Integration by parts (Section 8.1)

Idea similar to the l'Hospital's rule: **simplify by** d/dx

Problem: evaluate

$$\int h(x)dx.$$

but integral is unknown and there is no good substitution

Try to represent $h(x)$ as a product: $h(x) = f(x)g'(x)$.

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

or

$$f(x)g'(x) = (f(x)g(x))' - f'(x)g(x)$$

$$\int h(x)dx = \int f(x)g'(x)dx = \int (f(x)g(x))'\,dx - \int f'(x)g(x)dx =$$

$$= f(x)g(x)\ [+C] - \int g(x)f'(x)dx.$$

We can omit C because $\int g(x)f'(x)dx$ already includes an arbitrary constant.

Integration by parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

or, since $g'dx = dg$, $f'dx = df$,

$$\int fdg = fg - \int gdf.$$

Typical situations when differentiation simplify:

$$f(x) = \ln x, \quad f' = \frac{1}{x}, \quad f(x) = \sin^{-1} x, \quad f' = \frac{1}{\sqrt{1-x^2}},$$

$$f(x) = \tan^{-1} x, \quad f'(x) = \frac{1}{1+x^2}$$

$$f(x) = x, \quad f' = 1$$

Example

$$\int xe^x dx$$

we can simplify if remove x, we can do it by differentiation, integral of e^x does not make situation more complex.

So $g'(x) = e^x$, $g(x) = e^x$, $f(x) = x$, $f'(x) = 1$,

$$\int xe^x dx = \int x\,(e^x)'\,dx = xe^x - \int e^x\,(x)'\,dx =$$

$$= xe^x - \int e^x dx = xe^x - e^x + C.$$

Example. Here $g'(x) = 1$.

$$\int \ln x\,dx$$

we can simplify by differenting $f(x) = \ln x$. But there is no $g'(x)$. Assume that it is here but $g' = 1$, $g(x) = x$.

$$\int \ln x\,dx = \int \ln x\ (x)'\,dx = x\ln x - \int x\,(\ln x)'\,dx =$$

$$= x\ln x - \int x\frac{1}{x}dx = x\ln x - x + C.$$

Example. Apply method twice

$$\int x^2 \sin x dx$$

To remove x^2 we need to differentiate twice. This is possible because sin does not become more complicated after integration. So first time $f(x) = x^2$, $g'(x) = \sin x$, $g(x) = -\cos x$,

$$\int x^2 \sin x dx = \int x^2\,(-\cos x)'\,dx = -x^2\cos x - \int (-\cos x)\left(x^2\right)'\,dx =$$

$$= -x^2\cos x + 2\int x\cos x\,dx$$

Apply the second time, $f = x$, $g' = \cos x$, $g = \sin x$,

$$\int x\cos x dx = \int x\,(\sin x)'\,dx = x\sin x - \int \sin x\,(x)'\,dx =$$

$$= x \sin x - \int \sin x dx = x \sin x + \cos x + C$$

Finally,

$$\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$