Push Forward Measures And Concentration Phenomena

Joint work with C. Hugo Jiménez and Raffael Villa

Márton Naszódi

Dept. of Geometry, Eötvös University, Budapest

Notations

All measures regular Borel measures, not supported on a singleton.
(X, d, μ): a *metric probability space*.

ε-expansion: For $A \subseteq X$,

$$A_\varepsilon = \{x \in X : d(A, x) < \varepsilon\} = \{x \in X : \exists a \in A : d(a, x) < \varepsilon\}.$$

Concentration function: For $\varepsilon > 0$,

$$\alpha_\mu(\varepsilon) = \sup \left\{ 1 - \mu(A_\varepsilon) : A \subseteq X \text{ measurable }, \mu(A) \geq \frac{1}{2} \right\}.$$

$m_f \in \mathbb{R}$ is a *median* of a random variable $f : X \to \mathbb{R}$ if
$\mu(\{x \in X : f(x) \leq m_f\}), \mu(\{x \in X : f(x) \geq m_f\}) \geq 1/2$.

Motivation

Problem

Transfer a well concentrated measure from one finite dim. Banach space to another.

(UC) Uniformly convex space
(CM) Concentration of measure
(CD) Concentration of distance
(AE) Almost equilateral set of large cardinality

Swanepoel

Is there an almost equilateral set of exponential size in any normed space?
Is it true that for any ε there is a $c > 0 : \quad N(\varepsilon) \geq e^{cn}$?

Motivation

(UC) $\Rightarrow$ (CM) for normalized Lebesgue measure λ

[Gromov – V. Milman, '83]
$\alpha_\lambda(\varepsilon) \leq 2e^{-2n\delta(\varepsilon)}$, with the modulus of convexity
$\delta(\varepsilon) = 1 - \sup\{\|x+y\|/2 : \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\}$.

Motivation

(UC) $\Rightarrow$ (CM) for normalized Lebesgue measure λ

[Gromov – V. Milman, '83]
$\alpha_\lambda(\varepsilon) \leq 2e^{-2n\delta(\varepsilon)}$, with the modulus of convexity
$\delta(\varepsilon) = 1 - \sup\{\|x+y\|/2 : \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\}$.

(CM) $\Rightarrow$ (CD) for λ

[Arias-de-Reyna – Ball – Villa, '98]
If $\alpha_\lambda(\varepsilon) \leq e^{-n\phi(\varepsilon)}$, for some increasing function ϕ, then $\exists a \in [\frac{1}{2}, 2]$:

$$(\lambda \otimes \lambda)\Big\{(x,y) : \|x-y\| \in \big[a(1-\varepsilon), a(1+\varepsilon)\big]\Big\} \geq 1 - 4e^{-n\phi(\varepsilon/6)}.$$

Motivation

(UC) $\Rightarrow$ (CM) for normalized Lebesgue measure λ

[Gromov – V. Milman, '83]
$\alpha_\lambda(\varepsilon) \leq 2e^{-2n\delta(\varepsilon)}$, with the modulus of convexity
$\delta(\varepsilon) = 1 - \sup\{\|x+y\|/2 : \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\}$.

(CM) $\Rightarrow$ (CD) for λ

[Arias-de-Reyna – Ball – Villa, '98]
If $\alpha_\lambda(\varepsilon) \leq e^{-n\phi(\varepsilon)}$, for some increasing function ϕ, then $\exists a \in [\frac{1}{2}, 2]$:

$$(\lambda \otimes \lambda)\Big\{(x,y) : \|x-y\| \in \big[a(1-\varepsilon), a(1+\varepsilon)\big]\Big\} \geq 1 - 4e^{-n\phi(\varepsilon/6)}.$$

(CD) $\Rightarrow$ (AE) for any μ.
(CM) $\Rightarrow$ (AE) for any μ.

