Random almost spherical sections of centrally-symmetric convex sets

Konstantin Tikhomirov

University of Alberta

June 21, 2013
Dependence on $n$ and $\epsilon$ in Dvoretzky’s theorem

Given $n \in \mathbb{N}$ and $\epsilon > 0$, what is the largest $m$ such that any $n$-dimensional normed space contains a $(1 + \epsilon)$-Euclidean subspace?

The right dependence of $m$ on the dimension $n$ was found by V. Milman. The best known (so far) lower bound for $m$ is due to G. Schechtman:

**Theorem [G. Schechtman]**

$$m \geq c \frac{\epsilon}{\ln^2(1/\epsilon)} \ln n,$$

for an absolute constant $c$. 


Outline of the proof of the theorem of G. Schechtman

Assume that \( X = (\mathbb{R}^n, \| \cdot \|) \) is a normed space with the unit ball \( B_{\| \cdot \|} \) in John’s position. Let \( g = (g_1, g_2, \ldots, g_n) \) be the standard Gaussian vector. There are two possibilities:

**\( X \) is “far from \( l^\infty_n \)”**. Precisely, \( \mathbb{E}\|g\| \geq C \sqrt{\frac{\ln n}{\epsilon}} \). Then from the classical argument, it follows that \( X \) contains a \((1 + \epsilon)\)-Euclidean subspace of dimension \( c\epsilon \ln n/\ln \frac{1}{\epsilon} \).

Otherwise, by applying certain argument (which involves James’ blocking), one can find a subspace \( E \subset X \) with \( \dim E \geq n^{c\epsilon/\ln \frac{1}{\epsilon}} \), which is \((1 + \epsilon)\)-isometric to \( l^\dim E \). It is well known that \( l^k_\infty \) contains a \((1 + \epsilon)\)-Euclidean subspace of dimension \( c \frac{\ln k}{\ln(1/\epsilon)} \). Thus, \( X \) contains a \((1 + C\epsilon)\)-Euclidean subspace of dimension

\[
 c \frac{\ln(n^{\epsilon/\ln(1/\epsilon)})}{\ln(1/\epsilon)} = c \frac{\epsilon}{\ln^2(1/\epsilon)} \ln n.
\]
In this talk, a modified proof of the theorem of G. Schechtman is considered, without the construction of $l_\infty$-subspaces and not using James’ blocking.
The space $l^n_\infty$ contains $(1 + \epsilon)$-Euclidean subspaces of dimension $c \frac{\ln n}{\ln(1/\epsilon)}$. But how many almost Euclidean subspaces are there?

[G. Schechtman]

For any natural $n$ and $m$, if the Haar measure of “$(1 + \epsilon)$-spherical” $m$-dimensional sections of $l^n_\infty$ is greater than $1 - n^{-C\epsilon}$ then with necessity $m \leq c\epsilon \ln n$.

In other words, in case of $l^n_\infty$, if we put an additional restriction on the Haar measure of almost Euclidean subspaces then the dependence on $\epsilon$ becomes much worse.
In case of an arbitrary normed space \((\mathbb{R}^n, \| \cdot \|)\), the theorem of G. Schechtman tells us that there always exists a \((1 + \epsilon)\)-Euclidean subspace of dimension \(m = c \frac{\epsilon}{\ln^2(1/\epsilon)} \ln n\).

But, as in the case of \(l^n_\infty\), we can also ask how many \((1 + \epsilon)\)-spherical subspaces of dimension \(m\) the space contains.

For a normed space \((\mathbb{R}^n, \| \cdot \|)\), can we always find a bijective linear operator \(T : \mathbb{R}^n \to \mathbb{R}^n\) such that for the norm \(\| T \cdot \|\) defined as

\[
\| T \cdot \| : x \in \mathbb{R}^n \to \| Tx \|,
\]

the vast majority (with respect to the Haar measure) of subspaces of \((\mathbb{R}^n, \| T \cdot \|)\) of dimension \(c \frac{\epsilon}{\ln^2(1/\epsilon)} \ln n\) are \((1 + \epsilon)\)-Euclidean (and even “\((1 + \epsilon)\)-spherical”)?
A positive answer to the last question can be obtained if in the original proof of G. Schechtman we replace the construction of $(1 + \epsilon)$-isometric $l_\infty$-sections by a different procedure.

