

There is a simple idea: maybe we can improve accuracy by making better approximation for $f(x)$? This approach gives other integration rules. The trapezoidal rule corresponds to linear approximation.

The approach with approximations can be formulated as follows: if we cannot evaluate integral for $f(x)$, let us replace it by a close function which we can integrate, e.g. by a polynomial.

If at each interval $[x_{i-1}, x_i]$ we want to approximate $f(x) \approx A_i (x - z_i) + B_i$ we need to find two unknown coefficients A_i, B_i, and so we need two conditions. One of the choices may be to use the tangent line at some point, e.g. in the middle, such that $f(x) \approx f(z_i) + (x - z_i) f'(z_i)$. It is interesting that this choice again gives the midpoint rule — integral for the linear term will vanish due to symmetry! But the resulting approximation still may be discontinuous.

We can make a continuous approximation if we require that our approximation equals to $f(x)$ at the endpoints of each interval,

$$A_i (x_{i-1} - z_i) + B_i = -\frac{A_i h}{2} + B_i = f(x_{i-1}),$$

$$A_i (x_i - z_i) + B_i = \frac{A_i h}{2} + B_i = f(x_i),$$

$$2B_i = f(x_i) + f(x_{i-1}), \qquad B_i = \frac{f(x_i) + f(x_{i-1})}{2}.$$

$$\int_{x_{i-1}}^{x_i} (A_i (x - z_i) + B_i)\, dx = [u = x - z_i] \quad \int_{-h/2}^{h/2} (A_i u + B_i) = B_i h = \frac{f(x_i) + f(x_{i-1})}{2} h,$$

so we even don't need A_i.

$$\int_a^b f(x)dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)dx \approx \sum_{i=1}^{n} \frac{f(x_i) + f(x_{i-1})}{2} h.$$

This gives the **Trapezoidal rule**:

$$\int_a^b f(x)dx \approx h \left(\frac{1}{2} f(a) + f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + \frac{1}{2} f(b) \right).$$

The maximum error bound δ for this formula can be estimated in similar way through the Taylor polynomial, and

$$|E| \le \delta = \frac{(b-a)^3 K}{12 n^2} = \frac{(b-a) K h^2}{12}.$$

The error bound is twice greater than for the midpoint rule, so why this formula may be needed at all?

1. Actual error usually is not so big.
2. Sometimes the values of f are known only at the endpoints of the interval, then the midpoint rule cannot be applied.

Simpson's rule

Simpson's rule corresponds to approximation of $f(x) \approx Ax^2 + Bx + C$. To determine 3 coefficients we need to know $f(x)$ at three points. So we group intervals in pairs, find the coefficients for each pair, and then evaluate integral over this pair. Note that n must be even to do this in a simple way (for n odd this idea in principle can be used with overlapping intervals).

For $x \in [x_{i-1}, x_i] \cup [x_i, x_{i+1}]$ let $f(x) \approx A_i (x - x_i)^2 + B_i (x - x_i) + C_i$, then, using $(x_{i-1} - x_i) = -h$, $(x_{i+1} - x_i) = h$

$$f(x_{i-1}) = A_i h^2 - B_i h + C_i,$$

$$f(x_i) = C_i,$$

$$f(x_{i+1}) = A_i h^2 + B_i h + C_i.$$

$$f(x_{i+1}) + f(x_{i-1}) = 2A_i h^2 + 2C_i = 2A_i h^2 + 2f(x_i),$$

$$A_i = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{2h^2}.$$

$$\int_{x_{i-1}}^{x_{i+1}} \left(A_i (x - x_i)^2 + B_i (x - x_i) + C_i\right) = \int_{-h}^{h} \left(A_i u^2 + B_i u + C_i\right) du =$$

$$= \left(A_i \frac{u^3}{3} + B_i \frac{u^2}{2} + C_i u\right)\Bigg]_{-h}^{h} = \frac{2h^3}{3} A_i + 2hC_i =$$

$$= \frac{h}{3}\left(f(x_{i+1}) - 2f(x_i) + f(x_{i-1})\right) + 2hf(x_i) = \frac{h}{3}\left(f(x_{i+1}) + 4f(x_i) + f(x_{i-1})\right).$$