Motivation

Families of spaces: X_n

(UC) Uniformly convex spaces: $\delta_n(\varepsilon) \geq \delta(\varepsilon)$.
(CM) Concentration of measure for λ_n

$$\alpha_{\lambda_n}(\varepsilon) \leq Ce^{-n\phi(\varepsilon)}.$$

(CD) Concentration of distance for λ_n

$$\lambda_n \otimes \lambda_n\{(x, y) : \|x - y\| \in [a(1 - \varepsilon), a(1 + \varepsilon)]\} \geq 1 - Ce^{-n\phi(\varepsilon)}.$$

Motivation

Families of spaces: X_n

(UC) Uniformly convex spaces: $\delta_n(\varepsilon) \geq \delta(\varepsilon)$.
(CM) Concentration of measure for λ_n

$$\alpha_{\lambda_n}(\varepsilon) \leq Ce^{-n\phi(\varepsilon)}.$$

(CD) Concentration of distance for λ_n

$$\lambda_n \otimes \lambda_n\{(x, y) : \|x - y\| \in [a(1 - \varepsilon), a(1 + \varepsilon)]\} \geq 1 - Ce^{-n\phi(\varepsilon)}.$$

Neither of the three are preserved by isomorphisms.
Compare ℓ_2^{n+1} and $(\ell_2^n \oplus \ell_1^\infty)_\infty$.

Motivation

Families of spaces: X_n

(UC) Uniformly convex spaces: $\delta_n(\varepsilon) \geq \delta(\varepsilon)$.
(CM) Concentration of measure for λ_n

$$\alpha_{\lambda_n}(\varepsilon) \leq Ce^{-n\phi(\varepsilon)}.$$

(CD) Concentration of distance for λ_n

$$\lambda_n \otimes \lambda_n\{(x,y) : \|x-y\| \in [a(1-\varepsilon), a(1+\varepsilon)]\} \geq 1 - Ce^{-n\phi(\varepsilon)}.$$

Neither of the three are preserved by isomorphisms.
Compare ℓ_2^{n+1} and $(\ell_2^n \oplus \ell_1^\infty)_\infty$.

(AE) Almost equilateral set of large cardinality

$$N_n \geq Ce^{n\phi(\varepsilon)}$$

Is it preserved under isomorphism?

Push-forward measure

Definition

(X, μ) measure spaces, Y a measurable space, $\phi : X \to Y$ a measurable map. The *push-forward* of μ by ϕ is ν where

$$\nu(A) := \phi_*(\mu)(A) := \mu(\phi^{-1}(A))$$

for any measurable set $A \subseteq Y$.

The Lipschitz bound

For any (X, d_X, μ) metric prob. space, and (Y, d_Y) metric space, if $\phi : X \to Y$ is λ-Lipschitz then

$$\alpha_\nu(\varepsilon) \leq \alpha_\mu(\varepsilon/\lambda).$$

Concentration of λ and BM distance

Banach–Mazur distance

$K, L \subset \mathbb{R}^n$ o-symmetric convex bodies.

$$d_{BM}(K,L) = \inf\{\lambda > 0 : \; K \subset T(L) \subset \lambda K, T \in GL(\mathbb{R}^n)\}.$$

Central projection of K to L

$$\pi : \mathbb{R}^n \to \mathbb{R}^n; \pi(x) = \frac{\|x\|_K}{\|x\|_L} x$$

The Lipschitz bound

If K and L are in Banach–Mazur position and $d_{BM}(K,L) = \lambda$ then π is λ-Lipschitz, and hence $\alpha_\nu(\varepsilon) \leq \alpha_\mu(\varepsilon/\lambda)$ for $\nu = \pi_*(\mu)$ for any prob. measure μ on K.

A Positive Result

The Lipshcitz bound

If K and L are in Banach–Mazur position and $d_{\mathrm{BM}}(K, L) = \lambda$ then π is λ-Lipschitz, and hence $\alpha_\nu(\varepsilon) \leq \alpha_\mu(\varepsilon/\lambda)$ for $\nu = \pi_*(\mu)$ for any prob. measure μ on K.

A Positive Result

The Lipshcitz bound

If K and L are in Banach–Mazur position and $d_{\mathrm{BM}}(K, L) = \lambda$ then π is λ-Lipschitz, and hence $\alpha_\nu(\varepsilon) \leq \alpha_\mu(\varepsilon/\lambda)$ for $\nu = \pi_*(\mu)$ for any prob. measure μ on K.

Theorem 1

$K, L \subset \mathbb{R}^n$ o-symmetric convex bodies, $L \subseteq K \subseteq \lambda L, \mu$ a prob. measure on $\mathbb{R}^n$ and $\pi(x) = \frac{x}{\|x\|_L}\|x\|_K$ the central projection. $\nu := \pi_*(\mu)$. Then

$$\alpha_{(L,\nu)}(\varepsilon) \leq 16\alpha_{(K,\mu)}\left(\frac{\varepsilon}{14}\frac{m_L}{\lambda m_K}\right),$$

where
$m_K := \mathrm{median}_\mu(\|.\|_K : \mathbb{R}^n \to \mathbb{R})$ and $m_L := \mathrm{median}_\mu(\|.\|_L : \mathbb{R}^n \to \mathbb{R})$.

