SCHREIER SINGULAR OPERATORS

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Abstract. In this paper we further investigate Schreier singular operators introduced in [ADST]. If \( X \) and \( Y \) are two Banach spaces, a bounded operator \( T : X \to Y \) is Schreier singular if for every \( \varepsilon > 0 \) and every basic sequence \( (x_n) \) in \( X \) there is a vector of the form \( x = \sum_{i=1}^{n} a_i x_{k_i} \) for some \( a_1, \ldots, a_n \in \mathbb{R} \) and \( n \leq k_1 < \cdots < k_n \) such that \( \|Tx\| < \varepsilon \|x\| \). It was shown in [ADST] that the class of Schreier singular operators on a reflexive space is stable under left and right multiplication by bounded operators. We show that this remains valid for non-reflexive spaces. We also present a characterisation of Schreier singular operators in terms of spreading models.

It was shown in [ADST] that if \( X \) has “few” spreading models then the product of any “sufficiently many” Schreier singular operators is compact. We show that the conclusion remains valid if there are no “long” chains of spreading models. Finally, we show that this cannot be extended to arbitrary Banach spaces by presenting an example of a finitely strictly singular operator which is not even polynomially compact.

0. Introduction

This paper continues the study of Schreier singular operators initiated in [ADST]. If \( X \) and \( Y \) are two Banach spaces and \( T \in L(X, Y) \), we say that \( T \) is Schreier singular if for every \( \varepsilon > 0 \) and every basic sequence \( (x_n) \) in \( X \) there is a vector of the form \( x = \sum_{i=1}^{n} a_i x_{k_i} \) for some \( a_1, \ldots, a_n \in \mathbb{R} \) and \( n \leq k_1 < \cdots < k_n \) such that \( \|Tx\| < \varepsilon \|x\| \). We also consider \( S_\xi \)-singular operators which generalize Schreier singular operators.

Among other things, it was shown in [ADST] that if \( X \) is reflexive then the class of all \( S_\xi \)-singular operators on \( X \) is stable under left and right multiplications by bounded operators. We show in Section 1 that reflexivity of \( X \) is not necessary.

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In Section 2, we present a characterization of Schreier singular operators in terms of operators they induce on spreading models.

Milman showed in [Mil70] that the product of any two strictly singular operators on \( L_p[0, 1] \) is compact. In [ADST] this was extended in the following way: if \( X \) has “few” spreading models then the product of any “sufficiently many” Schreier operators on \( X \) is compact. In Section 3 we extend this result to spaces where the set of spreading models may be infinite, but the maximal chain length is finite. Combined with Lomonosov’s theorem [Lom73], this implies that every Schreier singular operator on such a space has an invariant subspace. Finally, we show in Section 4 that this result fails for arbitrary Banach spaces by presenting a finitely strictly singular operator such that no polynomial of it is compact.

1. Definition and basic properties of \( S_\xi \)-singular operators

Suppose that \( X \) and \( Y \) are two Banach spaces. We denote by \( L(X, Y) \) the space of all bounded linear operators from \( X \) to \( Y \). Recall that \( T \in L(X, Y) \) is said to be \textit{strictly singular} if the restriction of \( T \) to any infinite-dimensional subspace is not an isomorphism. We say that \( T \) is \textit{finitely strictly singular} if for every \( \varepsilon > 0 \) there exists \( n \in \mathbb{N} \) such that for every subspace \( W \subseteq X \) with \( \dim W \geq n \) there exists \( z \in W \) such that \( \|Tz\| < \varepsilon \|z\| \). See, e.g., [LT77, Mil70, Plic04, SSTT] for more details on these classes of operators.

We are going to study more classes of operators which sit between finitely strictly singular and strictly singular operators. We start by recalling the definition of \textit{Schreier families} \( S_\xi \) which were originally defined in [AA92]. We define \( S_\xi \) inductively for every ordinal \( \xi < \omega_1 \).

\[
S_0 = \{\{n\} \mid n \in \mathbb{N}\} \cup \{\emptyset\}.
\]

After defining \( S_\xi \) for some \( \xi < \omega_1 \), set

\[
S_{\xi +1} = \bigcup_{i=1}^n F_i \mid n \in \mathbb{N}, n \leq F_1 < \cdots < F_n, F_i \in S_\xi\}.
\]

Here by \( A < B \) where \( A \) and \( B \) are two finite subsets of \( \mathbb{N} \) we mean \( \max A < \min B \). Similarly, \( n \leq A \) means \( n \leq \min A \). We assume that \( \emptyset < F \) and \( F < \emptyset \) for any non-empty finite set \( F \subseteq \mathbb{N} \). If \( \xi < \omega_1 \) is a limit ordinal and \( S_\alpha \) has been defined for all \( \alpha < \xi \) then fix a sequence \( \xi_n \nearrow \xi \) and define

\[
S_\xi = \{F \mid n \leq F \text{ and } F \in S_{\xi_n} \text{ for some } n \in \mathbb{N}\}.
\]