The Result

Given any normed space $(\mathbb{R}^n, \| \cdot \|)$ and any $\epsilon > 0$, there exists a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ (depending on $\| \cdot \|$ and $\epsilon$) such that the Haar measure of $(1 + \epsilon)$-spherical sections of $(\mathbb{R}^n, \| T \cdot \|)$ of dimension $c \frac{\epsilon}{\ln^2(1/\epsilon)} \ln n$ is greater than $1 - n^{-c\epsilon/\ln \frac{1}{\epsilon}}$. 
Recall that $g = (g_1, g_2, \ldots, g_n)$ is the standard Gaussian vector in $\mathbb{R}^n$. For any subspace $E \subset \mathbb{R}^n$, let $\text{Proj}_E$ be the orthogonal projection onto $E$. The Result follows from

**Proposition**

Let $\| \cdot \|$ be a norm in $\mathbb{R}^n$ with the unit ball in John’s position. Then for any $\epsilon \in (0, 1/2]$ there is a subspace $E = E(\epsilon, \| \cdot \|) \subset \mathbb{R}^n$ of dimension at least $\sqrt{n}$ such that

$$
\mathbb{P} \left\{ \| \text{Proj}_E g \| - \text{Med} \| \text{Proj}_E g \| > \epsilon \text{Med} \| \text{Proj}_E g \| \right\} 
\leq 2 \exp\left( -c\epsilon \ln n / \ln \frac{1}{\epsilon} \right).
$$
So, we assume that $\| \cdot \|$ is a norm in $\mathbb{R}^n$ and the unit ball $B_{\| \cdot \|}$ is in John’s position. As in the original proof of G. Schechtman, we consider two possibilities:

* The space $(\mathbb{R}^n, \| \cdot \|)$ is “far” from $l^n_{\infty}$, precisely, $\text{Med} \| g \| \geq C \sqrt{\frac{\ln n}{\epsilon}}$. Then the standard concentration inequality for Gaussians gives

$$\mathbb{P} \left\{ \| g \| - \text{Med} \| g \| > \epsilon \text{Med} \| g \| \right\} \leq 2 \exp\left(-c\epsilon \ln n\right),$$

so in this case the Proposition holds with $E = \mathbb{R}^n$. 
Proof of the Proposition

* Otherwise, $\text{Med} \|g\| \leq C \sqrt{\frac{\ln n}{\epsilon}}$. Without loss of generality (rotation + Dvoretzky–Rogers), we can assume that the norm of the first $n/2$ standard unit vectors

$$\|e_1\|, \|e_2\|, \ldots, \|e_{n/2}\| \geq 1/4.$$  

For any $\delta > 0$ and any subset $J \subset \{1, 2, \ldots, n\}$, let $M(J, \delta)$ be the number such that

$$\mathbb{P}\{\|g\chi_J\| > M(J, \delta)\} = \delta.$$  

Let $k = \exp\left(c\epsilon \ln n / \ln \frac{1}{\epsilon}\right)$. Next, we apply some procedure to generate a subset $A \subset \{1, 2, \ldots, n/2\}$ of cardinality at least $\sqrt{n}$ such that there is a partition of $A$: $\{A_1, A_2, \ldots, A_k\}$ such that

$$M(A_i, k^{-1/2}) \geq (1 - \epsilon)M(A, k^{-1/2})$$  

for all $i = 1, 2, \ldots, k$.  

Proof of the Proposition

So, $A$ is of cardinality at least $\sqrt{n}$ and there is a partition 
\(\{A_1, A_2, \ldots, A_k\}\) of $A$ such that
\[
M(A_i, k^{-1/2}) \geq (1 - \epsilon)M(A, k^{-1/2}) \quad \text{for all } i = 1, 2, \ldots, k,
\]
where for any subset $J$: 
\[
P\{\|g\chi_J\| > M(J, k^{-1/2})\} = k^{-1/2}.
\]

Claim

\[
P \left\{ \|g\chi_A\| < (1 - 2\epsilon)M(A, k^{-1/2}) \right\} \leq \exp(-\sqrt{k}) + k^{-1/2}.
\]

