Affine invariant points

Mathieu Meyer, Carsten Schütt and Elisabeth Werner

Mathematisches Seminar
CAU Kiel

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- Then a map $p : \mathcal{K}_n \to \mathbb{R}^n$ is called an affine invariant point, if p is continuous and if for every nonsingular affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ one has,

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A point that is affine invariant, but not continuous

$$p(K) = \begin{cases} \frac{1}{|\operatorname{ext}(K)|} \sum_{v \in \operatorname{ext}(K)} v & K \text{ is a polytope} \\ g(K) & K \text{ is not a polytope} \end{cases}$$

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Theorem

Yes.

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Theorem

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Intuitively, there should be a lot of affine invariant points. Grünbaum lists only a few. For the proof of the theorem we have to construct enough affine invariant points.

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- Affine invariant point

$$g_\delta(K) = g(K \setminus K_\delta)$$

We introduce a norm on $V\mathcal{P}_n = \mathcal{P}_n - g$

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By compactness $V\mathcal{P}_n$ cannot be finite dimensional.

In order to show

$$\frac{1}{4} \leq \|v_{\delta_j} - v_{\delta_k}\|_{\mathcal{P}}.$$

we find for every pair $j \neq k$ a convex body K such that

$$\frac{1}{4} \leq \|v_{\delta_j}(K) - v_{\delta_k}(K)\|_2.$$

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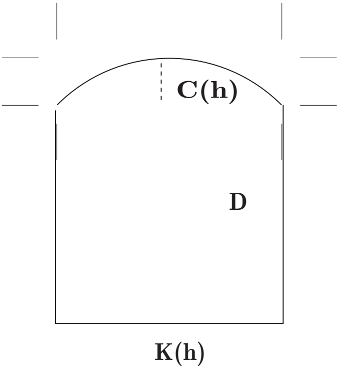

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Schütt and Werner

Let K be a convex body in $\mathbb{R}^n$. Then one has

$$c_n \lim_{\delta \to 0} \frac{|K| - |K_\delta|}{(\delta|K|)^{\frac{2}{n+1}}} = \int_{\partial K} \kappa^{\frac{1}{n+1}}(x)\ d\mu_K(x).$$

where $c_n = 2\left(\frac{|B^{n-1}|}{n+1}\right)^{\frac{2}{n+1}}$.

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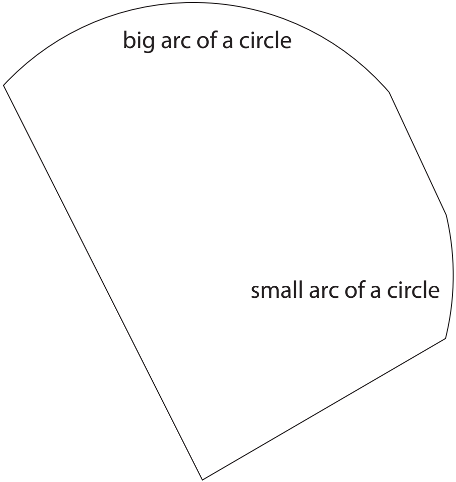
These convex bodies are actually dense in $\mathcal{K}_n$ with respect to the Hausdorff metric.


big arc of a circle
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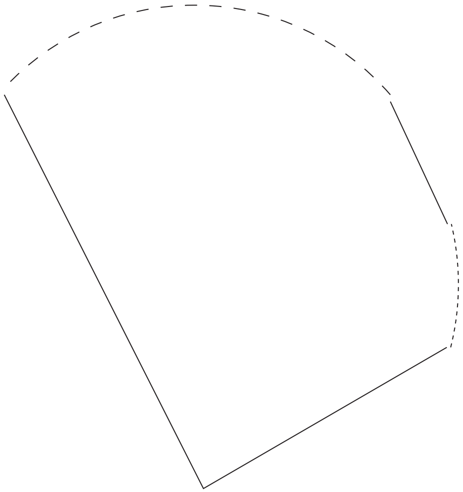

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Let K be a convex body and let $T:\mathbb{R}^n\to\mathbb{R}^n$ be an affine map with $T(K)=K$. Then we have for all affine invariant points p

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Grünbaum: Do we have $\mathcal{F}_n(K) = \mathcal{P}_n(K)$?

Theorem

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$$R(h + t\xi) = h - t\xi$$

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We have

$$H = \mathcal{P}_n(K) \qquad\qquad \xi \in \mathcal{P}_n(K^\circ)^\perp$$

Definition

A map $A : \mathcal{K}_n \to \mathcal{K}_n$ is called an affine invariant set mapping, if A is continuous and if for every nonsingular affine map T of $\mathbb{R}^n$, one has

$$A(TK) = T(A(K)).$$

We then call $A(K)$, or simply the map A, an affine invariant set mappings. We denote by $\mathfrak{S}_n$ the set of affine invariant set mappings,

$$\mathfrak{S}_n = \{A : \mathcal{K}_n \to \mathcal{K}_n \big| A \text{ is affine invariant and continuous}\}.$$

If $p \in \mathfrak{P}_n$ and $A \in \mathfrak{S}_n$, then $p \circ A \in \mathfrak{P}_n$.

Lemma

Let $p \in \mathfrak{P}_n$ *and let* g *be the centroid. For* $0 < \varepsilon < 1$*, define* $A_{p,\epsilon} : \mathcal{K}_n \to \mathcal{K}_n$ *by*

$$A_{p,\epsilon}(K) = \left\{ x \in K \,\middle|\, \langle x, p((K - g(K))^\circ)\rangle \geq \sup_{y \in K} \langle y, p((K - g(K))^\circ)\rangle - \varepsilon \right\}.$$

Then $A_{p,\epsilon}$ *is an affine invariant set map.*

Lemma

Let $K \in \mathcal{K}_n$ and let $P : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto $\mathfrak{P}_n((K - g(K))^\circ)$. Then the restriction of P to the subspace $\mathfrak{P}_n(K - g(K))$ is an isomorphismn between $\mathfrak{P}_n(K - g(K))$ and $\mathfrak{P}_n((K - g(K))^\circ)$.
In particular,

$$\dim(\mathfrak{P}_n(K - g(K))) = \dim(\mathfrak{P}_n((K - g(K)))^\circ).$$