Each class \( S_\xi \) is spreading, i.e., if \( \{n_1, \ldots, n_k\} \in S_\xi \) and \( n_i \leq m_i \) for \( i = 1, \ldots, k \) then \( \{m_1, \ldots, m_k\} \in S_\xi \). It is obvious that \( S_\xi \subseteq S_{\xi+1} \). We should warn the reader, however, that \( \xi < \zeta \) doesn’t generally imply \( S_\xi \subseteq S_\zeta \).
For a sequence \((x_n)\) in a Banach space and \(A \subseteq \mathbb{N}\) we will write \([x_i]_{i \in A}\) for the closed linear span of \(\{x_i\}_{i \in A}\). Let \(X\) and \(Y\) be two Banach spaces, \(T \in L(X, Y)\), and \(\xi < \omega_1\). Following [ADST] we will say that \(T\) is \(\xi\)-\textbf{singular} and write \(T \in SS_\xi(X, Y)\) if for every \(\varepsilon > 0\) and every basic sequence \((x_n)\) in \(X\) there exist a set \(F \in S_\xi\) and a vector \(z \in [x_i]_{i \in F}\) such that \(\|Tz\| < \varepsilon \|z\|\). If \(X = Y\) then we write \(T \in SS_\xi(X)\). If \(\xi = 1\) we will say that \(T\) is \(\textbf{Schreier singular}\). It is easy to see that for every \(0 < \xi < \omega_1\) we have the following hierarchy of properties:

\[
\text{compact} \Rightarrow \text{finitely strictly singular} \Rightarrow S_\xi\text{-singular} \Rightarrow \text{strictly singular}.
\]

**Remark 1.1.** It was pointed out in [ADST] that \(T\) is \(S_\xi\)-singular iff for every normalized basic sequence \((x_n)\) and \(\varepsilon > 0\) there exist a subsequence \((x_{n_k})\), \(F \in S_\xi\) and \(w \in [x_{n_k}]_{k \in F}\) such that \(\|T w\| < \varepsilon \|w\|\).

It is well known that strictly singular, finitely strictly singular, and compact operators form norm closed ideals. It was shown in [ADST] that \(SS_\xi(X, Y)\) is norm-closed and that if \(X\) is reflexive then \(S_\xi(X)\) is stable under right and left multiplications by operators in \(L(X)\). We will show that reflexivity is not necessary here.

We will make use of the following standard lemma, see e.g., [ADST, Remark 2.3].

**Lemma 1.2.** Let \((x_n)\) be a bounded sequence in a Banach space \(X\). Then there is a subsequence \((x_{n_k})\) such that one of the following conditions hold.

\begin{enumerate}[(i)]
  \item \((x_{n_k})\) converges;
  \item \((x_{n_k})\) is equivalent to the unit vector basis of \(\ell_1\);
  \item The difference sequence \((d_k)\) defined by \(d_k = x_{n_{k+1}} - x_{n_k}\) has a seminormalized weakly null basic subsequence. Moreover, if \(X\) has a basis then this subsequence can be chosen to be equivalent to a block sequence of the basis.
\end{enumerate}

We will also use the following technical lemma.

**Lemma 1.3.** Suppose that \(\xi < \omega_1\), \(A \subseteq S_\xi\), and put \(A^{x^2} = \{2i, 2i + 2 \mid i \in A\}\). Then \(A^{x^2}\) is also in \(S_\xi\).

**Proof.** The proof is by induction on \(\xi\). For \(\xi = 1\), if \(A \in S_1\) then \(|A| \leq \min A\). But then \(|A^{x^2}| \leq 2|A| \leq 2 \min A = \min A^{x^2}\), so that \(A^{x^2} \in S_\xi\). Suppose that we have already proved the statement for \(\xi\), and let \(A \in S_{\xi+1}\). Then

\[
A = \bigcup_{i=1}^n F_i \text{ where } n \in \mathbb{N}, \ n \leq F_1 < \cdots < F_n, \text{ and } F_i \in S_\xi \text{ for each } i.
\]
It follows that
\[ A^{\times 2} = \bigcup_{i=1}^{n} F_i^{\times 2}, \] and \( F_i^{\times 2} \in S_\xi \) for each \( i \) by the induction hypothesis.

Let \( G_1 = F_1^{\times 2}, \) \( G_i = F_i^{\times 2}\setminus F_{i-1}^{\times 2} \) for \( 2 \leq i \leq n, \) then
\[ A^{\times 2} = \bigcup_{i=1}^{n} G_i, \quad G_i \in S_\xi \] for each \( i, \) and \( n < 2n \leq G_1 < \cdots < G_n, \)
so that \( A^{\times 2} \in S_{\xi+1}. \) Finally, suppose that \( \xi \) is a limit ordinal and \( A \in S_\xi. \) Then \( A \in S_{\xi+\alpha} \) and \( A \leq A \) for some \( n \in \mathbb{N}. \) It follows from the induction hypothesis that \( n < 2n \leq A^{\times 2} \in S_{\xi+\alpha}, \) so that \( A^{\times 2} \in S_\xi. \)

\textbf{Theorem 1.4.} Suppose that \( X \) and \( Y \) are two Banach spaces and \( 1 \leq \xi < \omega_1. \) Then if \( T \in \mathcal{SS}_\xi(X,Y), A \in L(Y,V), \) and \( B \in L(U,X) \) for some Banach spaces \( U \) and \( V, \) then \( ATB \in \mathcal{SS}_\xi(U,V). \)

\textbf{Proof.} Assume that \( A \) and \( B \) are non-zero, as otherwise the statement is trivial.

Let \( (x_n) \) be a basic sequence in \( X, \) and \( \varepsilon > 0. \) As \( A \neq 0, \) there exists \( F \in S_\xi \) and \( z \in [x_n]_{n \in F} \) such that \( ||Tz|| < \frac{1}{\varepsilon} ||z||. \) Thus \( ||ATz|| \leq ||A|| ||Tz|| < ||A|| \frac{1}{\varepsilon} ||z|| = \varepsilon ||z||, \) so that \( AT \in \mathcal{SS}_\xi(X,V), \) hence \( \mathcal{SS}_\xi \) is stable under left multiplication.

