Exponential and logarithmic function (Sections 7.2–7.4)

Exponential function: $y = a^x$, $a > 0$, $a \neq 1$. Domain $(-\infty, \infty)$, range $(0, \infty)$, exponential function is continuous and differentiable. It is inverse to logarithmic function $y = \log_a x$.
Logarithmic function: $y = \log_a x$, $a > 0$, $a \neq 1$. Meaning: $y = \log_a x$ means that $x = a^y$. Domain $(0, \infty)$, range $(-\infty, \infty)$, exponential function is continuous and differentiable. It is inverse to exponential function $y = a^x$.
$a^0 = 1$, $\log_a 1 = 0$, so for any a the plot of a^x passes through the point $(0, 1)$ and the plot of $\log_a x$ passes through $(1, 0)$.
For $a > 1$ a^x and $\log_a x$ are increasing,

$$\lim_{x\to\infty} a^x = \infty, \qquad \lim_{x\to-\infty} a^x = 0, \qquad \lim_{x\to\infty} \log_a x = \infty, \qquad \lim_{x\to 0} \log_a x = -\infty.$$

For $0 < a < 1$ a^x and $\log_a x$ are decreasing,

$$\lim_{x\to\infty} a^x = 0, \qquad \lim_{x\to-\infty} a^x = \infty, \qquad \lim_{x\to\infty} \log_a x = -\infty, \qquad \lim_{x\to 0} \log_a x = \infty.$$

Properties:

Exponential	Logarithmic
$a^x a^y = a^{x+y}$	$\log_a(uv) = \log_a u + \log_a v$
$(a^x)^y = a^{xy}$	$\log_a(u^y) = y \log_a u$
$a^{-x} = 1/a^x$	$\log_a(1/u) = -\log u$
$a^x/a^y = a^{x-y}$	$\log_a(u/v) = \log_a u - \log_a v$

Exponential	Logarithmic
$a^0 = 1$	$\log_a 1 = 0$
$a^1 = a$	$\log_a a = 1$
$(ab)^x = a^x b^x$	

Popular bases a and special notation: $a = 10$ (decimal logarithms, $\log_{10} x = \lg x$) and $a = e$ =2.71828182845904523... (natural logarithms, $\log_e x = \ln x$).
Examples:

$$2^3 = 8, \quad \log_2 8 = 3, \qquad \left(\frac{1}{2}\right)^{-3} = 8, \quad \log_{\frac{1}{2}} 8 = -3,$$

$$2^{-4} = \frac{1}{16}, \quad \log_2 \frac{1}{16} = -4, \qquad \left(\frac{1}{16}\right)^{-\frac{1}{4}} = 2, \quad \log_{\frac{1}{16}} 2 = -\frac{1}{4},$$

$$\log_a\left(\sqrt{\frac{x^3y^4}{z^5}}\right) = \frac{1}{2}\log_a\left(\frac{x^3y^4}{z^5}\right) = \frac{1}{2}\left(\log_a\left(x^3\right) + \log\left(y^4\right) - \log\left(z^5\right)\right) =$$

$$= \frac{3}{2}\log_a x + 2\log_a y - \frac{5}{2}\log_a z$$

Example: find

$$\lim_{x\to\infty} \frac{\log_2(1+5^x)}{x+2}.$$

Here both numerator and denominator tend to ∞, this is an indeterminate form, to find the answer we have to transform the expression such that at least one of them has a finite limit. Note that $1 = 5^0 = 5^{x-x} = 5^x 5^{-x}$, $1 + 5^x = 5^x 5^{-x} + 5^x = 5^x(5^{-x} + 1)$, so

$$\frac{\log_2(1+5^x)}{x+2} = \frac{\log_2(5^x(5^{-x}+1))}{x+2} = \frac{\log_2 5^x + \log_2(1+5^{-x})}{x+2} =$$

$$= \frac{x\log_2 5 + \log_2\left(1+5^{-x}\right)}{x+2} = \frac{x}{x+2}\log_2 5 + \frac{1}{x+2}\log_2\left(1+5^{-x}\right).$$

Now we have an expression where each term has a finite limit:

$$\lim_{x\to\infty}\frac{x}{x+2} = 1, \quad \lim_{x\to\infty}\frac{1}{x+2} = 0, \quad \lim_{x\to\infty} 5^{-x} = 0,$$

$\log_2 z$ is a continuous function at $z = 1$, hence, according to the properties of limit,

$$\lim_{x\to\infty}\frac{\log_2\left(1+5^x\right)}{x+2} = (\log_2 5)\lim_{x\to\infty}\frac{x}{x+2} + \left(\lim_{x\to\infty}\frac{1}{x+2}\right)\log_2\left(1+\lim_{x\to\infty}5^{-x}\right) = \log_2 5.$$

Example: find

$$\lim_{x\to 0^+}\ln\left(\tan x\right)$$

From the properties of $\tan x$ we know, that $\lim_{x\to 0^+}\tan x = 0$. If we denote $u = \tan x$, then $x \to 0^+$ implies $u \to 0^+$ and since $e > 1$,

$$\lim_{x\to 0^+}\ln\left(\tan x\right) = \lim_{u\to 0^+}\ln\left(u\right) = -\infty.$$

Number $e = 2.71828182845904523...$

Consider the numbers $a_n = \left(1+\frac{1}{n}\right)^n$ for integer $n = 1, 2, 3, ...$ Here are the values of a_n for some n:

n :	1	2	3	4	5	10	100	1000	10^4	10^5
a_n :	2	2.25	2.37...	2.44...	2.49...	2.59...	2.705...	2.717...	2.71814...	2.718268...

It is possible to prove that 1) a_n increase with n, 2) $a_n < 3$ for all n. There is a theorem, that under such conditions there exists limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.71828182845904523...$$

Similarly for $x \to \infty$ but noninteger and for $h \to 0$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{h\to 0}\left(1+h\right)^{\frac{1}{h}} = e.$$

Example. Find

$$\lim_{x\to\infty}\left(1+\frac{2}{x^2}\right)^{5x^2}.$$

Let us transform

$$\left(1+\frac{2}{x^2}\right)^{5x^2} = \left(\left(1+\frac{2}{x^2}\right)^{\frac{x^2}{2}}\right)^{10}$$

and denote $u = x^2/2 \to \infty$ as $x \to \infty$. Now

$$\lim_{x\to\infty}\left(1+\frac{2}{x^2}\right)^{5x^2} = \lim_{u\to\infty}\left(\left(1+\frac{1}{u}\right)^u\right)^{10} = \left(\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^u\right)^{10} = e^{10}.$$

Derivative of $\ln x$ and e^x

Logarithmic function

$$\frac{d}{dx}(\ln x) = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\frac{1}{h}\ln\frac{x+h}{x} =$$

$$= \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right) = \lim_{h\to 0}\frac{1}{x}\frac{x}{h}\ln\left(1+\frac{h}{x}\right) = \lim_{h\to 0}\frac{1}{x}\ln\left(1+\frac{h}{x}\right)^{\frac{x}{h}} =$$

denote $u = h/x$, $u \to 0$ as $x \to 0$, and use continuity of $\ln x$

$$= \lim_{u\to 0}\frac{1}{x}\ln(1+u)^{\frac{1}{u}} = \frac{1}{x}\ln\lim_{u\to 0}(1+u)^{\frac{1}{u}} = \frac{1}{x}\ln e = \frac{1}{x}.$$

Exponential function. Let $y = e^x$, then $\ln y = x$,

$$\frac{d}{dx}\ln y = \left(\frac{d}{dy}\ln y\right)\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}x = 1, \qquad \frac{dy}{dx} = y = e^x.$$

Remember: Derivatives

$$\frac{d}{dx}\ln x = \frac{1}{x}, \quad x > 0, \qquad \frac{d}{dx}e^x = e^x, \quad -\infty < x < \infty.$$

Remember: Integrals

$$\int\frac{dx}{x} = \ln|x| + C, \quad x \neq 0, \qquad \int e^x dx = e^x + C.$$

Example: Substitution $u = kx$, $dx = \frac{1}{k}du$

$$\int e^{kx}dx = \frac{1}{k}\int e^u du = \frac{1}{k}e^{kx} + C.$$

Example: Substitution $u = x + a$, $dx = du$

$$\int\frac{dx}{x+a} = \int\frac{du}{u} = \ln|u| + C = \ln|x+a| + C$$

Base change for a^x and $\log_a x$

To apply the above formulas to a^x and $\log_a x$ we need to change to e^{kx} and $\ln x$.
$a = e^{\ln a}$, so $a^x = e^{x\ln a}$
$u = \log_a x$, hence $x = a^u$, $\ln x = \ln(a^u) = u\ln a = (\log_a x)(\ln a)$.
This gives **base change formulas** (remember or quickly derive)

$$a^x = e^{x\ln a}, \qquad \log_a x = \frac{\ln x}{\ln a}.$$

Example: Derivatives of a^x and $\log_a x$

$$\frac{d}{dx}a^x = \frac{d}{dx}\left(e^{x\ln a}\right) = \ln a e^{x\ln a} = a^x\ln a, \qquad \frac{d}{dx}\log_a x = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{x\ln a}.$$

Example: Integral for a^x (see example with $\int e^{kx}dx$, here $k = \ln a$)

$$\int a^x dx = \int e^{x\ln a}dx = \frac{1}{\ln a}e^{x\ln a} + C = \frac{1}{\ln a}a^x + C.$$

First step rule:

- If you have to integrate or to differentiate an expression containing a^x and/or $\log_a x$, first **change base** to e, and then use formulas for e^x and $\ln x$.
- If you have an expression containing $e^{k\ln x}$, **change** it to $e^{k\ln x} = \left(e^{\ln x}\right)^k = x^k$ (k does not depend on x, otherwise this rule may not work).

Example:

$$\int 2^{\log_5 x} dx = \int e^{(\ln 2)(\ln x)/(\ln 5)} dx = \int \left(e^{\ln x}\right)^{(\ln 2)/(\ln 5)} dx =$$

$$= \int x^a dx = \frac{1}{a+1} x^{a+1} + C, \qquad a = \frac{\ln 2}{\ln 5} = \log_5 2 \neq -1.$$

Example:

$$\int e^{x^2+\ln x} dx = \int e^{x^2} e^{\ln x} dx = \int e^{x^2} x dx =$$

substitution $u = x^2$, $du = 2xdx$, $xdx = \frac{1}{2}du$,

$$= \frac{1}{2}\int e^u du = \frac{1}{2} e^{x^2} + C$$

Logarithmic differentiation (Section 7.4)

Sometimes it is easier to differentiate $\ln y$ than y. Then it is possible to use the following trick: $(\ln y)' = \frac{1}{y} y'$, or

$$\frac{dy}{dx} = y \frac{d}{dx} (\ln y).$$

This formula is most useful in two cases:

- when $y = f(x)^{g(x)}$;
- when $y = (f(x))^{p_1} (g(x))^{p_2} \ldots (h(x))^{p_k}$.

Example: differentiate $y = x^x$.

$$y' = y \frac{d}{dx} (\ln x^x) = y \frac{d}{dx} (x \ln x) = y \cdot (\ln x + 1) = x^x (\ln x + 1).$$

Example: differentiate

$$y = \sqrt{\frac{x^3 \left(1+x^2\right)^4}{(1+x^5)}}.$$

$$\ln y = \frac{1}{2} \ln \left(\frac{x^3 \left(1+x^2\right)^4}{(1+x^5)} \right) = \frac{3}{2} \ln x + 2 \ln \left(1+x^2\right) - \frac{1}{2} \ln \left(1+x^5\right),$$

$$\frac{d}{dx} \ln\left(1+x^2\right) = \frac{1}{1+x^2} \left(1+x^2\right)' = \frac{2x}{1+x^2}, \qquad \frac{d}{dx} \ln \left(1+x^5\right) = \frac{5x^4}{1+x^5},$$

$$y' = y \cdot (\ln y)' = y \cdot \left(\frac{3}{2}\ln x + 2\ln\left(1+x^2\right) - \frac{1}{2}\ln\left(1+x^5\right)\right)' =$$

$$= y \cdot \left(\frac{3}{2x} + \frac{4x}{1+x^2} - \frac{5x^4}{2\left(1+x^5\right)}\right).$$

Separable differential equations (Sections 10.3–10.5)

Differential equation (DE) is an equation containing derivative of the unknown function. Separable DE has the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}.$$

It can be solved in the following steps:
1. Separate variables

$$g(y)dy = f(x)dx.$$

2. Change equality of differentials into equality of integrals

$$\int g(y)dy = \int f(x)dx.$$

3. Let $G(y)$ and $F(y)$ be antiderivatives for g and f respectively (**any** antiderivatives). Then the solution is

$$G(y) = F(x) + C,$$

here C is an arbitrary constant (Note: indefinite integrals are on both sides, but only one arbitrary constant may be retained). This is an implicit form of solution. **Don't forget about** C.
4. Try to express y through x: $y = G^{-1}\left(F(x) + C\right)$. This is not a single curve $y(x)$, but **a family of curves** — for each value of C there is a different solution. If no additional data are provided, this is the answer. Sometimes the final answer may contain a function $\phi(C)$, not just C. Then it may be better to introduce another arbitrary constant $A = \phi(C)$, this may simplify the answer (see examples below).
5. To determine the value of C it is necessary to know a point on the curve: for $x = a$ $y = b$. If such information is provided, then, for example, we can write

$$G(b) = F(a) + C, \qquad C = G(b) - F(a).$$

Example. Solve

$$\frac{dy}{dx} = k\left(y - b\right).$$

$$\int \frac{dy}{y-b} = \int k dx, \qquad \ln|y-b| = kx + C, \qquad y - b = \pm e^C e^{kx}.$$

Let us denote $A = \pm e^C$, this is also an arbitrary constant, so the solution is $y = b + Ae^{kx}$.
Example. Find solution of the previous problem if $b = 1$, $k = -1$, and $y(0) = -2$.
The solution is $y = 1 + Ae^{-x}$. To determine A we use the additional condition: $-2 = 1 + Ae^0 = 1 + A$, so $A = -3$. Answer: $y = 1 - 3e^{-x}$.
Example. Solve

$$\frac{dy}{dx} = x\left(e^y + 1\right), \qquad y(0) = 1.$$

$$\int \frac{dy}{e^y+1} = \int x dx = \frac{1}{2}x^2 + C.$$

$$\int \frac{dy}{e^y+1} = \int \frac{e^{-y}dy}{e^{-y}\left(e^y+1\right)} = \int \frac{e^{-y}dy}{1+e^{-y}} =$$

substitution $u = 1 + e^{-y}$, $du = -e^{-y}dy$,

$$= -\int \frac{du}{u} = -\ln\left|1+e^{-y}\right| + C_1 = -\ln\left(1+e^{-y}\right) + C_1.$$

Since we need any antiderivative $G(y)$, we may set $C_1 = 0$, so

$$\ln\left(1+e^{-y}\right) = -\frac{1}{2}x^2 - C, \quad 1 + e^{-y} = e^{-C}e^{-x^2/2},$$

$$y = -\ln\left(e^{-C}e^{-x^2/2} - 1\right).$$

$y(0) = 1$, $1 = -\ln\left(e^{-C} - 1\right)$, $e^{-C} = 1 + e^{-1}$, the solution is

$$y(x) = -\ln\left(\left(1+e^{-1}\right)e^{-x^2/2} - 1\right).$$

Example. Find a curve $y(x)$ passing through the point $(1,1)$ such that its slope is $-y$ for all x.
Since the slope is y', $y(x)$ must satisfy equation $y' = -y$. Here we can use the result for the solution of $y' = k\left(y-b\right)$ (see above) with $k = -1$, $b = 0$, $y(x) = Ae^{-x}$. Since for $x = 1$ $y = 1 = Ae^{-1}$ we have $A = e$, so the answer is $y(x) = e \cdot e^{-x} = e^{1-x}$.

Newton's law of cooling

According to this law, the heat flow from or to an object is proportional to the temperature difference between the object T and its environment T_E. The heat flow determines the rate of change of T, T_E is assumed constant. Therefore, the objects's temperature satisfies differential equation

$$\frac{dT}{dt} = -k\left(T - T_E\right).$$

The coefficient k depends on many details (material, shape, properties of the environment), and usually it is hard to find it theoretically, better to measure it from the experiment. To do it we need to know two points on the curve $T(t)$ — one to find the constant in the solution of DE, another — to find k.
Example. A medical thermometer which was in the room with the temperature $T_R = 20^o$C is used to measure the temperature of an ill person. The temperature of the person $T_E = 38^o$C. After 300 seconds thermometer showed 37.9^oC. Find k and the dependence $T(t)$.
Here the initial temperature of the thermometer $T(0) = T_R$, and the ill person plays the role of its "environment". A general form of the solution of the DE for $T(t)$ we again can take from the example $y' = k\left(y-b\right)$, so $T(t) = T_E + Ae^{-kt}$. Now let us substitute additional data:
$t = 0$: $20 = 38 + Ae^0$, $A = -18$;
$t = 300$: $37.9 = 38 - 18e^{-k300}$, $18e^{-300k} = 0.1$, $-300k = \ln\left(\frac{1}{180}\right) = -\ln 180$,

$$k = \frac{\ln\left(180\right)}{300} \approx 0.017, \qquad T(t) = 38 - 18e^{-0.017t},$$

t is in seconds.
Note: in principle, T_E also may be unknown, and it may be necessary to determine from experimental data A, k, and T_E. Then one needs to know three points on the solution curve: measurements for $t = 0$, $t = t_1$, and $t = t_2$.

Exponential, blow-up and logistic (limited) growth

1. For many processes the rate of change of a certain variable is proportional to this variable. In other words, they satisfy DE

$$\frac{dy}{dt} = ky, \qquad y(t) = Ae^{kt}.$$

This kind of solution is called exponential growth. For $k > 0$ the process accelerates itself (positive feedback), for $k < 0$ the process decelerates itself (negative feedback).
2. Quite rarely the growth for big y can accelerate even faster. For example, the Earth human population during last 50000 years quite satisfactorily can be described by the equation

$$\frac{dy}{dt} = ky^2.$$

Let us solve it:

$$\int \frac{dy}{y^2} = \int k dt, \qquad -\frac{1}{y} = kt - C, \quad y(t) = \frac{1}{C - kt}.$$

The constant C can be expressed as $C = kt_f$ where t_f is another constant. It can be determined from additional data $y(0)$. Now

$$y(t) = \frac{1}{k\left(t_f - t\right)}.$$

This means that at $t = t_f$ the solution becomes infinite. This growth is faster than exponential, sometimes this kind of solutions is called "blow-up". It is interesting that in this case for small y the growth may be slower that exponential.
3. In reality nothing grows to infinity, the assumptions made during the derivation of the model become invalid, and the solution of such a simple model ceases to agree with reality. To achieve agreement at this later stage additional terms are added to the model. For example, in biology sometimes it is assumed that small populations grow faster than large. The resulting equation for the population $P(t)$ has the form

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

(logistic equation). Solution of this equation can be simplified if we introduce new variable $y = K/P$, $P = K/y$, $P' = -y'K/y^2$, so the equation for y is

$$-\frac{K}{y^2}\frac{dy}{dt} = r\frac{K}{y}\left(1 - \frac{1}{y}\right) = r\frac{K}{y^2}\left(y - 1\right)$$

or

$$\frac{dy}{dt} = -r\left(y - 1\right).$$

This equation is familiar, again we can use the result for $y' = k\left(y - b\right)$ and write $y(t) = 1 + Ae^{-rt}$, so

$$P(t) = \frac{K}{1 + Ae^{-rt}} = \frac{Ke^{rt}}{A + e^{rt}}, \quad P(0) = p = \frac{K}{A+1}, \quad A = \frac{K}{p} - 1.$$

If initial population $p \ll K$, then A is very big, and during some time e^{rt} is inessential compared to A, and $P(t) \approx \left(K/A\right)e^{rt} \approx pe^{rt}$. This means that for some time the population grows exponentially, as if it were described by the equation $P' = rP$. But when P becomes closer to K, the growth slows down and as t grows further $P(t)$ slowly approaches K. This is a more realistic model showing logistic or limited growth.