Note that $\|.\|_K \leq \|.\|_L$, so $m_K \leq m_L$.

A Distance-like Quantity: β

$m_K := \text{median}_{\mu}(\|.\|_K : \mathbb{R}^n \to \mathbb{R})$ and $m_L := \text{median}_{\mu}(\|.\|_L : \mathbb{R}^n \to \mathbb{R})$.

- If $supp(\mu) \subseteq \partial K$ then $supp(\nu) \subseteq \partial L$ and $m_K = 1$.
- If μ is the normalized Lebesgue measure restricted to K then $m_K \approx 1$.

A Distance-like Quantity: β

$m_K := \text{median}_\mu(\|.\|_K : \mathbb{R}^n \to \mathbb{R})$ and $m_L := \text{median}_\mu(\|.\|_L : \mathbb{R}^n \to \mathbb{R})$.

- If $supp(\mu) \subseteq \partial K$ then $supp(\nu) \subseteq \partial L$ and $m_K = 1$.
- If μ is the normalized Lebesgue measure restricted to K then $m_K \approx 1$.
- A natural quantity to study:

$$\beta((K,\mu),L) = \inf\left\{\lambda \frac{m_K}{m_{TL}} : TL \subseteq K \subseteq \lambda TL, T \in GL(\mathbb{R}^n)\right\}$$

A Distance-like Quantity: β

$m_K := \text{median}_\mu(\|.\|_K : \mathbb{R}^n \to \mathbb{R})$ and $m_L := \text{median}_\mu(\|.\|_L : \mathbb{R}^n \to \mathbb{R})$.

- If $supp(\mu) \subseteq \partial K$ then $supp(\nu) \subseteq \partial L$ and $m_K = 1$.
- If μ is the normalized Lebesgue measure restricted to K then $m_K \approx 1$.
- A natural quantity to study:

$$\beta((K,\mu),L) = \inf\left\{\lambda \frac{m_K}{m_{TL}} : TL \subseteq K \subseteq \lambda TL, T \in GL(\mathbb{R}^n)\right\}$$

- $\beta((K,\mu),L) \leq d_{\text{BM}}(K,L)$.

A Distance-like Quantity: β

$m_K := \text{median}_\mu(\|.\|_K : \mathbb{R}^n \to \mathbb{R})$ and $m_L := \text{median}_\mu(\|.\|_L : \mathbb{R}^n \to \mathbb{R})$.

- If $supp(\mu) \subseteq \partial K$ then $supp(\nu) \subseteq \partial L$ and $m_K = 1$.
- If μ is the normalized Lebesgue measure restricted to K then $m_K \approx 1$.
- A natural quantity to study:

$$\beta((K,\mu),L) = \inf\left\{\lambda\frac{m_K}{m_{TL}} : TL \subseteq K \subseteq \lambda TL, T \in GL(\mathbb{R}^n)\right\}$$

- $\beta((K,\mu),L) \leq d_{\text{BM}}(K,L)$.
- So, Theorem 1 replaces the obvious bound
  $\alpha_\nu(\varepsilon) \leq \alpha_\mu\left(\frac{\varepsilon}{d_{\text{BM}}}\right)$
  by
  $\alpha_\nu(\varepsilon) \leq 16\alpha_\mu\left(\frac{\varepsilon}{14\beta}\right)$.

More on β

- β is not new. Let K be the Euclidean ball with the normalized Lebesgue measure, and $k(L,\varepsilon)$ denote the dimension of an ε-almost Euclidean section of L. Milman's Theorem: $k(L,\varepsilon) \geq c(\varepsilon)\frac{n}{\beta^2}$.
- β can be far below d_{BM}: while $d_{\mathrm{BM}}(B_2^n, B_1^n) = \sqrt{n}$, we have $\beta(B_2^n, B_1^n) \leq c$.

More on β

- β is not new. Let K be the Euclidean ball with the normalized Lebesgue measure, and $k(L,\varepsilon)$ denote the dimension of an ε-almost Euclidean section of L. Milman's Theorem: $k(L,\varepsilon) \geq c(\varepsilon)\frac{n}{\beta^2}$.
- β can be far below d_{BM}: while $d_{\mathrm{BM}}(B_2^n, B_1^n) = \sqrt{n}$, we have $\beta(B_2^n, B_1^n) \leq c$.
- We can replace the medians in the definition of β by means to obtain $\tilde{\beta}$. If μ is a log-concave measure, then β and $\tilde{\beta}$ are equivalent.