Proof of the claim is elementary when the norm $\| \cdot \|$ is unconditional: Since $A_1, A_2, \ldots, A_k$ are pairwise disjoint, the random variables $\|g\chi_{A_1}\|, \|g\chi_{A_2}\|, \ldots, \|g\chi_{A_k}\|$ are independent, so
\[
P \left\{ \|g\chi_A\| < (1 - \epsilon)M(A, k^{-1/2}) \right\} \leq P \left\{ \max_{i} \|g\chi_{A_i}\| < \ldots \right\}
\]
\[
\leq \prod_{i=1}^{k} P \left\{ \|g\chi_{A_i}\| < M(A_i, k^{-1/2}) \right\} = \left(1 - k^{-1/2}\right)^k \leq \exp(-\sqrt{k}).
\]

In fact, the unconditionality is unnecessary.
Proof of the Proposition

Thus, $A$ is a set of cardinality at least $\sqrt{n}$, and, by the claim,

$$\mathbb{P}\left\{ \|g\chi_A\| < (1 - 2\epsilon)M(A, k^{-1/2}) \right\} \leq 2k^{-1/2}. $$

On the other hand, by the definition of $M(A, k^{-1/2})$,

$$\mathbb{P}\left\{ \|g\chi_A\| > M(A, k^{-1/2}) \right\} = k^{-1/2}. $$

Recall that we defined $k$ as $k = \exp\left( c\epsilon \ln n / \ln \frac{1}{\epsilon} \right)$. Then from the above estimates we get for $L = (1 - \epsilon)M(A, k^{-1/2})$

$$\mathbb{P}\left\{ \|g\chi_A\| - L > \epsilon M(A, k^{-1/2}) \right\} \leq 3 \exp\left( -c\epsilon \ln n / \ln \frac{1}{\epsilon} \right). $$

It is easy to derive from the last inequality

$$\mathbb{P}\left\{ \|g\chi_A\| - \text{Med}\|g\chi_A\| > C\epsilon \text{Med}\|g\chi_A\| \right\}
\leq 2 \exp\left( -\tilde{c}\epsilon \ln n / \ln \frac{1}{\epsilon} \right),$$

So the Proposition holds with $E = \text{span}\{e_i : i \in A\}$. 
We’ve just proved

**Proposition**

Let $\| \cdot \|$ be a norm in $\mathbb{R}^n$ with the unit ball in John’s position. Then for any $\epsilon \in (0, 1/2]$ there is a subspace $E = E(\epsilon, \| \cdot \|) \subset \mathbb{R}^n$ of dimension at least $\sqrt{n}$ such that

$$
P \{ |\| \text{Proj}_E g \|- \text{Med} \|\text{Proj}_E g\| | > \epsilon \text{Med} \|\text{Proj}_E g\| \} \leq 2 \exp(-c\epsilon \ln n / \ln \frac{1}{\epsilon}).$$

Clearly, we can find a bijective linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\| Tx \| \approx \|\text{Proj}_E x\|, \quad x \in S^{n-1}.$$  

Then, in particular, we obtain

$$
P \{ |\| Tg \|- \text{Med} \| Tg\| | > \epsilon \text{Med} \| Tg\| \} \leq 2 \exp(-c\epsilon \ln n / \ln \frac{1}{\epsilon}).$$
The last identity implies The Result:

Given any normed space \((\mathbb{R}^n, \| \cdot \|)\) and any \(\epsilon > 0\), there exists a linear operator \(T : \mathbb{R}^n \rightarrow \mathbb{R}^n\) (depending on \(\| \cdot \|\) and \(\epsilon\)) such that the Haar measure of \((1 + \epsilon)\)-spherical sections of \((\mathbb{R}^n, \|T \cdot\|)\) of dimension \(c \frac{\epsilon}{\ln^2(1/\epsilon)} \ln n\) is greater than \(1 - n^{-c\epsilon/\ln \frac{1}{\epsilon}}\).

The unit ball of the norm \(\|T \cdot\|\) —

\[\{x \in \mathbb{R}^n : \|Tx\| \leq 1\}\]

— looks like this: