Improper integrals (Section 8.8)

Previous lecture

Improper integral

$$\int_a^b f(x)dx$$

(definte only!) — one which does not exist according to the definition due to infinite domain ($a = -\infty$, $b = \infty$, or both, type 1) of infinite discontinuity at some $x = c \in [a, b]$ (type 2). Nonetheless, some of improper integrals can be evaluated with the help of fundamental theorem of calculus — the antidefivative $F(x)$ has a finite limit at $+\infty$ or $-\infty$, or $F(x)$ does not have an infinite discontinuity at $x = c$.

How to work with improper integrals:

1) Represent I.I. as a **limit** of usual definite ("proper") integrals that exist (e.g. $b \to \infty$ or $a \to c$).

2) Do operations like substitutions or integration by parts **only** on the "proper" integrals.

3) **Evaluate** the definite integral (via F) or construct a **bound** for it (comparison theorem).

4) Determine whether the limit exists. If it does and is finite, then declare the improper integral **convergent** and take the limit for its value. If the limit does not exist or is infinite, then declare the improper integral **divergent**.

Improper integrals of type 1

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

if the definite integral under the limit exists for all $t > a$ or all $t < b$. If for some a both

$$\int_{-\infty}^a f(x)dx \qquad \text{and} \qquad \int_a^\infty f(x)dx$$

are convergent then it is possible to define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx,$$

which is convergent. If at least one of $\int_{-\infty}^a f(x)dx$, $\int_a^\infty f(x)dx$ is divergent, $\int_{-\infty}^\infty f(x)dx$ is divergent.

Example 1.

$$\int_0^{\infty} xe^{-x}dx = \lim_{t\to\infty}\int_0^t xe^{-x}dx$$

$$\int xe^{-x}dx = \int x\left(-e^{-x}\right)'dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C$$

$$\lim_{t\to\infty}\int_0^t xe^{-x}dx = \lim_{t\to\infty}\left(-xe^{-x} - e^{-x}\right)\Big]_0^t = \lim_{t\to\infty}\left(1-(t+1)\,e^{-t}\right) = .$$

$$= 1 - \lim_{t\to\infty}\frac{t+1}{e^t} = 1 - \lim_{t\to\infty}\frac{1}{e^t} = 1$$

There the improper integral is convergent and equals 1.

Example 2.

$$\int_{-\infty}^{0} e^{-x}dx = \lim_{t\to-\infty}\int_t^0 e^{-x}dx = \lim_{t\to-\infty}\left(-e^{-x}\right)\Big]_t^0 = \lim_{t\to-\infty}\left(e^{-t}-1\right) = \infty,$$

therefore the improper integral is divergent.

Example 3. Determine for which p values the integral

$$\int_1^{\infty}\frac{1}{x^p}dx$$

is convergent.

If $p \neq 1$,

$$\int_1^t \frac{1}{x^p}dx = \int_1^t x^{-p}dx = \frac{1}{1-p}x^{1-p}\Big]_1^t = \frac{1}{1-p}\left(t^{1-p}-1\right),$$

$$\int_1^{\infty}\frac{1}{x^p}dx = \lim_{t\to\infty}\int_1^t\frac{1}{x^p}dx = \frac{1}{1-p}\lim_{t\to\infty}t^{1-p} - \frac{1}{1-p}.$$

If $1-p>0$, $\lim_{t\to\infty}t^{1-p}=\infty$, and hence the integral diverges. If $1-p<0$ then $\lim_{t\to\infty}t^{1-p}=0$, and hence the integral converges. Now what happens if $p=1$,

$$\int_1^{\infty}\frac{1}{x}dx = \lim_{t\to\infty}\int_1^t\frac{1}{x}dx = \lim_{t\to\infty}\ln x\Big]_1^t = \lim_{t\to\infty}\ln t = \infty,$$

and the improper integral diverges too. So, the improper integral converges if $p>1$, and diverges if $p\le 1$.

Example 4. Convergent or divergent?

$$\int_{-\infty}^{\infty}\frac{x}{1+x^2}dx$$

Argument 1: divergent

$$\text{a)} \quad \int_0^\infty \frac{x}{1+x^2}dx = \lim_{t\to\infty}\int_0^t \frac{x}{1+x^2}dx = \lim_{t\to\infty}\frac{1}{2}\ln\left(1+x^2\right)\Big]_0^t = \infty.$$

Argument 2: convergent and equals to 0

$$\text{b)} \quad \int_{-\infty}^\infty \frac{x}{1+x^2}dx = \lim_{t\to\infty}\int_{-t}^t \frac{x}{1+x^2}dx = \lim_{t\to\infty} 0 = 0.$$

Which is correct? Remember the definition...

Improper integrals of type 2.

If $f(x)$ has an infinite discontinuity at $x = a$ and is continuous on $(a, b]$

$$\int_a^b f(x)dx = \lim_{t\to a}\int_t^b f(x)dx.$$

If $f(x)$ has an infinite discontinuity at $x = b$ and is continuous on $[a, b)$

$$\int_a^b f(x)dx = \lim_{t\to b}\int_a^t f(x)dx.$$

If $f(x)$ has an infinite discontinuity at $x = c \in (a, b)$ and is continuous on $[a, c) \cup (c, b]$, and **both**

$$\int_a^c f(x)dx \qquad and \qquad \int_c^b f(x)dx$$

are convergent, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

is convergent. If at least one of $\int_a^c f(x)dx$, $\int_c^b f(x)dx$ is divergent, then $\int_a^b f(x)dx$ is divergent.

Example 5. Determine for which p values the integral

$$\int_0^1 \frac{1}{x^p}dx$$

is convergent.

Let $p \neq 1$,

$$\int_0^1 \frac{1}{x^p}dx = \lim_{t\to 0}\int_t^1 x^{-p}dx = \frac{1}{1-p}\lim_{t\to 0}\left(x^{1-p}\right)\Big]_t^1 = \frac{1}{1-p}\lim_{t\to 0}\left(1-t^{1-p}\right).$$

If $p > 1$, then $\lim_{t\to 0} t^{1-p} = \infty$ and hence the improper integral diverges. If $p < 1$, $\lim_{t\to 0} t^{1-p} = 0$, and the improper integral diverges. If $p = 1$

$$\int_0^1 \frac{1}{x}dx = \lim_{t\to 0}\int_t^1 x^{-1}dx = \lim_{t\to 0} \ln x]_t^1 = \lim_{t\to 0}\ln t = -\infty,$$

so the improper integral diverges.

Answer: the improper integral converges for $p < 1$, and diverges for $p \geq 1$.

Example 6. Evaluate if possible

$$\int_{-1}^1 x^{-1}dx.$$

Here infinite discontinuity is in the middle, so we need to study two integrals, $\int_{-1}^0 x^{-1}dx$ and $\int_0^1 x^{-1}dx$. In the previous example we have shown that the second integral is divergent, hence $\int_{-1}^1 x^{-1}dx$ is divergent too.

Mixed type

Very rarely both infinities can coexist, e.g.

$$\int_0^\infty \frac{1}{\sqrt{x}\,(x+1)}dx.$$

Then the integral has to be split into two, one of type 2, and the other of type 1.

Comparison theorem, type 1 integrals

Here we shall consider only continuous $f(x) \geq 0$. In some cases it $F(x)$ cannot be expressed in elementary functions, and it is impossible to evaluate the integral. But it may be necessary to determine, whether the integral is convergent and divergent. Then it is necessary to find another continuous function $g(x)$ for which the antiderivative can be found and such that $0 \leq f(x) \leq g(x)$ (if it is necessary to proove convergence) or $0 \leq g(x) \leq f(x)$ (if it is necessary to prove divergence). We use the property of definite integrals that if

$$f(x) \leq g(x) \quad \text{then} \quad \int_a^b f(x)dx \leq \int_a^b g(x)dx, \quad a < b.$$

Comparison theorems:

- If $0 \leq f(x) \leq g(x)$ and $\int_a^\infty g(x)dx$ is convergent, then $\int_a^\infty f(x)dx$ is convergent too.

- If $0 \le g(x) \le f(x)$ and $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent too.