Next, we will show that that \( TB \in \mathcal{SS}_\xi(U,Y), \) so that \( \mathcal{SS}_\xi(X,Y). \) Again, let \( (x_n) \) be a basic sequence in \( U \) and \( \varepsilon > 0. \) We can assume by Remark 1.1 that \( (x_n) \) is normalized. First, suppose that \( (Bx_n) \) has a basic subsequence, say, \( (Bx_{n_k}) \).

Then there exists \( G \in S_\xi \) and \( y = \sum_{k \in G} \alpha_k Bx_{n_k} \) such that \( ||Ty|| \leq \frac{\varepsilon}{||B||} ||y||. \) Put \( x = \sum_{k \in G} \alpha_k x_{n_k}, \) then \( Bx = y \) and \( x \in [x_i]_{i \in F} \) where \( F = \{n_k \mid k \in G\}, \) so that \( F \in S_\xi. \) It follows that
\[ ||TxB|| = ||Ty|| \leq \frac{\varepsilon}{||B||} ||y|| \leq \varepsilon ||x||. \]

Now suppose that \( (Bx_n) \) and, therefore, \( (Bx_{2n}) \), has no basic subsequences. Lemma 1.2 yields that after passing to a further subsequence of \( (x_n) \) we have that either \( (y_n) \to 0 \) or \( (y_n) \) has a basic subsequence, where \( y_n = Bx_{2n+2} - Bx_{2n}. \)

Suppose that \( (y_{n_k}) \) is basic. Then there exists \( G \in S_\xi \) and \( z = \sum_{k \in G} \alpha_k y_{n_k} \) such that \( ||Tz|| \leq \frac{\varepsilon}{||B||} ||z||. \) Put \( x = \sum_{k \in G} \alpha_k (x_{2n+2} - x_{2n_k}), \) then \( z = Bx \) and \( x \in [x_i]_{i \in F^{\times 2}} \) where \( F = \{n_k \mid k \in G\}. \) It follows from Lemma 1.3 that \( F^{\times 2} \in S_\xi \) and
\[ ||TxB|| = ||Tz|| \leq \frac{\varepsilon}{||B||} ||z|| \leq \varepsilon ||x||. \]

Finally, suppose that \( y_n \to 0. \) Then we can find \( m \in \mathbb{N} \) such that \( \{m\} \in S_\xi \) and
\[ ||y_m|| < \frac{\varepsilon}{C||T||}, \] where \( C \) is the basis constant of \( (x_n). \) Put \( x = x_{2m+2} - x_{2m}. \) Note
that \( \{2m, 2m+2\} = \{m\}^{\times 2} \in S_\xi \) and
\[
\|TBx\| = \|Ty_m\| \leq \frac{\varepsilon}{C} \leq \varepsilon \|x_{2m+2} - x_{2m}\| = \varepsilon\|x\|.
\]

This gives a short proof of [ADST, Remark 2.8].

Corollary 1.5. [ADST] Suppose that \( X \) and \( Y \) are Banach spaces, \( 1 \leq \xi < \omega_1 \) and \( T \in SS_\xi(X, Y) \). Let \( \tilde{T} \in L(X \oplus Y) \) given by \( (x, y) \mapsto (0, Tx) \), that is, \( \tilde{T} = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \). Then \( \tilde{T} \in SS_\xi(X \oplus Y) \).

Proof. Note that \( \tilde{T} = JTP \) where \( P: X \oplus Y \to X \) and \( J: Y \to X \oplus Y \) given by \( P(x, y) = x \) and \( Jy = (0, y) \). The result now follows from Theorem 1.4.

Remark 1.6. It is not known whether \( SS_\xi(X) \) is an algebraic ideal in \( L(X) \). Theorem 1.4 guarantees that \( SS_\xi(X) \) is stable under left and right multiplication by operators in \( L(X) \). It is not known whether \( SS_\xi(X) \) is closed under addition. It was shown in [ADST], however, that if \( S \) and \( T \) are in \( SS_\xi(X) \) then \( S + T \in SS_{\xi+1}(X) \).

Remark 1.7. It is known (see, for instance, the remark before Lemma 5 in [Plic04]) that there exists a finitely strictly singular operator whose adjoint is not strictly singular. This already shows that the class of \( S_\xi \)-singular operators is not stable under conjugation.

2. Schreier singular operators and spreading models

Recall that given two sequences \( (x_n) \) and \( (\tilde{x}_n) \) in Banach spaces \( X \) and \( Y \) respectively, we say that \( (\tilde{x}_n) \) is a spreading model for \( (x_n) \) if there exists a sequence of reals \( (\varepsilon_n) \) such that \( \varepsilon_n \searrow 0 \) and
\[
\left| \sum_{i=1}^{n} a_i x_k \right| - \left| \sum_{i=1}^{n} a_i \tilde{x}_i \right| < \varepsilon_n
\]
whenever \( n \leq k_1 < \cdots < k_n \) and \( a_1, \ldots, a_n \in [-1, 1] \).

We will often use the following classical result.

Theorem 2.1 ([BL84, BS74, BS75]). If \( X \) is a Banach space then every bounded sequence in \( X \) with no convergent subsequences has a subsequence with a (unique) spreading model.
We will use the following standard lemma, see e.g., [ADST], Lemma 3.1. By $a \approx b$ we mean $\frac{1}{K} a \leq b \leq K a$.

**Lemma 2.2.** Suppose that $(x_n)$ is a seminormalized basic sequence with a spreading model $(\tilde{x}_n)$. Then for every $K > 1$ there exists $n_0 \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^{n} a_i \tilde{x}_i \right\| \sim K \left\| \sum_{i=1}^{n} a_i x_i \right\|$$

whenever $n_0 \leq n \leq k_1 < \cdots < k_n$ and $a_1, \ldots, a_n \in \mathbb{R}$.

Suppose that $X$ and $Y$ are Banach spaces, $T \in L(X, Y)$, and $(x_n)$ is a sequence in $X$ with a spreading model $(\tilde{x}_n)$. We say that $T$ and $(x_n)$ **induce** an operator $\tilde{T}$ if

- either $(T x_n)$ converges, then we define $\tilde{T} x_n = \lim_k T x_k$; or
- $(T x_n)$ has a spreading model $(\tilde{y}_n)$, then we define $\tilde{T} x_n = \tilde{y}_n$.

This defines a linear operator on $\text{span}\{\tilde{x}_n \mid n \geq 1\}$. To extend it to $[\tilde{x}_n]_{n \in \mathbb{N}}$, we will show that $\tilde{T}$ is bounded. Take an arbitrary $\tilde{x} = \sum_{i=1}^{N} a_i \tilde{x}_i$ with $\|\tilde{x}\| = 1$. We consider two cases.

Suppose first that $(T x_n)$ converges, i.e., $\tilde{T} x_n = \lim_k T x_k$. Denote $y = \lim_k T x_k$. Since $T x_n \to y$, there is $M \in \mathbb{N}$ such that $\|T x_n - y\| < \frac{1}{N \max|a_i|}$ whenever $n \geq M$. Let $n_0$ be as in Lemma 2.2 with $K = 2$. Suppose that $\max\{n_0, M\} \leq n_1 < \cdots < n_N$, then

$$\|\tilde{T} \tilde{x}\| = \left\| \sum_{i=1}^{N} a_i y \right\| \leq \left\| \sum_{i=1}^{N} a_i y - \sum_{i=1}^{N} a_i T x_n \right\| + \left\| \sum_{i=1}^{N} a_i T x_n \right\|$$

$$\leq \sum_{i=1}^{N} |a_i| \|y - T x_n\| + \|T\| \sum_{i=1}^{N} a_i x_n \| \leq 1 + 2\|T\|.$$

Suppose now that $(\tilde{y}_n)$ is a spreading model for $(T x_n)$, and $\tilde{T} x_n = \tilde{y}_n$ for every $n$. By Lemma 2.2 we can find $n_0$ such that

$$\left\| \sum_{i=1}^{N} a_i \tilde{x}_i \right\| \approx 2 \left\| \sum_{i=1}^{N} a_i x_n \right\| \quad \text{and} \quad \left\| \sum_{i=1}^{N} a_i \tilde{y}_i \right\| \approx 2 \left\| \sum_{i=1}^{N} a_i T x_n \right\|$$

whenever $\max\{n_0, N\} \leq n_1 < \cdots < n_N$. It follows that

$$\|\tilde{T} \tilde{x}\| = \left\| \sum_{i=1}^{N} a_i \tilde{y}_i \right\| \leq 2 \left\| \sum_{i=1}^{N} a_i T x_n \right\| \leq 2\|T\| \left\| \sum_{i=1}^{N} a_i x_n \right\| \leq 4\|T\|.$$
It follows now that \( \tilde{T} \) can be extended to \([\tilde{x}_n]\). From now on we will consider \( \tilde{T} \) as an operator on \([\tilde{x}_n]\).

**Remark 2.3.** It should be clear to the reader that an induced operator \( \tilde{T} \) need not exist for every sequence \((x_n)\) in \(X\) and operator \(T \in \mathcal{L}(X,Y)\). However, it follows immediately from Theorem 2.1 that if \((x_n)\) has no convergent subsequences then there is a subsequence \((z_n)\) of \((x_n)\) with a spreading model such that \((Tz_n)\) either converges or has its own spreading model. In either case, \((z_n)\) and \(T\) induce an operator \(\tilde{T}\) on the closed span of spreading model of \((z_n)\).

**Theorem 2.4.** Let \(T \in \mathcal{L}(X,Y)\). Then \(T\) is Schreier singular if and only if every operator \(\tilde{T}\) induced by \(T\) and a seminormalized basic sequence in \(X\) is not an isomorphism.

**Proof.** Suppose that \(T\) is Schreier singular. Let \((x_n)\) be a seminormalized basic sequence in \(X\) with spreading model \((\tilde{x}_n)\) such that \((x_n)\) and \(T\) induce \(\tilde{T}\).

If \((Tx_n)\) converges to some element \(y \in X\) then \(\tilde{T}\tilde{x}_n = y\) for every \(n \in \mathbb{N}\), hence \(\tilde{T}\) is of rank one, hence not bounded below.

Now suppose that \((\tilde{y}_n)\) be the spreading models for \((Tx_n)\) and \(\tilde{T}: \tilde{x}_n \mapsto \tilde{y}_n\). By Lemma 2.2 we can find \(N\) such that

\[
\left\| \sum_{i=1}^{n} a_i x_{k_i} \right\| \geq \frac{1}{2} \left\| \sum_{i=1}^{n} a_i \tilde{x}_i \right\| \quad \text{and} \quad \left\| \sum_{i=1}^{n} a_i T x_{k_i} \right\| \geq \frac{1}{2} \left\| \sum_{i=1}^{n} a_i \tilde{y}_i \right\|
\]

whenever \(N \leq n \leq k_1 < \cdots < k_n\) and \(a_1, \ldots, a_n \in \mathbb{R}\). Fix \(\varepsilon > 0\). Since \(T \in \mathcal{SS}_1(X)\), there exists \(F \in S_1\) with \(\min F \geq N\) and \(x \in [x_i]_{i \in F}\) with \(\|Tx\| < \varepsilon\|x\|\). Then \(F = \{k_1, \ldots, k_n\}\) with \(N \leq n \leq k_1 < \cdots < k_n\) and \(x = \sum_{i=1}^{n} a_i x_{k_i}\) for some \(a_1, \ldots, a_n \in \mathbb{R}\). Put \(\tilde{x} = \sum_{i=1}^{n} a_i \tilde{x}_i\), then

\[
\left\| \tilde{T}\tilde{x} \right\| = \left\| \sum_{i=1}^{n} a_i \tilde{y}_i \right\| \leq 2 \left\| \sum_{i=1}^{n} a_i T x_{k_i} \right\| \leq 2\varepsilon \left\| \sum_{i=1}^{n} a_i x_{k_i} \right\| \leq 4\varepsilon \left\| \sum_{i=1}^{n} a_i \tilde{x}_i \right\| = 4\varepsilon\|\tilde{x}\|.
\]

It follows that \(T\) is not bounded below.

To prove the converse, let \((x_n)\) be a normalized basic sequence. By Remark 2.3, there is a subsequence \((z_n)\) of \((x_n)\) such that \((z_n)\) has a spreading model and \((z_n)\) and \(T\) induce an operator \(\tilde{T}\). Fix \(\varepsilon > 0\). By Remark 1.1, it suffices to find \(z \in [z_n]_{n \in F}\) for some \(F \in S_1\) satisfying \(\|Tz\| < \varepsilon\|z\|\). Let \((\tilde{z}_n)\) be the spreading models of \((z_n)\), and let \(C\) be the basis constant of \((z_n)\).

If \((Tz_n)\) converges then there exist \(m, n \geq 3\) such that \(\|Tz_m - Tz_n\| < \varepsilon/C\). Then \((m,n) \in S_1\) and \(z = z_m - z_n\) satisfies \(\|Tz\| \leq \varepsilon/C \leq \varepsilon\|z\|\).

Suppose now that \((Tz_n)\) has a spreading models of \((\tilde{y}_n)\). By assumption, there exists \(\tilde{z}\) in \([\tilde{z}_n]\) such that \(\|T\tilde{z}\| < \varepsilon\|\tilde{z}\|\). Since \(\tilde{T}\) is bounded, we can find such
\( \tilde{z} \) in \( \text{span}\{\tilde{z}_n\} \), i.e., \( \tilde{z} = \sum_{i=1}^{N} a_i \tilde{z}_n \) for some \( N \in \mathbb{N} \) and \( a_1, \ldots, a_N \in \mathbb{R} \). By Lemma 2.2, we can find \( k_1, \ldots, k_N \) such that \( N \leq k_1 < \cdots < k_N \) and

\[
\left\| \sum_{i=1}^{N} a_i \tilde{z}_{k_i} \right\| \geq 2 \left\| \sum_{i=1}^{N} a_i \tilde{z}_i \right\| \quad \text{and} \quad \left\| \sum_{i=1}^{N} a_i T \tilde{z}_{k_i} \right\| \geq 2 \left\| \sum_{i=1}^{N} a_i \tilde{y}_i \right\|.
\]

Put \( z = \sum_{i=1}^{N} a_i z_{k_i} \). Then \( z \in [z_n]_{n \in F} \) where \( F = \{k_1, \ldots, k_N\} \in \mathcal{S}_1 \), and

\[
\|Tz\| \leq 2 \left\| \sum_{i=1}^{N} a_i \tilde{y}_i \right\| = 2 \|\tilde{T}z\| \leq 2 \varepsilon \|z\|.
\]

\[\square\]

### 3. Products of \( SS_\xi \)-operators

It was shown in [ADST] that if \( X \) has finitely many non-equivalent spreading models or Schreier-spreading sequences, then the product of sufficiently many strictly singular or \( S_\xi \)-singular operators is compact. Using techniques similar to those in [ADST], we extend these results to the case when the set of equivalence classes of spreading models or Schreier spreading sequences is infinite but contains no infinite chains. This approach was motivated by the studies of the order structure of the set of spreading models in a given Banach space in [AOST05] and [DOS].

Let \( X \) be a Banach space. Following [ADST], we say that a seminormalized basic sequence \((x_n)\) in \( X \) is \textbf{Schreier spreading}, if there exists \( 1 \leq C < \infty \) such that for every \( F \in \mathcal{S}_1 \), increasing sequence \( n_1 < n_2 < \cdots \) of positive integers, and scalars \((a_i)_{i \in F}\) we have

\[
\left\| \sum_{i \in F} a_i x_i \right\| \geq C \left\| \sum_{i \in F} a_i x_{n_i} \right\|.
\]

We denote by \( \text{SP}_{1,w}(X) \) the set of seminormalized weakly null basic sequences in \( X \) which are Schreier spreading.

Recall that two basic sequences \((x_n)\) and \((y_n)\) in Banach spaces \( X \) and \( Y \) respectively are said to be \textbf{equivalent} and write \((x_n) \approx (y_n)\) if \( \sum_{n=1}^{\infty} a_n x_n \) converges if and only if \( \sum_{n=1}^{\infty} a_n y_n \) converges. Equivalently, if the operator \( T: [x_n] \to [y_n], T: x_n \mapsto y_n \), is a surjective isomorphism. We will write \((x_n) \preceq (y_n)\) if the convergence of \( \sum_{n=1}^{\infty} a_n x_n \) implies the convergence of \( \sum_{n=1}^{\infty} a_n y_n \). This defines a partial order relation on the set of \( \approx \)-equivalence classes of basic sequences.
Remark 3.1. Observe that \((x_n) \preceq (y_n)\) if and only if there is a constant 
\(C > 0\) such that for every sequence \((a_n)\) of reals and for every 
\(N \in \mathbb{N}\) we have 
\[
\|\sum_{i=1}^{N} a_i x_i\| \leq C \|\sum_{i=1}^{N} a_i y_i\|.
\]
Indeed, the “if” implication is obvious. The converse follows easily from 
the Closed Graph Theorem. Indeed, let the operator \(T\) : \([y_n] \to [x_n]\) 
be defined by 
\[
T(\sum_{i=1}^{\infty} a_i y_i) = \sum_{i=1}^{\infty} a_i x_i.
\]
Since the convergence of \(\sum_{i=1}^{\infty} a_i y_i\) implies the convergence 
of \(\sum_{i=1}^{\infty} a_i x_i\), this operator is defined for every 
\(y \in [y_n]\). The conclusion will hold if we prove that \(T\) is bounded.

Let \((z_n)\) be a sequence in \([y_n]\) such that \(z_n \to 0\) and 
\(Tz_n \to w\) for some \(w \in [x_n]\).
Write \(z_n = \sum_{i=1}^{\infty} a_i^{(n)} y_i\). Since \(z_n \to 0\) we have that 
\(a_i^{(n)} \to 0\) as \(n \to \infty\), because 
\((y_n)\) is a basic sequence.

Let \(w = \sum_{i=1}^{\infty} b_i x_i\), then, since \((x_n)\) is a basic sequence, we also have 
\(a_i^{(n)} \to b_i\) as \(n \to \infty\). Hence, \(b_i = 0\) for every \(i \in \mathbb{N}\), so that 
\(w = 0\). Now the Closed Graph theorem yields that \(T\) is a bounded operator.

The following lemma is a part of the proof of Theorem 4.1 in [ADST].

Lemma 3.2. Let \(X\) be a Banach space and \(N \geq 0\). Let 
\(T_1, \ldots, T_N \in L(X)\) and 
\(S \in SS(X)\). If \(T_N T_{N-1} \ldots T_1 S\) is not compact or \(\ell_1 \not\hookrightarrow X\) and 
\(T_N T_{N-1} \ldots T_1\) is not compact then there are \(N + 1\) sequences 
\((x_n^{(1)}), \ldots, (x_n^{(N+1)}) \in SP_{1,w}(X)\) such that 
\(x_n^{(k)} = T_{k-1} \ldots T_1 x_n^{(1)}\), for each \(k \leq N + 1, n \in \mathbb{N}\).

In the rest of this section we present several extensions of Theorem 4.1 of 
[ADST]. Put \(SP_{1,w,\approx}(X) = SP_{1,w}(X)/\approx\).

Theorem 3.3. Let \(X\) be a Banach space and \(N \geq 0\) such that 
\((SP_{1,w,\approx}(X), \preceq)\) contains no chains of length greater than \(N\). Then the product of any \((N + 1)\) 
strictly singular operators on \(X\) is compact. Moreover, if \(\ell_1\) does not isomorphically 
embed in \(X\) then the product of any \(N\) strictly singular operators on \(X\) is 
compact.

Proof. When \(N = 0\), we just simply have that \(SP_{1,w} = \emptyset\), and this case 
was considered in [ADST].

Assume that \(N \neq 0\). By Lemma 3.2, there are \(N + 1\) sequences 
\((x_n^{(1)}), \ldots, (x_n^{(N+1)}) \in SP_{1,w}(X)\) such that 
\(x_n^{(k)} = T_{k-1} \ldots T_1 x_n^{(1)}\), for each \(k \leq N + 1, n \in \mathbb{N}\). It is easy 
to see that 
\((x_n^{(N+1)}) \prec (x_n^{(N)}) \prec \cdots \prec (x_n^{(1)})\) in \(SP_{1,w,\approx}\) because 
\(T_1, \ldots, T_N\) are strictly singular. This gives a chain of length \(N + 1\) in 
\(SP_{1,w,\approx}\), contradiction. \(\square\)

The following equivalence relations on the set \(SP_{1,w}(X)\) was introduced in [ADST].
**Definition 1.** Let $X$ be a Banach space and $1 \leq \xi < \omega_1$. Suppose that $(x_n)$ and $(y_n)$ are two Schreier sequences in $X$. For $K \geq 1$ we write $(x_n) \approx^K (y_n)$ if for every $F \in S_\xi$ and scalars $(a_i)_{i \in F}$ we have
\[
\left\| \sum_{i \in F} a_i x_i \right\| \approx^K \left\| \sum_{i \in F} a_i y_i \right\|.
\]

We write $(x_n) \approx^\xi (y_n)$ if $(x_n) \approx^K (y_n)$ for some $K \geq 1$. Similarly, we write $(x_n) \preceq^\xi (y_n)$ if there is a constant $K > 0$ such that
\[
\left\| \sum_{i \in F} a_i x_i \right\| \leq K \left\| \sum_{i \in F} a_i y_i \right\|
\]
for every $F \in S_\xi$ and every sequence $(a_n)$ of reals. Put $SP_{1,w,\xi}(X) = SP_{1,w}(X) / \approx^\xi$. It is easy to see that “$\preceq^\xi$” induces a partial order relation on $SP_{1,w,\xi}(X)$, which we will still denote “$\preceq^\xi$.”

**Remark 3.4.** Suppose that $T \in L(X)$ and $(x_n)$ is in $SP_{1,w}(X)$ such that $(Tx_n)$ is also in $SP_{1,w}(X)$. Then $(Tx_n) \preceq^\xi (x_n)$ as $\| \sum_{i \in F} a_i Tx_i \| \leq \| T \| \sum_{i \in F} a_i x_i \|$ for every $F \in S_\xi$. Furthermore, if $T \in SS_\xi(X)$ then $(Tx_n) \preceq^\xi (x_n)$ as otherwise there would exist $C > 0$ such that $\| \sum_{i \in F} a_i Tx_i \| \geq C \| \sum_{i \in F} a_i x_i \|$ for every $F \in S_\xi$ and real $(a_n)$, which would contradict $T$ being $S_\xi$-singular.