More on β

- β is not new. Let K be the Euclidean ball with the normalized Lebesgue measure, and $k(L, \varepsilon)$ denote the dimension of an ε-almost Euclidean section of L. Milman's Theorem: $k(L, \varepsilon) \geq c(\varepsilon)\frac{n}{\beta^2}$.
- β can be far below d_{BM}: while $d_{\mathrm{BM}}(B_2^n, B_1^n) = \sqrt{n}$, we have $\beta(B_2^n, B_1^n) \leq c$.
- We can replace the medians in the definition of β by means to obtain $\tilde{\beta}$. If μ is a log-concave measure, then β and $\tilde{\beta}$ are equivalent.
- For any o-symmetric convex body L,

$$\tilde{\beta}(B_2^n, L) \leq C\sqrt{\frac{n}{\log n}}.$$

Examples: ℓ_p spaces

- Let $1 \leq p < 2$, consider B_2^n with λ. Then

$$\tilde{\beta}(B_2^n, B_p^n) \leq b_p,$$

where b_p depends only on p. Better than the Lipschitz bound.

- Let $2 \leq p < \infty$, consider B_2^n with λ. Then

$$\tilde{\beta}(B_2^n, B_p^n) \leq C_p n^{\frac{1}{2} - \frac{1}{p}}$$

implies

$$\alpha_{(B_p^n, \nu)}(\varepsilon) \leq C_3 \exp\{-c_3 \varepsilon^2 n^{2/p}\},$$

Not better than the Lipschitz bound.

A Negative Result

- B_∞^n has no concentration w.r.t. the Lebesgue measure: $\alpha_\lambda(\varepsilon) \approx 1/2 - \varepsilon$.
- $\tilde{\beta}(B_2^n, B_\infty^n) \approx \sqrt{\frac{n}{\log n}}$. This won't yield a well-concentrated measure on B_∞^n.

A Negative Result

- B_∞^n has no concentration w.r.t. the Lebesgue measure: $\alpha_\lambda(\varepsilon) \approx 1/2 - \varepsilon$.
- $\tilde{\beta}(B_2^n, B_\infty^n) \approx \sqrt{\frac{n}{\log n}}$. This won't yield a well-concentrated measure on B_∞^n.

Theorem 2

For any ν o-symmetric prob. measure on B_∞^n,

$$\alpha_\nu(\varepsilon) \geq \frac{1}{2n}(1 - \nu(\varepsilon B_\infty^n)).$$

A Stronger Negative Result

Definition

$T : X \to Y$ linear map between two normed spaces is a *d-embedding*, if $a\|x\|_X \le \|Tx\|_Y \le b\|x\|_X$ with $b/a \le d$.

Theorem 3

Let $T : (\mathbb{R}^n, \|.\|_K) \to \ell_\infty^N$ be a d-embedding, μ an o-symmetric prob. measure on $(\mathbb{R}^n, \|.\|_K)$. Then for any $0 < \varepsilon < 1/d$, if $\alpha_\mu(\varepsilon) > 0$ then

$$N \ge \frac{1 - \mu(d\varepsilon K)}{2\alpha_\mu(\varepsilon)}.$$

Applying this to $T = id_{\ell_\infty^n}$ yields Theorem 2.

Combining the results

Theorem 4

$K \subset \mathbb{R}^n$ a convex body, μ an o-symmetric probability measure on $(\mathbb{R}^n, \|.\|_K)$. Assume that for some $0 < s$,

$$\alpha_{(K,\mu)}(\varepsilon) \leq Ce^{-c\varepsilon^s n},$$

and $(\mathbb{R}^n, \|.\|_L) \overset{d}{\hookrightarrow} \ell_\infty^N$.
Then

$$\beta((K,\mu), L) \geq \frac{m_K}{14d} \left(\frac{cn}{\log(64CN)} \right)^{1/s}.$$

Proof of Theorem 3

Theorem 3

Let $T : (\mathbb{R}^n, \|.\|_K) \to \ell_\infty^N$ be a d-embedding, μ an o-symmetric prob. measure on K. Then for any $0 < \varepsilon < 1/d$, if $\alpha_\mu(\varepsilon) > 0$ then

$$N \geq \frac{1 - \mu(d\varepsilon K)}{2\alpha_\mu(\varepsilon)}.$$

$T : X \to \ell_\infty^N$, assume that $\frac{1}{d}\|x\| \leq \|Tx\|_\infty \leq \|x\|$.

