Some estimates for Random polytopes and their perturbations

David Alonso Gutiérrez.

Joint work with N. Dafnis, J. Prochno and M.A. Hernández-Cifre

Universidad de Murcia

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A convex body $K \subseteq \mathbb{R}^n$ is isotropic if it has volume 1 and

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- $\int_K \langle x, \theta \rangle^2 dx = L_K^2 \quad \forall \theta \in S^{n-1}$.

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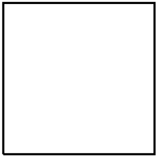

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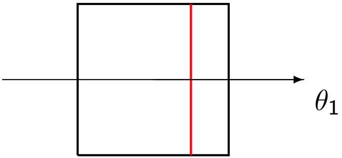

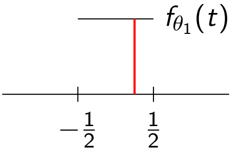

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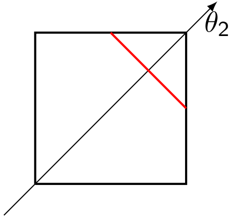

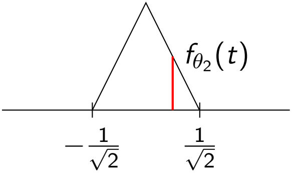

K is isotropic if all the $\langle X, \theta\rangle$ are centered and have the same variance.

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Does there exist an **absolute** constant C such that for every $K \subseteq \mathbb{R}^n$

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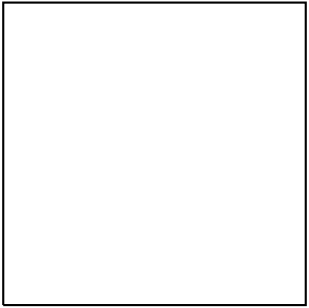
Random polytopes.

Theorem (Milman, Pajor 1989)

Let $K \subseteq \mathbb{R}^n$ be an isotropic body and $X_1, \ldots, X_n$ independent random vectors uniformly distributed in K. Then

$$L_K^{2n} = n! \mathbb{E} |\mathrm{conv}\{0, X_1, \ldots, X_n\}|^2.$$



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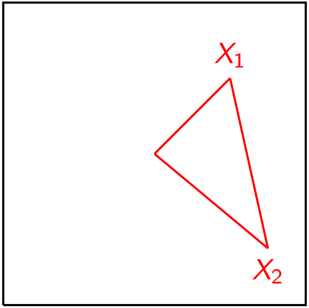

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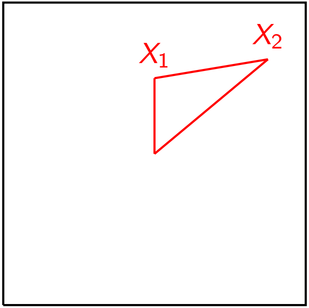

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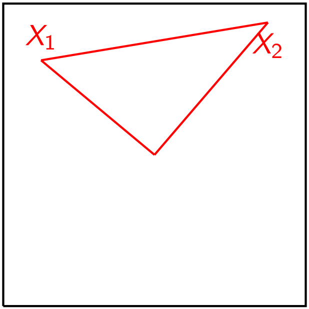

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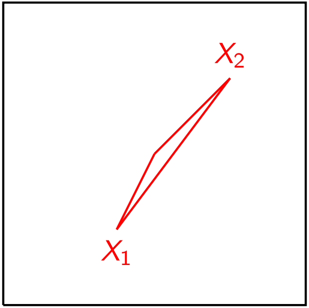

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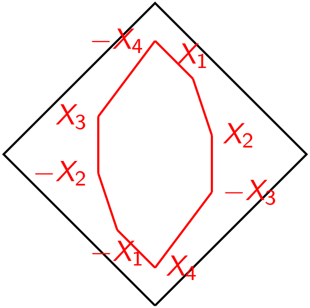

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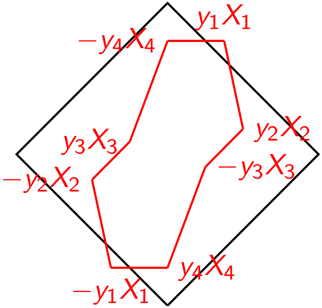

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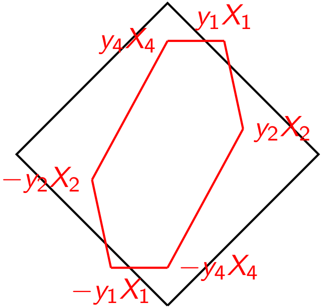

Quermaßintegrals.