This gives **Simpson's rule**

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f dx + \int_{x_2}^{x_4} f dx + \int_{x_4}^{x_6} f dx + \ldots + \int_{x_{n-2}}^{x_n} f dx \approx$$

$$\approx \frac{h}{3}\left(f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(b)\right)$$

— the endpoints appear with the factor 1, even points with 2, odd points with 4.

If $f(x)$ has continuous derivative $f^{(4)}(x)$ on $[a, b]$, and $M = \max_{[a,b]} \left|f^{(4)}(x)\right|$ then the accuracy of the Simpson's rule is

$$|E| \le \delta = \frac{M(b-a)^5}{180n^4} = \frac{M(b-a)h^4}{180},$$

so the Simpson's rule has the 4-th order of accuracy: increasing n 10 times we obtain 10000 times more accurate estimate.

Example 5. Write trapezoidal and Simpson's rules for

$$S = \int_1^3 \frac{\sin x}{x} dx$$

with $n = 6$, evaluate the sums and find error bound for them.

Solution. First we find all necessary derivatives.

$$f\left(x\right)=\frac{\sin x}{x},\qquad f'\left(x\right)=\frac{\cos x}{x}-\frac{\sin x}{x^2},$$

$$f''\left(x\right)=-\frac{\sin x}{x}-2\frac{\cos x}{x^2}+2\frac{\sin x}{x^3},$$

$$f'''\left(x\right)=-\frac{\cos x}{x}+3\frac{\sin x}{x^2}+6\frac{\cos x}{x^3}-6\frac{\sin x}{x^4},$$

$$f^{(4)}\left(x\right)=\frac{\sin x}{x}+4\frac{\cos x}{x^2}-12\frac{\sin x}{x^3}-24\frac{\cos x}{x^4}+24\frac{\sin x}{x^5}.$$

$$K=\max_{[1,3]}\left|f''\left(x\right)\right|\leq\max_{[1,3]}\left(\left|\frac{\sin x}{x}\right|+2\left|\frac{\cos x}{x^2}\right|+2\left|\frac{\sin x}{x^3}\right|\right)\leq 1+2+2=5$$

$$M=\max_{[1,3]}\left|f^{(4)}\left(x\right)\right|\leq\max_{[1,3]}\left(\left|\frac{\sin x}{x}\right|+4\left|\frac{\cos x}{x^2}\right|+12\left|\frac{\sin x}{x^3}\right|+24\left|\frac{\cos x}{x^4}\right|+24\left|\frac{\sin x}{x^5}\right|\right)\leq$$

$$\leq 1+4+12+24+24=65.$$

$$n=6,\qquad h=\frac{2}{6}=\frac{1}{3},$$

$$x_0=1,\quad x_1=\frac{4}{3},\quad x_2=\frac{5}{3},\quad x_3=2,\quad x_4=\frac{7}{3},\quad x_5=\frac{8}{3},\quad x_6=3$$

Trapezoidal rule:

$$S\approx\frac{1}{6}\left(\frac{\sin\left(1\right)}{1}+2\frac{3}{4}\sin\left(\frac{4}{3}\right)+2\frac{3}{5}\sin\left(\frac{5}{3}\right)+2\frac{\sin\left(2\right)}{2}+2\frac{3}{7}\sin\left(\frac{7}{3}\right)+2\frac{3}{8}\sin\left(\frac{8}{3}\right)+\frac{\sin\left(3\right)}{3}\right)=$$

$$=\frac{1}{6}\left(\sin 1+\frac{3}{2}\sin\left(\frac{4}{3}\right)+\frac{6}{5}\sin\left(\frac{5}{3}\right)+\sin 2+\frac{6}{7}\sin\left(\frac{7}{3}\right)+\frac{3}{4}\sin\left(\frac{8}{3}\right)+\frac{\sin 3}{3}\right)\approx 0.902$$

$$\delta=\frac{\left(b-a\right)^3 K}{12n^2}=\frac{8\cdot 5}{12\cdot 36}\approx 0.093,$$

The true value should belong to the interval $0.809<S<0.995$.