$f_i : X \to \mathbb{R}$ the i^{th} coordinate of T. Let $A_i := \{x \in \mathbb{R}^n : |f_i(x)| \leq \varepsilon\}$.

Proof of Theorem 3

Theorem 3

Let $T : (\mathbb{R}^n, \|.\|_K) \to \ell_\infty^N$ be a d-embedding, μ an o-symmetric prob. measure on K. Then for any $0 < \varepsilon < 1/d$, if $\alpha_\mu(\varepsilon) > 0$ then

$$N \geq \frac{1 - \mu(d\varepsilon K)}{2\alpha_\mu(\varepsilon)}.$$

$T : X \to \ell_\infty^N$, assume that $\frac{1}{d}\|x\| \leq \|Tx\|_\infty \leq \|x\|$.
$f_i : X \to \mathbb{R}$ the i^{th} coordinate of T. Let $A_i := \{x \in \mathbb{R}^n : |f_i(x)| \leq \varepsilon\}$.
$\{x \in \mathbb{R}^n : f_i(x) \leq \varepsilon\} \supseteq \underbrace{\{x \in \mathbb{R}^n : f_i(x) \leq 0\}}_{\text{of measure } \geq 1/2} + \varepsilon K.$

Thus, $\mu(A_i^C) \leq 2\alpha_\mu(\varepsilon)$, and hence

$$\mu\left(\bigcap_{i=1}^{N} A_i\right) \geq 1 - 2N\alpha_\mu(\varepsilon).$$

Proof of Theorem 3

Theorem 3

Let $T : (\mathbb{R}^n, \|.\|_K) \to \ell_\infty^N$ be a d-embedding, μ an o-symmetric prob. measure on K. Then for any $0 < \varepsilon < 1/d$, if $\alpha_\mu(\varepsilon) > 0$ then

$$N \geq \frac{1 - \mu(d\varepsilon K)}{2\alpha_\mu(\varepsilon)}.$$

$T : X \to \ell_\infty^N$, assume that $\frac{1}{d}\|x\| \leq \|Tx\|_\infty \leq \|x\|$.
$f_i : X \to \mathbb{R}$ the i^{th} coordinate of T. Let $A_i := \{x \in \mathbb{R}^n : |f_i(x)| \leq \varepsilon\}$.
$\{x \in \mathbb{R}^n : f_i(x) \leq \varepsilon\} \supseteq \underbrace{\{x \in \mathbb{R}^n : f_i(x) \leq 0\}}_{\text{of measure } \geq 1/2} + \varepsilon K$.

Thus, $\mu(A_i^C) \leq 2\alpha_\mu(\varepsilon)$, and hence

$$\mu\left(\bigcap_{i=1}^N A_i\right) \geq 1 - 2N\alpha_\mu(\varepsilon).$$

Let $r \in (0,1)$ be such that $\mu(rK) < 1 - 2N\alpha_\mu(\varepsilon)$. We have $\bigcap_{i=1}^N A_i \not\subseteq rK$.

Proof of Theorem 3

Let $r \in (0,1)$ be such that $\mu(rK) < 1 - 2N\alpha_\mu(\varepsilon)$. We have $\bigcap_{i=1}^N A_i \not\subseteq rK$.

$$\exists x \in \bigcap_{i=1}^{N} A_i \text{ with } \|x\|_K \geq r.$$

$$r < \|x\|_K \leq d\|Tx\|_\infty \leq d\varepsilon.$$

Proof of Theorem 3

Let $r \in (0,1)$ be such that $\mu(rK) < 1 - 2N\alpha_\mu(\varepsilon)$. We have $\bigcap_{i=1}^N A_i \not\subseteq rK$.

$$\exists x \in \bigcap_{i=1}^{N} A_i \text{ with } \|x\|_K \geq r.$$

$$r < \|x\|_K \leq d\|Tx\|_\infty \leq d\varepsilon.$$

So we have: if $d\varepsilon < r$ then $\mu(rK) \geq 1 - 2N\alpha_\mu(\varepsilon)$.