- Let L be a convex body,

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- $Q_1(L) = w(L).$ $\qquad w(K_N) = \int_{S^{n-1}} \max_{1 \leq i \leq N} |\langle X_i, \theta \rangle| d\mu(\theta).$

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Theorem (Dafnis, Giannopoulos, Tsolomitis)

Let K be an isotropic body and K_N a random polytope in K. If $cn \leq N \leq e^{\sqrt{n}}$

$$c_1\sqrt{\log\frac{N}{n}}L_K \leq \mathbb{E}Q_n(K_N) \leq \mathbb{E}Q_1(K_N) \leq c_2\sqrt{\log N}L_K.$$

Corollary

Let K be an isotropic body and K_N a random polytope in K. If $n^2 \leq N \leq e^{\sqrt{n}}$

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Let K be an isotropic body and K_N a random polytope in K. If $n^2 \leq N \leq e^{\sqrt{n}}$, then with probability greater than $1 - \frac{1}{N}$

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Expectation of the support function.

Theorem (A., Prochno)

Let K be a symmetric isotropic body and K_N ($n \leq N \leq n^\delta$) a random polytope in K. Then, there exists a set $\Theta \subseteq S^{n-1}$, with $\mu(\Theta) \geq 1 - Ce^{-\sqrt{n}}$, such that for every $\theta \in \Theta$

$$\mathbb{E} h_{K_N}(\theta) = \mathbb{E} \max_{1 \leq i \leq N} |\langle X_i, \theta \rangle| \geq \frac{c}{\sqrt{\delta}} \sqrt{\log N} L_K$$

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$$\begin{aligned}
\mathbb{E}w(K_N) &= \mathbb{E}\int_{S^{n-1}} \max_{1\leq i\leq N} |\langle X_i, \theta\rangle| d\mu(\theta) \\
&= \int_{S^{n-1}} \mathbb{E} \max_{1\leq i\leq N} |\langle X_i, \theta\rangle| d\mu(\theta) \\
&\geq \mu(\Theta)\frac{c}{\sqrt{\delta}}\sqrt{\log N} L_K \geq \frac{c'}{\sqrt{\delta}}\sqrt{\log N} L_K
\end{aligned}$$

Expectation of the support function and Orlicz norms.

Theorem (Gordon, Litvak, Schütt, Werner)

Let $X_1, \ldots, X_N$ be independent, identically distributed random variables with finite first moment. Let

$$M(s) = \int_0^s \int_{\{\frac{1}{t} \leq |X_1|\}} |X_1| d\mathbb{P} dt.$$

Then, for every $x \in \mathbb{R}^N$,

$$\mathbb{E} \max_{1 \leq i \leq N} |x_i X_i| \sim \|x\|_M = \inf \left\{ s : \sum_{i=1}^{N} M\left(\frac{|x_i|}{s}\right) \leq 1 \right\}.$$

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Corollary

Let $X_1, \ldots, X_N$ be independent random vectors uniformly distributed in K and $K_N = \mathrm{conv}\{\pm X_1, \ldots, \pm X_N\}$. Given $\theta \in S^{n-1}$ let

$$M_\theta(s) = \int_0^s \int_{\{\frac{1}{t} \leq |\langle X_1, \theta \rangle|\}} |\langle X_1, \theta \rangle| d\mathbb{P} dt.$$

Then

$$\mathbb{E} h_{K_N}(\theta) = \mathbb{E} \max_{i=1,\ldots,N} |\langle X_i, \theta \rangle| \sim \|(1, \ldots, 1)\|_{M_\theta} = \inf \left\{ s \, : \, M_\theta\left(\frac{1}{s}\right) \leq \frac{1}{N} \right\}.$$

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Corollary

Let $X_1, \dots, X_N$ be independent random vectors uniformly distributed in K, $y \in \mathbb{R}^N$, and $K_{N,y} = \mathrm{conv}\{\pm y_1 X_1, \dots, \pm y_N X_N\}$. Given $\theta \in S^{n-1}$ let

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Then

$$\begin{aligned} \mathbb{E} h_{K_{N,y}}(\theta) &= \mathbb{E} \max_{i=1,\dots,N} |\langle y_i X_i, \theta\rangle| \sim \|(y_1, \dots, y_N)\|_{M_\theta} \\ &= \inf\left\{ s : \sum_{i=1}^N M_\theta\left(\frac{|y_i|}{s}\right) \leq 1 \right\}. \end{aligned}$$

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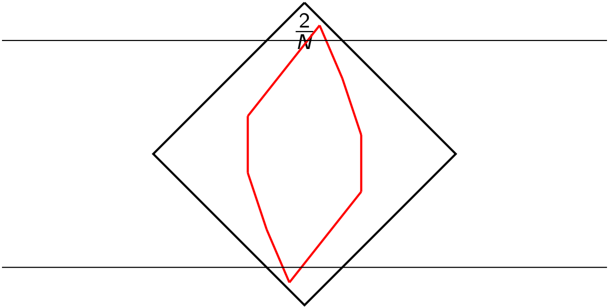


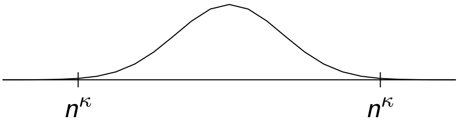
Central limit theorem

Theorem (Klartag, 2007)

Let $K \subseteq \mathbb{R}^n$ be an isotropic body and Y a random vector uniformly distributed in $\frac{K}{L_K}$. There exists a set $\Theta \subseteq S^{n-1}$ with $\mu(\Theta) \geq 1 - Ce^{-\sqrt{n}}$ such that if $\theta \in \Theta$

- $\int_{-\infty}^{\infty} |f_\theta(t) - \gamma(t)| dt \leq \frac{1}{n^\kappa}$
- For every $|t| \leq n^\kappa$ $\left|\frac{f_\theta(t)}{\gamma(t)} - 1\right| \leq \frac{1}{n^\kappa}$

f_θ is the density of $\langle Y, \theta \rangle$ and $\gamma(t) = \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}$ the density of $\mathcal{N}(0,1)$.



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- If $n \leq N \leq n^{\delta}$, for any $\theta \in \Theta$,

$$|\{x \in K \,:\, |\langle x, \theta \rangle| \geq \alpha \sqrt{\log N} L_K\}| \geq \frac{c}{\alpha N^{\frac{\alpha^2}{2}} \sqrt{\log N}},$$

provided $\frac{\alpha^2}{2} < \frac{\kappa}{\delta}$.

Expectation of the support function of random polytopes.

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- If $n \le N \le n^{\delta}$ and $\theta \in \Theta$, then $\mathbb{E} h_{K_N}(\theta) \ge \frac{c}{\sqrt{\delta}}\sqrt{\log N} L_K$.

Expectation of the mean width of perturbations of random polytopes

- What can we say about $\mathbb{E}w(K_{N,y})$?

Expectation of the mean width of perturbations of random polytopes

- What can we say about $\mathbb{E}w(K_{N,y})$?

Theorem (A., Prochno)

Let $X_1, \ldots, X_N$ be independent random vectors in an isotropic convex body K, $y \in \mathbb{R}^N$, if we define

$$I(y) := \left\{ k \in \{1, \ldots, n\} \, : \, \frac{1}{|y_k^*| \sqrt{\frac{1}{k} \sum_{i=1}^k \frac{1}{|y_i^*|^2}}} \left(\leq \frac{|y_1^*|}{|y_k^*|} \right) \leq n^{c_1} \right\},$$

where y^* denotes the decreasing rearrangement of y, then

$$\mathbb{E}w(K_{N,y}) \geq \sup_{k \in I(y)} \frac{c_2 \sqrt{\log(k+1)}}{\sqrt{\frac{1}{k} \sum_{i=1}^k \frac{1}{|y_k^*|^2}}} L_K .$$

Expectation of the mean width of perturbations of random polytopes

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Expectation of the mean width of perturbations of random polytopes

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Theorem (A., Prochno)

Let $X_1, \dots, X_N$ be independent random vectors in an isotropic convex body K ($n \le N \le e^{\sqrt{n}}$) and let G be a Gaussian random vector in $\mathbb{R}^N$ independent of $X_1, \dots, X_N$. Then there exist absolute constants c, c_1, c_2 such that for every $t > 0$

$$\mathbb{P}_G\left(c_1(1-t) \le \frac{\mathbb{E}_{X_1,\dots,X_N} w(K_{N,G})}{(\log N) L_K} \le c_2(1+t)\right) \ge 1 - \frac{1}{N^{ct^2}}.$$

Expectation of the mean width of random perturbations of random polytopes

Sketch of the proof:

$\mathbb{E}w(K_{N,G})$. For any $\theta \in S^{n-1}$, $\mathbb{E}h_{K_{N,G}}(\theta) = \mathbb{E}\max_{1\le i\le N} |\langle G_i X_i, \theta\rangle|$

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  - If $n \leq N \leq n^\delta$, for every $\theta \in S^{n-1}$

$$\mathbb{E}h_{K_{N,G}}(\theta) \sim \mathbb{E}_G \|(G_1, \ldots, G_N)\|_{M_\theta}$$

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$$\|(1,\dots,1)\|_{N_\theta} \sim \mathbb{E}h_{K_{N,G}}(\theta) \sim \mathbb{E}_G\|(G_1,\dots,G_N)\|_{M_\theta}$$

$$N_\theta(s) = 2\int_0^s\int_{-\infty}^{\infty}\int_{\frac{1}{|a|t}}^{\infty}|a||K\cap\{\langle x,\theta\rangle = a\}|b\frac{e^{-\frac{b^2}{2}}}{\sqrt{2\pi}}\,db\,da\,dt.$$