This remark leads to the following theorem.

**Theorem 3.5.** Let $X$ be a Banach space, $1 \leq \xi < \omega_1$, and $N \geq 0$ be such that $(SP_{1,w,\xi}, \preceq^\xi)$ contains no chains of length greater than $N$. Then for any $T_1, \ldots, T_N \in SS_\xi(X)$ and every strictly singular operator $S$ the composition $T_N T_{N-1} \ldots T_1 S$ is compact. Moreover, if $\ell_1$ does not isomorphically embed in $X$ then $T_N T_{N-1} \ldots T_1$ is compact.

**Proof.** The proof is analogous to that of Theorem 3.3. \qed

Sometimes it may be more convenient to work with spreading models over $X$ instead of sequences inside of $X$. Recall that $SP_{w}(X)$ stands for the set of all spreading models of weakly null seminormalized basic sequences in $X$. Any spreading model is in fact a semi-normalized basic sequence. We will often identify $\approx$-equivalent spreading models: put $SP_{w,\approx}(X) = SP_{w}(X) / \approx$.

Theorem 2.1 together with Lemma 2.2 imply that every seminormalized basic sequence has a Schreier spreading subsequence with a spreading model.
**Lemma 3.6.** Suppose that \((x_n)\) and \((y_n)\) are two Schreier spreading sequences in \(SP_{w,1}(X)\) with spreading models \((\tilde{x}_n)\) and \((\tilde{y}_n)\), respectively. Then \((\tilde{x}_n) \preceq (\tilde{y}_n)\) if and only if \((x_n) \preceq_1 (y_n)\).

**Proof.** Let \((\tilde{x}_n) \preceq (\tilde{y}_n)\). Then there is \(0 < C < \infty\) such that

\[
\left\| \sum_{i=1}^{n} a_i \tilde{x}_i \right\| \leq C \left\| \sum_{i=1}^{n} a_i \tilde{y}_i \right\|
\]

for every \(a_i \in \mathbb{R}\). By Lemma 2.2, there is \(n_0 \in \mathbb{N}\) such that

\[
\left\| \sum_{i=1}^{n} a_i x_i \right\| \leq C \left\| \sum_{i=1}^{n} a_i x_i \right\| \quad \text{and} \quad \left\| \sum_{i=1}^{n} a_i y_i \right\| \leq C \left\| \sum_{i=1}^{n} a_i y_i \right\|
\]

whenever \(n_0 \leq n \leq k_1 < \cdots < k_n\) and \(a_1, \ldots, a_n \in \mathbb{R}\).

Since \((x_i)\) and \((y_i)\) are Schreier spreading, we have

\[
(x_i) \approx_1 (x_{n_0+i}) \quad \text{and} \quad (y_i) \approx_1 (y_{n_0+i})
\]

for some \(K\).

Let \(F = \{n_1, \ldots, n_m\} \in S_1\), and \(a_1, \ldots, a_m \in \mathbb{R}\). Then

\[
\left\| \sum_{i=1}^{m} a_i x_{n_i} \right\| \leq K \left\| \sum_{i=1}^{m} a_i x_{n_{n_0+i}} \right\| \leq 2K \left\| \sum_{i=1}^{m} a_i \tilde{x}_i \right\| \leq 2CK \left\| \sum_{i=1}^{m} a_i \tilde{y}_i \right\| \leq 4CK \left\| \sum_{i=1}^{m} a_i y_{n_{n_0+i}} \right\| \leq 4CK^2 \left\| \sum_{i=1}^{m} a_i y_{n_i} \right\|
\]

For the converse, let now \((x_n) \preceq_1 (y_n)\). Again, using Lemma 2.2, we get: there is \(n_0 \in \mathbb{N}\) such that

\[
\left\| \sum_{i=1}^{m} a_i \tilde{x}_i \right\| \leq 2 \left\| \sum_{i=1}^{m} a_i x_{n_i} \right\| \leq 2C \left\| \sum_{i=1}^{m} a_i y_{n_i} \right\| \leq 4C \left\| \sum_{i=1}^{m} a_i \tilde{y}_i \right\|
\]

for every \(m \in \mathbb{N}\), every \(a_1, \ldots, a_m \in \mathbb{R}\), and \(n_0 \leq m \leq n_1 < \cdots < n_m\). \(\square\)

**Corollary 3.7.** Let \(X\) be a Banach space, and \(n \geq 0\) be such that \((SP_{w,1}, \preceq)\) contains no chains of length greater than \(n\). Then for any \(T_1, \ldots, T_n \in SS_1(X)\) and every strictly singular operator \(S\) the composition \(T_n T_{n-1} \cdots T_1 S\) is compact. Moreover, if \(\ell_1\) does not isomorphically embed in \(X\) then \(T_n T_{n-1} \cdots T_1\) is compact.
4. Example of a finitely strictly singular operator which is not polynomially compact

In view of the results of Section 3 one may ask whether in every Banach space the product of a certain number of strictly singular or $\mathcal{S}^\infty$-singular operators is compact, or, at least, some power of such an operator is compact. It is easy to see that this fails for strictly singular operators, as Read constructed in [Read91] an example of a strictly singular operator with no invariant subspaces. It follows now from Lomonosov's theorem [Lom73] that no power of such an operator is compact. We will show in this section that the conjecture fails even for finitely strictly singular operators.

Example 4.1. There is a separable Banach space $X$ and a finitely strictly singular operator $T: X \to X$ on $X$ such that no non-zero polynomial of $T$ is compact.