Hyperbolic functions (Section 7.6)

Hyperbolic sine and cosine are

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2} \ge 1.$$

Like $\sin x$ and $\cos x$, $\sinh x$ is an odd function, $\cosh x$ is an even function. Their domain D is $(-\infty, \infty)$, ranges are $R(\sinh x) = (-\infty, \infty)$, $R(\cosh x) = [1, \infty)$. They satisfy

$$\cosh^2 x - \sinh^2 x = 1, \quad \cosh x = \sqrt{1 + \sinh^2 x}, \quad \sinh x = \pm\sqrt{\cosh^2 x - 1}.$$

(Here is the origin of the name: $y^2 - x^2 = 1$ is an equation of hyperbola).
There are many analogies between trigonometric and hyperbolic functions. For example there is hyperbolic tangent $\tanh x = \sinh x / \cosh x$, formulas for $\sinh(x + y)$, $\cosh(2x)$ and so on.
Derivatives:

$$\frac{d}{dx} \sinh x = \cosh x, \qquad \frac{d}{dx} \cosh x = \sinh x.$$

Integrals:

$$\int \sinh x dx = \cosh x + C, \qquad \int \cosh x dx = \sinh x + C.$$

$\sinh x$ is one-to-one and its inverse can be expressed in terms of logarithms. Let $y = \sinh^{-1} x$, $x = \sinh y$, $2x = e^y - e^{-y}$. Denote $z = e^y$, then

$$2x = z - \frac{1}{z}, \quad z^2 - 2xz - 1 = 0, \quad z = x \pm \sqrt{x^2 + 1}.$$

We need a positive number, therefore we need sign "+". Now $y = \ln z$ or

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right).$$

If we restrict the domain of $\cosh x$ to $x \ge 0$, it also has an inverse, and similarly

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right).$$

Why do we need hyperbolic functions? They are convenient as substitutions if we need to get rid of expressions like $\sqrt{x^2 + 1}$ (after $x = \sinh t$ it becomes $\cosh t$) or $\sqrt{x^2 - 1}$ (after $x = \cosh t$ it becomes $\sinh t$). But the same thing can be done with substitutions $x = \tan t$ and $x = \sec t$, though the latter may require more work. So hyperbolic functions **may be helpful** in this course, but all problems in the course can be solved without them, so they are **not absolutely necessary**. Even if you use sinh or cosh as substitution, it is always possible to express them through e^t and do all calculations in terms of exponents and logarithms.
Example where substitution $x = \sinh t$ is more efficient than $x = \tan t$

$$\text{a)} \int \frac{dx}{\sqrt{1 + x^2}} = \int \frac{d(\sinh t)}{\sqrt{1 + \sinh^2 t}} = \int \frac{\cosh t \, dt}{\cosh t} = \int dt = t + C = \sinh^{-1} x + C = \ln\left(x + \sqrt{x^2 + 1}\right) + C;$$

$$\text{b)} \int \frac{dx}{\sqrt{1 + x^2}} = \int \frac{d(\tan t)}{\sqrt{1 + \tan^2 t}} = \int \frac{\frac{1}{\cos^2 t} dt}{\frac{1}{\cos t}} = \int \frac{dt}{\cos t} = \int \frac{\cos t \, dt}{\cos^2 t} = [u = \sin t] = \int \frac{du}{1 - u^2} =$$

$$= \int \frac{du}{(1 - u)(1 + u)} = \frac{1}{2} \int \left(\frac{1}{1 - u} + \frac{1}{1 + u}\right) du = \frac{1}{2} \ln \frac{1 + u}{1 - u} + C = \ln \frac{1 + \sin t}{1 - \sin t} + C =$$

$$= \ln\left(\sqrt{1 + \tan^2 t} + \tan t\right) + C = \ln\left(x + \sqrt{x^2 + 1}\right) + C.$$