Expectation of the mean width of random perturbations of random polytopes

- $N_\theta\left(\frac{1}{\alpha^2 \log N}\right) \geq \frac{1}{N^{2\alpha^2}\sqrt{2\pi\alpha^2 \log N}} \left|\{x \in K \,:\, |\langle X, \theta\rangle| \geq \alpha\sqrt{\log N} L_K\}\right|$

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- By CLT there exists $\Theta\subseteq S^{n-1}$ with $\mu(\Theta)\geq 1-Ce^{-\sqrt{n}}$ such that

$$N_\theta\left(\frac{1}{\alpha^2\log N}\right)\geq \frac{c}{N^{\frac{5}{2}\alpha^2}\alpha^2\log N} > \frac{1}{N}$$

for all $\theta\in\Theta$ if α small enough.

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Concentration of measure

Theorem (Concentration of measure)

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz function with constant L. Then

$$\mathbb{P}_G\left(\left|\frac{f(G)}{\mathbb{E}f(G)} - 1\right| \geq t\right) \leq e^{-\frac{ct^2(\mathbb{E}f(G))^2}{L^2}}.$$

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Expectation of the mean width of perturbations of random polytopes

Theorem (A., Prochno)

Let $X_1, \ldots, X_N$ be independent random vectors in an isotropic convex body K ($n \leq N \leq e^{\sqrt{n}}$) and let u be a random vector uniformly distributed on S^{N-1} independent of $X_1, \ldots, X_N$. Then there exist absolute constants c, c_1, c_2 such that for every $t > 0$

$$\sigma\left(u \in S^{N-1} \,:\, c_1(1-t) \leq \frac{\mathbb{E}_{X_1,\ldots,X_N} w(K_{N,u})}{\frac{\log N}{\sqrt{N}} L_K} \leq c_2(1+t)\right) \geq 1 - \frac{1}{N^{ct^2}}.$$

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$$\mathbb{P}_y \left(c_1(1-t) \le \frac{\mathbb{E}_{X_1,\dots,X_N} w(K_{N,y})}{\frac{(\log N)^{\frac{1}{p}+\frac{1}{2}}}{N^{\frac{1}{p}}} L_K} \le c_2(1+t) \right) \ge 1 - \frac{1}{N^{\frac{(ct)^p}{p}}}.$$

Mean outer radii of a convex body

$K \subseteq \mathbb{R}^n$ symmetric convex body

- $w(K) = \int_{S^{n-1}} h_K(\theta) d\sigma(\theta) = \int_{G_{n,1}} R(P_F K) d\nu(F)$.

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- $\mathbb{E}\widetilde{R}_k(K_N)$?

Mean outer radii of random polytopes.

Theorem (A., Dafnis, Prochno, Hernández-Cifre)

Let $X_1, \dots, X_N$ be independent random vectors in an isotropic convex body K ($n \leq N \leq e^{\sqrt{n}}$)

$$\mathbb{E}\widetilde{R}_k(K_N) \sim \max\left\{\sqrt{k}, \sqrt{\log N}\right\} L_K.$$

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Theorem (A., Dafnis, Prochno, Hernández-Cifre)

Let $X_1, \ldots, X_N$ be independent random vectors in an isotropic convex body K ($n \leq N \leq e^{\sqrt{n}}$), $s > 0$

$$c_1(s) \max\left\{\sqrt{k}, \sqrt{\log \frac{N}{n}}\right\} L_K \leq \widetilde{R}_k(K_N) \leq c_2(s) \max\left\{\sqrt{k}, \sqrt{\log N}\right\} L_K$$

with probability greater than $1 - N^{-s}$.

Mean outer radii of random polytopes.

Sketch of the proof:

- $\widetilde{R}_k(K_N) \leq ct \left(\int_{G_{n,k}} I_q(K,F)^q \, d\nu(F) \right)^{1/q}$ with probability greater than $1 - t^{-q}$.

$$I_q(K,F) = \left(\int_K |P_F x|^q dx \right)^{\frac{1}{q}} .$$

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- $q = \log N,\ t = e^s$ and $I_{-q}(K) \sim \sqrt{n} L_K$

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Mean outer radii of isotropic bodies.

Theorem

Let $K \subseteq \mathbb{R}^n$ be an isotropic convex body, then

$$w(K) \leq Cn^{\frac{3}{4}} L_K$$

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Pivovarov gave a proof using random polytopes in K.

Theorem (A., Dafnis, Prochno, Hernández-Cifre)

Let $K \subseteq \mathbb{R}^n$ be an isotropic convex body, then

$$\widetilde{R}_k(K) \leq C \max\{\sqrt{k}, n^{\frac{1}{4}}\}\sqrt{n} L_K.$$