Proof. Fix a strictly increasing sequence of reals $1 \leq p_1 < p_2 < \ldots$, and put $X = \bigoplus_{n=1}^{\infty} \ell_{p_n}$. It can be easily verified that $X$ is separable. Throughout the proof, whenever we consider $x \in X$, we assume that $x = (x_1, x_2, \ldots)$ where $x_k \in \ell_{p_k}$ as $k \in \mathbb{N}$.

Let $F \subseteq X$ be a subspace of $X$ with $\dim F = M$. We will show that there exists $x \in F$ such that $\|Tx\| \leq \varepsilon \|x\|$, hence $T$ is finitely strictly singular.

Consider the projection $P: X \to X$ defined by

$$P: (x_1, x_2, \ldots) \mapsto (x_1, \ldots, x_N, 0, \ldots).$$

Let $F_1 = P(F)$. Consider the following two cases.

Case $\dim F_1 < M$. Let $b_1, \ldots, b_m$ be a basis of $F$ and put $\tilde{b}_i = P(b_i), i = 1, \ldots, M$. The set $\{b_i \mid i = 1, \ldots, M\}$ is linearly dependent because $\dim F_1 < M$, hence there is a nontrivial linear combination $\sum_{i=1}^{M} \alpha_i \tilde{b}_i = 0$. Take $x = \sum_{i=1}^{M} \alpha_i b_i$. Then $x \neq 0$ because the combination is nontrivial. Obviously, $x_1 = \cdots = x_N = 0$ since $Px = 0$. Then

$$\|Tx\| = \sup_{n>N} \frac{1}{n} \|x_n\|_{p_n+1} \quad \text{and} \quad \|x\| = \sup_{n>N} \|x_n\|_{p_n}.$$
Case dim \( F_1 = M \). Every \( x \in X \) is a tuple \( x = (x_1, x_2, \ldots) \), and for every \( k \in \mathbb{N} \), \( x_k = (x_{k1}, x_{k2}, \ldots) \in \ell_{p_k} \), so we can view \( x \) as a double sequence \( (x_{ki}) \). This way we can view \( F_1 \) as a linear subspace of \( c_0 \). By Lemma 3.4 of [SSTT] there exists \( x \in F_1 \) such that it attains its sup \( \delta = \sup_{k,i \in \mathbb{N}}|x_{ki}| \) at least at \( M \) pairs \((k, i)\). Without loss of generality, we can assume that \( \|x\| = 1 \).

Let \( A_k = \{i \in \mathbb{N} \mid |x_{ki}| = \delta\} \). It follows from \( x \in F_1 \) that \( A_k = \emptyset \) for every \( k > N \), so \( \sum_{k=1}^{N}|A_k| \geq M \). Therefore, \( |A_{k_0}| \geq \frac{M}{N} \) for some \( k_0 \leq N \). We have

\[
1 = \|x\| = \left( \sum_{i \in A_{k_0}} \delta^{\frac{1}{k_0}} \right)^{\frac{1}{p_{k_0}}},
\]

so that \( \delta \leq \left( \frac{N}{M} \right)^{\frac{1}{p_{k_0}}} \). For every \( 1 \leq k \leq N \) we have

\[
\|x_k\|_{p_k+1} = \sum_{i=1}^{\infty} |x_{ki}|^{p_{k+1}} \leq \delta^{p_{k+1}-p_k} \|x_k\|_{p_k}^{p_{k+1}-p_k} \leq \delta^{p_{k+1}-p_k} \|x\| = \delta^{p_{k+1}-p_k},
\]

hence

\[
\|x_k\|_{p_k+1} \leq \delta^{(1-\frac{p_k}{p_{k+1}})} \leq \left( \frac{N}{M} \right)^{(1-\frac{1}{p_{k+1}})} < \varepsilon.
\]

Now let \( y \in F \) be such that \( P(y) = x \), and put \( z = y - x \). Then \( z_k = 0 \) for all \( k \leq N \). It follows that \( \|y\| \geq \|x\| = 1 \) and \( \|y\| \geq \|z\| \). The structure of the norm in \( X \) and the operator \( T \) guarantees that \( \|Ty\| = \max\{\|Tx\|, \|Tz\|\} \). Observe that

\[
\|Tx\| = \max_{k \leq N} \|x_k\|_{p_{k+1}} < \varepsilon < \varepsilon \|y\|.
\]

On the other hand, \( z_k = 0 \) for all \( k \leq N \) implies \( \|Tz\| \leq \frac{1}{N} \|z\| \leq \varepsilon \|y\| \), so that \( \|Ty\| \leq \varepsilon \|y\| \). This shows that \( T \) is finitely strictly singular.

Let \( \ell(t) = \sum_{k=1}^{n} a_k t^k \) be a non-zero polynomial. Without loss of generality, \( a_0 \neq 0 \). Suppose for the sake of contradiction that \( Q(T) \) is compact. Consider bounded operators \( A: \ell_{p_1} \to X \) and \( B: X \to \ell_{p_{n+1}} \) given by \( A: h \mapsto (h, 0, 0, \ldots) \) and \( B: x \mapsto x_{n+1} \). For every \( h \in \ell_{p_1} \), we have

\[
Q(T): (h, 0, 0, \ldots) \mapsto (a_0 h, \frac{a_1}{a_0} h, \frac{a_2}{a_0} h, \ldots, \frac{a_n}{a_0} h, 0, 0, \ldots),
\]

so that \( BQ(T)A(h) = \frac{a_n}{a_0} h \). It follows that the compact operator \( \frac{a_n}{a_0} BQ(T)A \) equals the formal identity operator from \( \ell_{p_1} \) to \( \ell_{p_{n+1}} \), which is not compact since \( p_1 < p_{n+1} \), contradiction.

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