Indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ (Section 7.7)

Problem: find

$$\lim_{x\to a}\frac{f(x)}{g(x)}, \quad \text{if} \quad \lim_{x\to a} f(x)=0, \quad \lim_{x\to a} g(x)=0 \qquad \left(\frac{0}{0}\right)$$
$$\lim_{x\to a}\frac{f(x)}{g(x)}, \quad \text{if} \quad \lim_{x\to a} f(x)=\infty, \quad \lim_{x\to a} g(x)=\infty \qquad \left(\frac{\infty}{\infty}\right)$$

This information is insufficient to find the answer:

$$\lim_{x\to\infty}\frac{x^2+1}{x}=\infty, \quad \lim_{x\to\infty}\frac{x^2+1}{x^2}=1, \quad \lim_{x\to\infty}\frac{x^2+1}{x^3}=0.$$

General idea, finite a. Let be $f(x)$, $g(x)$ differentiable near a. For x close to a, $f(x)$ is close to linear function:

$$f(x)=f(a)+f'(a)\,(x-a)+h_1(x)\,(x-a)\,, \quad h_1(a)=0,$$

$$g(x)=g(a)+g'(a)\,(x-a)+h_2(x)\,(x-a)\,, \quad h_2(a)=0.$$

Suppose that $f(a)=g(a)=0$, $g'(a)\neq 0$,

$$\frac{f(x)}{g(x)}=\frac{f'(a)\,(x-a)+h_1(x)\,(x-a)}{g'(a)\,(x-a)+h_2(x)\,(x-a)}=\frac{f'(a)+h_1(x)}{g'(a)+h_2(x)},$$
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(a)+h_1(x)}{g'(a)+h_2(x)}=\frac{f'(a)}{g'(a)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

This is **l'Hospital's rule**.

If $f(x)$, $g(x)$ differentiable at a and $g'(x)\neq 0$ near a (except possibly at a). If

$$\lim_{x\to a} f(x)=0, \quad \lim_{x\to a} g(x)=0,$$

or

$$\lim_{x\to a} f(x)=\pm\infty, \quad \lim_{x\to a} g(x)=\pm\infty,$$

and there exists finite or infinite $\lim_{x\to a}\frac{f'(x)}{g'(x)}$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

- why does it work for $x \to \infty$: Let $t = 1/x$, $t \to 0$, $f(x) = f(1/t)$, $g(x) = g(1/t)$,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0} \frac{f(\frac{1}{t})}{g(\frac{1}{t})} = \lim_{t\to 0} \frac{f'(\frac{1}{t})\left(-\frac{1}{t^2}\right)}{g'(\frac{1}{t})\left(-\frac{1}{t^2}\right)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

- why does it work for $f, g \to \pm\infty$. Suppose both limits $L = \lim_{x\to a} \frac{f(x)}{g(x)}$, $\lim_{x\to a} \frac{f'(x)}{g'(x)}$, exist, $L \neq 0$.

  $1/f$ and $1/g \to 0$,

$$\frac{1}{L} = \lim_{x\to a} \frac{g(x)}{f(x)} = \lim_{x\to a} \frac{1/f(x)}{1/g(x)} =$$

$$= \lim_{x\to a} \frac{-\left(1/f(x)\right)^2 f'(x)}{-\left(1/g(x)\right)^2 g'(x)} = \lim_{x\to a} \left(\frac{g}{f}\right)^2 \frac{f'(x)}{g'(x)},$$

$$L = \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

  (this is not a proof, just explanation).

- what if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ again is an indeterminate form? Apply l'Hospital's rule again, $\lim_{x\to a} \frac{f''(x)}{g''(x)}$, $\lim_{x\to a} \frac{f'''(x)}{g'''(x)}$...

Example, 0/0, finite limit

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = 1.$$

Example, 0/0, infinite limit

$$\lim_{x\to 1^+} \frac{\ln x}{(x-1)^2} = \lim_{x\to 1^+} \frac{(\ln x)'}{\left((x-1)^2\right)'} = \lim_{x\to 1^+} \frac{1/x}{2(x-1)} = \infty.$$

Example, 0/0, apply the rule twice

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

Example, ∞/∞

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{(x)'}{(e^x)'} = \lim_{x\to\infty} \frac{1}{e^x} = \lim_{x\to\infty} e^{-x} = 0.$$

Example, ∞/∞, apply the rule n times

$$\lim_{x\to\infty}\frac{x^n}{e^x}=\lim_{x\to\infty}\frac{nx^{n-1}}{e^x}=\lim_{x\to\infty}\frac{n\,(n-1)\,x^{n-2}}{e^x}=\ldots=\lim_{x\to\infty}\frac{n!}{e^x}=0.$$

Exponent grows faster than any power of x

Example, ∞/∞, $a>0$

$$\lim_{x\to\infty}\frac{\ln x}{x^a}=\lim_{x\to\infty}\frac{(\ln x)'}{(x^a)'}=\lim_{x\to\infty}\frac{1/x}{ax^{a-1}}=\lim_{x\to\infty}\frac{1}{ax^a}=0.$$

Logarithm grows slower than any power.

When the l'Hospital's rule **may not work**?

When differentiation does not simplify things...

$$\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

or makes things even worse...

$$\frac{d}{dx}\left(\frac{1}{\ln x}\right)=-\frac{1}{(\ln x)^2}\frac{1}{x}.$$

In such cases — transform the expression.

Other types of indeterminate forms:

$$0\cdot\infty,\ 0^0,\ 1^\infty,\ \infty^0,\ \infty-\infty.$$

They should be transformed to $0/0$ or ∞/∞, then l'Hospital's rule applied.

type $0\cdot\infty$

let $f\to 0$, $g\to\infty$ as $x\to a$. Then $1/f\to\infty$ or $1/g\to 0$

$$\lim_{x\to a}f(x)g(x)=\lim_{x\to a}\frac{f(x)}{1/g(x)}=\lim_{x\to a}\frac{g(x)}{1/f(x)}$$

Example: ∞/∞ works, 0/0 does not

$$\lim_{x\to\infty}e^{-x}\ln x=\lim_{x\to\infty}\frac{\ln x}{e^x}=\lim_{x\to\infty}\frac{1/x}{e^x}=0.$$

$$\lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty} \frac{e^{-x}}{1/\ln x} =$$

$$= \lim_{x\to\infty} \frac{-e^{-x}}{-\left(\frac{1}{\ln x}\right)^2 \frac{1}{x}} = \lim_{x\to\infty} \left(x\left(\ln x\right)^2 e^{-x}\right) =?$$

Example: 0/0 works, ∞/∞ does not

$$\lim_{x\to 0}\left(\sinh x \cot x\right) = \lim_{x\to 0}\frac{\sinh x}{\tan x} =$$

$$= \lim_{x\to 0}\frac{(\sinh x)'}{(\tan x)'} = \lim_{x\to 0}\frac{\cosh x}{\sec^2 x} = \frac{1}{1} = 1$$

$$\lim_{x\to 0}\left(\sinh x \cot x\right) = \lim_{x\to 0}\frac{\cot x}{1/\sinh x} =$$

$$= \lim_{x\to 0}\frac{-\csc^2 x}{-\frac{\cosh x}{\sinh^2 x}} = \lim_{x\to 0}\frac{\sinh^2 x}{\cosh x \sin^2 x} =?$$

type 1^∞, 0^0, ∞^0: $f(x)^{g(x)} = e^{g(x)\ln f(x)}$,

$g(x)\ln f(x)$ is of $0\cdot\infty$ type

Example: 0^0

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\ln x},$$

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0}\frac{\ln x}{1/x} = \lim_{x\to 0}\frac{1/x}{-1/x^2} = \lim_{x\to 0}(-x) = 0,$$

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\ln x} = e^{\lim_{x\to 0} x\ln x} = e^0 = 1.$$

Example: 1^∞

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)},$$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{1/(1+x)}{1} = 1,$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)} = e^{\lim_{x\to 0}\frac{1}{x}\ln(1+x)} = e$$

Example: ∞^0

$$\lim_{x\to\infty}(\ln x)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln\ln x},$$

$$\lim_{x\to\infty}\frac{\ln\ln x}{x} = \lim_{x\to\infty}\frac{(1/\ln x)(1/x)}{1} = \lim_{x\to\infty}\frac{1}{x\ln x} = 0,$$

$$\lim_{x\to\infty}(\ln x)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln\ln x} = e^{\lim_{x\to\infty}\frac{1}{x}\ln\ln x} = e^0 = 1$$

type $\infty - \infty$

Example. Bring to common denominator

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x-\sin x}{x\sin x} = \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x} =$$

$$= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = 0$$

Example. Multiply and divide by simplifying factor

$$\lim_{x\to\infty}\left(\sqrt{x^2+x} - \sqrt{x^2+1}\right) =$$

$$= \lim_{x\to\infty}\frac{\left(\sqrt{x^2+x} - \sqrt{x^2+1}\right)\left(\sqrt{x^2+x} + \sqrt{x^2+1}\right)}{\left(\sqrt{x^2+x} + \sqrt{x^2+1}\right)} =$$

$$= \lim_{x\to\infty}\frac{(x^2+x-x^2-1)}{\left(\sqrt{x^2+x} + \sqrt{x^2+1}\right)} = \lim_{x\to\infty}\frac{(x-1)}{\left(\sqrt{x^2+x} + \sqrt{x^2+1}\right)} =$$

$$= \lim_{x\to\infty}\frac{(x-1)/x}{\left(\frac{\sqrt{x^2+x}}{x} + \frac{\sqrt{x^2+1}}{x}\right)} = \lim_{x\to\infty}\frac{1-x^{-1}}{\left(\sqrt{1+x^{-1}} + \sqrt{1+x^{-2}}\right)} =$$

$$= \frac{1-\lim_{x\to\infty}x^{-1}}{\left(\sqrt{1+\lim_{x\to\infty}x^{-1}} + \sqrt{1+\lim_{x\to\infty}x^{-2}}\right)} = \frac{1}{2}.$$

Example. Factor and change variable

$$\lim_{x\to\infty}\left(xe^{1/x} - x\right) = \lim_{x\to\infty} x\left(e^{1/x} - 1\right)$$

Let $t = 1/x$, $t \to 0$ as $x \to \infty$. Substitute $x = 1/t$,

$$= \lim_{t\to 0}\frac{1}{t}\left(e^t - 1\right) = \lim_{t\to 0}\frac{e^t-1}{t} = \lim_{t\to 0}\frac{e^t}{1} = 1$$

Simplifying idea: factor out a part which has a finite limit

$$\lim_{x\to a} h(x) = H, \quad \lim_{x\to a} f(x) = 0, \quad \lim_{x\to a} g(x) = 0 \quad (\text{or} \quad \infty,\ \infty),$$

$H \neq 0$, $H \neq \infty$. Assume that $\lim_{x\to a} \frac{f(x)}{g(x)}$, exists, then we can use the properties of limits:

$$\lim_{x\to a} \frac{h(x)f(x)}{g(x)} = \lim_{x\to a} h(x)\frac{f(x)}{g(x)} = \lim_{x\to a} h(x) \lim_{x\to a} \frac{f(x)}{g(x)} = H \lim_{x\to a} \frac{f(x)}{g(x)}.$$

$$\lim_{x\to a} \frac{f(x)}{h(x)g(x)} = \lim_{x\to a} \frac{1}{h(x)}\frac{f(x)}{g(x)} = \lim_{x\to a} \frac{1}{h(x)} \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{1}{H} \lim_{x\to a} \frac{f(x)}{g(x)}.$$

No need to differentiate $h(x)$ in this case!

Example.

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{1-\cos^2 x}{\cos^2 x\, 3x^2} =$$

$$= \lim_{x\to 0} \frac{1}{3\cos^2 x}\frac{\sin^2 x}{x^2} == \lim_{x\to 0} \frac{1}{3\cos^2 x} \lim_{x\to 0} \frac{\sin^2 x}{x^2} =$$

$$= \frac{1}{3} \lim_{x\to 0} \frac{2\sin x \cos x}{2x} = \frac{1}{3} \lim_{x\to 0} \cos x \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{3}.$$