(Why the condition ≥ 0 is important? If we drop it, we may have situation when, for example, $g(x) \le f(x)$, $\int_a^\infty g(x)dx = -\infty$, that is divergent, but $\int_a^\infty f(x)dx$ may still be convergent. Say, $g = -x^{-1}$, $f = x^{-2}$. If $g(x) \ge 0$, then integral of g can not be negative, it can diverge only in the "positive side".)

Example 7. Determine, convergent or divergent is the integral

$$\int_1^\infty e^{-x^2} dx.$$

So $f(x) = e^{-x^2} > 0$. What should we try to prove here? We know that $\int_1^\infty e^{-x}dx$ is convergent, e^{-x^2} decreases even faster, so it is reasonable to conjecture that the integral is convergent. Let's try to prove convergence. We need $g(x) \ge f(x)$, for which integral is convergent. For $x \ge 1 \quad x^2 \ge x$, so $e^{-x^2} \le e^{-x}$. Therefore we can take $g(x) = e^{-x}$, and hence the integral $\int_1^\infty e^{-x^2}dx$ is convergent.

Example 8. Determine, convergent or divergent is the integral

$$\int_0^\infty \frac{x}{\sqrt{x^p+1}}dx$$

for $p = 6$ and $p = 4$.

First note that if the lower limit is inconvenient for our proof we can change it, because

$$\int_0^\infty \frac{x}{\sqrt{x^p+1}}dx = \lim_{t\to\infty}\int_0^t \frac{x}{\sqrt{x^p+1}}dx = \int_0^a \frac{x}{\sqrt{x^p+1}}dx + \lim_{t\to\infty}\int_a^t \frac{x}{\sqrt{x^p+1}}dx =$$

$$= \int_0^a \frac{x}{\sqrt{x^p+1}}dx + \int_a^\infty \frac{x}{\sqrt{x^p+1}}dx,$$

that is $\int_0^\infty \frac{x}{\sqrt{x^p+1}}dx$ and $\int_a^\infty \frac{x}{\sqrt{x^p+1}}dx$ converge or diverge together.

For our purpose it may be convenient to use comparison functions of the form $g(x) = x^{-q}$, for this reason the lower limit $a = 1$ is better.

What should we prove, convergence or divergence? For x big $\sqrt{x^p+1} = \sqrt{x^p\left(1+x^{-p}\right)} = x^{p/2}\sqrt{1+x^{-p}} \approx x^{p/2}$, and we can conjecture that our integral is close to

$$\int_1^\infty \frac{x}{x^{p/2}}dx = \int_1^\infty x^{1-p/2}dx,$$

that is it converges for $1 - p/2 < -1$, that is $p > 4$, and diverges for $p \le 4$. This means that for $p = 6$ it is reasonable to try to prove convergence, for $p = 4$ — divergence.

But in this specific example we can prove many things at once. Let us show that our integral indeed behaves like $\int_1^\infty x^{1-p/2} dx$. For $x \ge 1$ and $p > 0$ $x^p \ge 1$, hence we can write that $x^p < x^p + 1 \le x^p + x^p = 2x^p$. This means that

$$\frac{x}{\sqrt{x^p}} = x^{1-p/2} > \frac{x}{\sqrt{x^p + 1}} \ge \frac{x}{\sqrt{2x^p}} = \frac{1}{\sqrt{2}} \frac{x}{\sqrt{x^p}} = \frac{1}{\sqrt{2}} x^{1-p/2}.$$

1) Suppose that $\int_1^\infty x^{1-p/2} dx$ is convergent, then we can set $g(x) = x^{1-p/2} > f(x)$, and according to the comparison theorem $\int_1^\infty \frac{x}{\sqrt{x^p+1}} dx$ is convergent too.

2) Suppose that $\int_1^\infty x^{1-p/2} dx$ is divergent, then we can set $g(x) = \frac{1}{\sqrt{2}} x^{1-p/2} < f(x)$, and according to the comparison theorem $\int_1^\infty \frac{x}{\sqrt{x^p+1}} dx$ is divergent too.

Therefore our conjecture was right, and the integral is convergent for $p = 6$ and divergent for $p = 4$.