Volumes by cross-sections (Sections 6.2, 6.3)

Volumes of solids: general idea

1. We choose a direction along a line in 3D space and introduce a coordinate that measures position along this line — now we denote it by z, in tasks usually this will be x- or y-axis.
2. For each value of z we consider a plane orthogonal to z-line. The position of the plane is defined by the value of z.
3. For some interval of z values $a \leq z \leq b$ these planes intersect the solid for which we need to find its volume V.
4. In the intersection of a plane and the solid there is a planar figure. We assume that we know its area A. Since the position of the plane depends on z, this area is a function of z. We consider only examples where $A(z)$ is a continuous function.
5. The volume of the solid is

$$V = \int_a^b A(z)dz.$$

The meaning of this formula: we approximate the solid by many "orthogonal cylinders" with the base $A(z)$ and very small height Δz, and then consider the limit $\Delta z \to 0$. Because of continuity of $A(z)$ the errors of approximation tend to zero as $\Delta z \to 0$.
6. The function $A(z)$ depends on how we choose the direction of z. It is necessary to choose it such that this function is as simple as possible.
Several examples with good pictures are in Section 6.2.
7. (supplementary) Cross-section by a plane is an orthogonal section in Cartesian coordinate system: we fix one of the coordinate and vary other two. There are other coordinate systems, for example, cylindric: it consists of polar coordinates on the plane (radius r + angle θ, if you are interested, you can find description in Chapter 11) and z-coordinate along the axis orthogonal to the plane.
In these coordinates surface orthogonal to z is a plane, we fix z and vary r and θ. But surface orthogonal to radius is surface of an infinite cylinder: we fix r and vary z and angle. This kind of orthogonal cross-section is used in the method of shells.

Solids of revolution: method of washers

Now we consider a special type of solids, obtained by rotating a planar figure around a line belonging to the same plane. The cross-sections are orthogonal to the axis of rotation. The coordinate z is measured along the axis of rotation.
Because we consider a solid of revolution, the figure in the cross-section is a washer with inner radius $r(z)$ and outer radius $R(z)$ — in each section they may be different. The area of the washer is $A(z) = \pi R^2 - \pi r^2$, so

$$V = \int_a^b \left(\pi R\left(z\right)^2 - \pi r\left(z\right)^2\right) dz.$$

How to apply the method.
1. Draw a picture with the axis of rotation and the rotated figure.
2. Determine, which coordinate (x or y) goes along the axis of rotation, this coordinate will play the role of z. We shall call it "axis-coordinate" and denote by z, and the other one call "radius-coordinate" and denote by u.
3. Fix some value of axis-coordinate z and draw a line orthogonal to the axis of rotation. This line goes along the radius-coordinate u. Find, where this line crosses the boundaries of the figure, and coordinates of these points along the radius-coordinate u.
4. Find the distance from these points to the axis of rotation. The smallest gives r, the biggest gives R.
5. Substitute R and r into the formula and determine a and b along z-coordinate.
Typical problems for which this method is **appropriate** (easy to apply).

1) Domain bounded by $y = f(x)$ and $y = g(x)$, $a \le x \le b$, is rotated about the line $y = c$ (parallel to x-axis). The position of the cross-section is defined by x-coordinate. Fix some value of x and find for this x the distances from the line $y = c$ to the curves $y = f(x)$ and $y = g(x)$, $|f(x) - c|$ and $|g(x) - c|$. The greatest of them is $R(x)$, the smallest is $r(x)$. Now $V = \int_a^b \left(\pi R^2(x) - \pi r^2(x)\right) dx$.
2) Domain bounded by $x = f(y)$ and $x = g(y)$, $a \le y \le b$, is rotated about the line $x = c$ (parallel to y-axis). The position of the cross-section is defined by y-coordinate. Fix some value of y and find for this y the distances from the line $x = c$ to the curves $x = f(y)$ and $x = g(y)$, $|f(y) - c|$ and $|g(y) - c|$. The greatest of them is $R(y)$, the smallest is $r(y)$. Now $V = \int_a^b \left(\pi R^2(y) - \pi r^2(y)\right) dy$.
Typical problems for which this method **may be inappropriate**.
3) If we have $y = f(x)$ and $y = g(x)$ rotated about $x = c$, to apply this method we need to express x through y and then do like in case 2).
4) If we have $x = f(y)$ and $x = g(y)$ rotated about $y = c$, to apply this method we need to express y through x and then do like in case 1).
Resume: the axis of rotation determines, along which coordinate we must integrate in the method of washers. When we are finding expressions for R and r, this coordinate must be fixed.
See pictures and examples in Section 6.2.