$$\mu(d\varepsilon K) \geq 1 - 2N\alpha_\mu(\varepsilon).$$

Proof of Theorem 1

Theorem 1

$L \subseteq K \subseteq \lambda L, \mu$ a prob. measure on $(\mathbb{R}^n, \|.\|_K)$. $\pi(x) = \frac{x}{\|x\|_L}\|x\|_K$ the central projection, $\nu := \pi_*(\mu)$. Then

$$\alpha_{(L,\nu)}(\varepsilon) \leq 16\alpha_{(K,\mu)}\left(\frac{\varepsilon}{14}\frac{m_L}{\lambda m_K}\right),$$

where

$m_K := \text{median}_\mu(\|.\|_K : \mathbb{R}^n \to \mathbb{R})$ and $m_L := \text{median}_\mu(\|.\|_L : \mathbb{R}^n \to \mathbb{R})$.

Let $A \subset \mathbb{R}^n$ with $1/2 \leq \nu(A) = \mu(\pi^{-1}(A))$.
Goal: $\mu(\pi^{-1}(A^L_\varepsilon))$ is large.
Let $\delta := \frac{\varepsilon}{7m_K}$.

$$G_L := \{x \in \mathbb{R}^n : (1-\delta)m_L < \|x\|_L < (1+\delta)m_L\}$$

$$G_K := \{x \in \mathbb{R}^n : (1-\delta)m_K < \|x\|_K < (1+\delta)m_K\}.$$

Proof of Theorem 1

$$G_L := \{x \in K : (1-\delta)m_L < \|x\|_L < (1+\delta)m_L\},$$
$$G_K := \{x \in K : (1-\delta)m_K < \|x\|_K < (1+\delta)m_K\},$$
$$J := \pi^{-1}(A) \cap G_L \cap G_K.$$

$$L \subseteq K \subseteq \lambda L \Longrightarrow \|.\|_K \leq \|.\|_L \leq \lambda\|.\|_K \Longrightarrow m_K \leq m_L \leq \lambda m_K.$$

By the Lipschitz bound

$$\mu(J) \geq 1/2 - 2\alpha_{(K,\mu)}(\delta m_L/\lambda) - 2\alpha_{(K,\mu)}(\delta m_K) \geq 1/2 - 4\alpha_{(K,\mu)}(\delta m_L/\lambda).$$

By computation:

$$J^K_{\frac{\delta m_L}{\lambda}} \subset \pi^{-1}\left(A^L_\varepsilon\right) \tag{1}$$

Proof of Theorem 1

$$J^K_{\frac{\delta m_L}{\lambda}} \subset \pi^{-1}\left(A^L_\varepsilon\right) \tag{1}$$

$$\nu\left(\left(A^L_\varepsilon\right)^c\right) = \mu\left(\pi^{-1}\left(\left(A^L_\varepsilon\right)^c\right)\right) \leq \mu\left(\left(J^K_{\frac{\delta m_L}{\lambda}}\right)^c\right) \leq \dots$$

Proof of Theorem 1

$$J^K_{\frac{\delta m_L}{\lambda}} \subset \pi^{-1}\left(A^L_\varepsilon\right) \tag{1}$$

$$\nu\left(\left(A^L_\varepsilon\right)^c\right) = \mu\left(\pi^{-1}\left(\left(A^L_\varepsilon\right)^c\right)\right) \le \mu\left(\left(J^K_{\frac{\delta m_L}{\lambda}}\right)^c\right) \le \dots$$

Lemma (Ledoux): $\mu(J)\mu(F) \le 4\alpha_{(X,d,\mu)}(dist(J,F)/2)$.

$$\dots \le \frac{4\alpha_{(K,\mu)}(\frac{\delta m_L}{2\lambda})}{\mu(J)} \le \frac{4\alpha_{(K,\mu)}(\frac{\delta m_L}{2\lambda})}{1/2 - 4\alpha_{(K,\mu)}(\frac{\delta m_L}{\lambda})}.$$

Hence, $\forall \varepsilon > 0$ such that $16\alpha_{(K,\mu)}\left(\varepsilon m_L/(7\lambda m_K)\right) \le 1$,

$$\nu\left(\left(A^L_\varepsilon\right)^c\right) \le 16\alpha_{(K,\mu)}\left(\frac{\varepsilon m_L}{14\lambda m_K}\right).$$

And so,

$$\alpha_{(L,\nu)}(\varepsilon) \le 16\alpha_{(K,\mu)}\left(\frac{\varepsilon m_L}{14\lambda m_K}\right).$$