Simpson's rule:

$$S\approx\frac{1}{9}\left(\frac{\sin 1}{1}+4\frac{3}{4}\sin\left(\frac{4}{3}\right)+2\frac{3}{5}\sin\left(\frac{5}{3}\right)+4\frac{\sin\left(2\right)}{2}+2\frac{3}{7}\sin\left(\frac{7}{3}\right)+4\frac{3}{8}\sin\left(\frac{8}{3}\right)+\frac{\sin\left(3\right)}{3}\right)=$$

$$=\frac{1}{9}\left(\sin 1+3\sin\left(\frac{4}{3}\right)+\frac{6}{5}\sin\left(\frac{5}{3}\right)+2\sin 2+\frac{6}{7}\sin\left(\frac{7}{3}\right)+\frac{3}{2}\sin\left(\frac{8}{3}\right)+\frac{\sin 3}{3}\right)\approx 0.903$$

$$\delta=\frac{\left(b-a\right)^5 M}{180n^4}=\frac{32\cdot 65}{180\cdot 1296}\approx 0.009$$

The true value should belong to the interval $0.893<S<0.912$.

Other rules

It is possible to approximate $f(x)$ by polinomials of higher orders and obtain more and more accurate rules, but they are too complicated. The Simpson's rule appears to be a good tradeoff between accuracy and complexity.

What if the necessary derivative do not exist or the integral is improper?

Sometimes there may be situations when the necessary derivative does not exist. For example, it may be necessary to integrate $f(x) = x^{3/2}\sin x$ from 0 to 1: $f''(x)$ at $x = 0$ becomes infinite. Then the dependence of the error on n is different, the error diminishes slower. In such situation the midpoint or trapezoid rule may be enough.

Sometimes it is possible to transform an integral via substitutions such that the new function under integral has more derivatives. For example, if we use $x = u^2$, then

$$\int_0^1 x^{3/2}\sin x\,dx = 2\int_0^1 u^4 \sin\left(u^2\right)du,$$

and for the new function the Simpson's rule will have much better accuracy.

Sometimes even improper integrals can be transformed such that they may be evaluated numerically.

Example. For the convergent improper integral on an infinite domain we can use substitution $u = x^{-1}$, $x = u^{-1}$, $dx = -u^{-2}du$. This substitution transform infinite interval $[1, \infty)$ into a finite one $[0, 1]$:

$$\int_1^\infty \frac{x}{\sqrt{x^5+1}}dx = \lim_{t\to\infty}\int_1^t \frac{x}{\sqrt{x^5+1}}dx = \lim_{t\to\infty} -\int_1^{1/t}\frac{u^{-1}}{\sqrt{u^{-5}+1}}u^{-2}du =$$

$$= \lim_{t\to\infty}\int_{1/t}^1 \frac{u^{-3}}{u^{-5/2}\sqrt{1+u^5}}du = \lim_{s\to 0}\int_s^1 \frac{1}{\sqrt{u}\sqrt{1+u^5}}du =$$

this is again an improper integral, now of type 2, substitute $v = \sqrt{u}$, $u = v^2$, $du = 2vdv$

$$= \lim_{s\to 0}\int_{\sqrt{s}}^1 \frac{1}{v\sqrt{1+v^{10}}}2vdv = 2\int_0^1 \frac{1}{\sqrt{1+v^{10}}}dv.$$

We transformed an improper integral into a ususal definite integral which can be evaluated numerically.