Solids of revolution: method of shells

Now we do cross-sections orthogonal to radius and approximate a solid by a number of very thin cylindrical shells of radius r and height h. In this method each shell is defined by a fixed value of radius, that is by a value of radius coordinate u. The height is measured along the axis-coordinate z. Then summing up all shells we obtain

$$V = \int_a^b 2\pi r(u)h(u)du.$$

So integration is performed *across* the axis of rotation. (**Note**: in the method of washers intergration is performed *along* the axis of rotation)
How to apply the method.
1. Draw a picture with the axis of rotation and the rotated figure.
2. Determine, which coordinate (x or y) goes along the axis of rotation (z), and which goes across it (u). Find which value $u = c$ defines the axis of rotation. Then $r(u) = |u - c|$
3. Fix some value of radius-coordinate u and draw a line parallel to the axis of rotation. This line goes along the axis-coordinate z. Find, where this line crosses the boundaries of the figure, and coordinates of these points along the radius-coordinate z, $z_1(u)$ and $z_2(u)$.
4. Find the distance between these two points, it gives the height of shell $h(u) = |z_1(u) - z_2(u)|$.
5. Substitute r and h into the formula and determine a and b along u-coordinate.

Typical problems for which this method is **appropriate** (easy to apply).
1) Domain bounded by $y = f(x)$ and $y = g(x)$, $a \le x \le b$, is rotated about the line $x = c$ (parallel to y-axis). The position of the shell is defined by x-coordinate and $r(x) = |x - c|$. Fix some value of x and find for this x the distance between curves $y = f(x)$ and $y = g(x)$, $h(x) = |f(x) - g(x)|$. Now $V = \int_a^b 2\pi\,|x - c|\,|f(x) - g(x)|\,dx$.
2) Domain bounded by $x = f(y)$ and $x = g(y)$, $a \le y \le b$, is rotated about the line $y = c$ (parallel to x-axis). The position of the shell is defined by y-coordinate and $r(y) = |y - c|$. Fix some value of y and find for this y the distance between curves $x = f(y)$ and $x = g(y)$, $h(y) = |f(y) - g(y)|$. Now $V = \int_a^b 2\pi\,|y - c|\,|f(y) - g(y)|\,dy$.
Typical problems for which this method **may be inappropriate**.
3) If we have $y = f(x)$ and $y = g(x)$ rotated about $y = c$, to apply this method we need to express x through y and then do like in case 2).
4) If we have $x = f(y)$ and $x = g(y)$ rotated about $x = c$, to apply this method we need to express y through x and then do like in case 1).
See pictures and examples in Section 6.3.
Note: when washers method may fail, shells method works and vice versa.

Inverse problem: solid from integral

If we are given an integral for which it is known that it gives volume of a solid of revolution, and it is necessary to describe this solid, this problem usually has infinitely many solutions.
Example. Describe the solid for which the volume is

$$\int_0^1 2\pi\left(x^3-x^7\right)dx.$$

1) Most obvious solution. The formula resembles shells method. Let $r(x)=x$, then $h(x)=x^2-x^6$. So we have the domain bounded by $f(x)=x^2$, $g(x)=x^6$, rotated about $x=0$ (y-axis).
2) Same as above, but $f(x)=x^2-x^6$, $g(x)=0$.
3) Same as above, but $f(x)=x^2-x^6+\sin x$, $g(x)=\sin x$.
4) Same as above, but $f(x)=x^2-x^6+q(x)$, $g(x)=q(x)$, where $q(x)$ is any function continuous on $[0,1]$.
5) $x^3-x^7=x^3\left(1-x\right)\left(x+1\right)\left(x^2+1\right)$. Let $r(x)=1-x$ (rotated about $x=1$), $f(x)=x^3\left(x+1\right)\left(x^2+1\right)$, $g(x)=0$.
6) Let $r(x)=1+x$ (rotated about $x=-1$), $f(x)=x^3\left(1-x\right)\left(x^2+1\right)$, $g(x)=0$.
7) Let $r(x)=3-x$ (rotated about $x=3$),

$$f(x)=\frac{x^3-x^7}{3-x},\quad g(x)=0.$$

8) Let $R(x)=f(x)=\sqrt{2\left(x^3-x^7\right)}$, then we have the washers method, domain bounded by $f(x)$ and $g(x)=0$ rotated about x-axis ($y=0$).
And so on...

Inverse function (Section 7.1)

Function

Definition of function includes two sets A, B, and a rule f that assigns some $y\in B$ for every $x\in A$. This is denoted as $y=f(x)$. For every x there must be one and only one y.
The set of all possible values of x is called domain of f. We shall denote it as $D(f)$.
The set of $y=f(x)$ where x takes all possible values from $D(f)$ is called range of f. We shall denote it as $R(f)$.
If both x and y are real numbers, function may be represented as a curve in the plane. This curve must pass vertical line test (VLT): vertical line through every $x\in D(f)$ on x-axis must cross this curve only once (otherwise there will be two or more y for this x).

One-to-one function

Function is called **one-to-one** (**OTO**) if it takes every value only once: for all $x_1\neq x_2$ $f(x_1)\neq f(x_2)$. The plot of OTO function must pass VLT and horizontal line test (HLT): horizontal line through every $y\in R(f)$ on y-axis must cross this plot only once (otherwise there will be two equal y for different x).
Important property. If function f is continuous on an interval and OTO, it is either increasing or decreasing on this interval.
Suppose that f is continuous and OTO on $[a,b]$, it is increasing on $[a,c]$, $a<c<b$, and decreasing on $[c,b]$. For $x>c$ $f(x)$ must be less than $f(c)$, but since f is OTO, it cannot take values between $f(a)$ and $f(c)$, and since $f(x)$ is continuous it cannot jump through this interval. Therefore, it cannot be decreasing for $x>c$.
How to check that differentiable $f(x)$ is OTO. For increasing functions $f'(x)\geq 0$, and $f'(x)$ may turn to zero only in single points, like for the function $f(x)=x^3$. For decreasing functions $f'(x)\leq 0$, and it may turn

to zero only in single points, like for the function $f(x) = -x^3$. **Note**, if $f'(x) = 0$ not in a single point, but on an interval $[a,b]$, this means that $f(x) = c$ for all $x \in [a,b]$, and therefore it is not OTO.

How to make a function OTO. If $f(x)$ is not OTO, often it is possible to change its domain such that it becomes OTO. According to the property, on the new domain f must be increasing or decreasing. Therefore, we may take for the new domain any interval between points of maximum and minimum, or between $-\infty$ and the first extremum, or between the last extremum and ∞.

Example. How to change the domain of $y = x^2$ such that it becomes OTO.

$f(x) = x^2$ is decreasing on $(-\infty, 0]$ and increasing on $[0,\infty)$. Therefore if we choose $D(f) = (-\infty, 0]$ or $D(f) = [0,\infty)$, it becomes OTO.

Example. Check whether $f(x) = 3x - x^3$ is OTO. If not, define its domain such that it becomes OTO and $x = 2$ belongs to the domain.

$f'(x) = 3 - 3x^2 = 3\,(1-x)\,(1+x)$, there is minimum at $x = -1$ and maximum at $x = 1$. For $x < -1$ $f'(x) < 0$; for $-1 < x < 1$ $f'(x) > 0$; for $1 < x$ $f'(x) < 0$. Therefore there are three possible choices for the domain to make $f(x)$ OTO: $D_1 = (-\infty, -1]$, $D_2 = [-1, 1]$, $D_3 = [1,\infty)$. Since $x = 2$ belongs to D_3 the answer is: $D(f) = [1,\infty)$.

Inverse function $f^{-1}(x)$

If $f(x)$ with domain $D(f)$ and range $R(f)$ is OTO, then its inverse function f^{-1} is defined as $x = f^{-1}(y)$ if $y = f(x)$. $D(f^{-1}) = R(f)$, $R(f^{-1}) = D(f)$.

- $f^{-1}(x)$ is a special notation, it does not mean $1/f(x) = [f(x)]^{-1}$. So $\sin^2 x = \sin x \times \sin x$, $\sin^{-1} x$ is inverse of $\sin x$, $(\sin x)^{-1} = 1/\sin x$.
- If $f(x)$ is not OTO, it does not have inverse.
- If we need to define an inverse for $f(x)$ which is not OTO, first we must change its domain to make it OTO.
- If $f(x)$ is continuous at $x = a$, $f^{-1}(x)$ is continuous at $x = b = f(a)$.

Cancellation equalities:

$$f^{-1}(f(x)) = x, \quad x \in D(f),$$
$$f(f^{-1}(x)) = x, \quad x \in R(f).$$

Plot and derivative of $f^{-1}(x)$

If point (a,b), $b = f(a)$, belongs to the plot of $f(x)$, then (b,a) belongs to the plot of $f^{-1}(x)$ since $a = f^{-1}(b)$ — by definition of f^{-1}.

The plot of $f^{-1}(x)$ can be obtained from the plot of $f(x)$ by symmetry about the line $y = x$.

If $f(x)$ is differentiable at $x = a$, $y = b = f(a)$, and $f'(a) \neq 0$, then $g(x) = f^{-1}(x)$ is differentiable at $x = b$, $y = a = g(b)$ and

$$g'(b) = \frac{1}{f'(a)}, \qquad g'(f(a)) = \frac{1}{f'(a)}, \qquad g'(b) = \frac{1}{f'(g(b))}.$$

These formulas can be derived by differentiating cancellation equalities $g(f(x)) = x$ and $f(g(x)) = x$.

Consequence. If $f(x)$ is increasing (decreasing), then $g(x)$ is increasing (decreasing) too — they have the same sign of derivative.

Example. For $f(x) = 3x - x^3$, $x \in [1,\infty)$, find $g'(b)$ where $b = f(2)$.

Here we cannot find an expression for $g(x)$. But we don't need it. $f'(x) = 3 - 3x^2$, $f'(2) = -9$, $g'(b) = g'\,(f(2)) = 1/f'(2) = -1/9$.