# Helly's Theorem and translates of convex sets.

## Geometrical, vector and set-theoretic differences. Covers, intersections. Support function.

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# Outline

# Definitions

- Denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of natural, real and complex numbers respectively.
- Let $S_1, S_2 \subset \mathbb{R}^n$, $n \in \mathbb{N}$. Denote by $S_1 + S_2 := \{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}$ *the Minkowski sum* of $S_1$ and $S_2$.
- For $S \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, the set $S + x := S + \{x\}$ is a parallel translation, i. e. *translate,* of set $S$ on the vector $x$.

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# Vincensini–Klee–Edelstein Theorem

respectively, 1939, 1953, 1958

**Theorem VKE.**

*Suppose $\mathcal{S}$ is a family of at least $n+1$ convex sets in $\mathbb{R}^n$, $C$ is a convex set in $\mathbb{R}^n$, and $\mathcal{S}$ is finite or $C$ and all members of $\mathcal{S}$ are compact. Then the existence of some translate of $C$ which intersects [is contained in; contains] all members of $\mathcal{S}$ is guaranteed by the existence of such a translate for each $n+1$ members of $\mathcal{S}$.*

This result is a corollary of Helly's Theorem on intersection of convex sets. Conversely, classical Helly's Theorem follows from Theorem VKE in the part "intersects", if $C := \{0\}$.

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# Again definitions

- A vector $y \in \mathbb{R}^n$ is a *direction of recession* of the set $C \subset \mathbb{R}^n$ iff for $\forall c \in C$ , $\forall \lambda > 0$ we have $c + \lambda y \in C$.
- A vector $y \in \mathbb{R}^n$ is a *direction of linearity* of the set $C \subset \mathbb{R}^n$ iff both $y$ and $-y$ are direction of recession.
- A set $C \subset \mathbb{R}^n$ is *polyhedral* iff $C$ is a intersection of a finite number of the closed half-spaces.

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# Helly's Theorem

E. Helly, 1930, R. Rockafellar, 1965

## Theorem HR

*Let* $\mathcal{C} := \{C_\alpha \subset \mathbb{R}^n : \alpha \in \mathrm{A}\}$ *be a family of convex sets, where* $\mathrm{A}$ *is an index set. If* $\mathrm{A}$ *is finite set or*

(d) *all set* $C_\alpha$, $\alpha \in \mathrm{A}$, *are closed, there exists a finite subset* $\mathrm{A}_0 \subset \mathrm{A}$ *such that all* $C_\alpha$ *are polyhedral for* $\alpha \in \mathrm{A}_0$, *and each common direction of recession for all* $C_\alpha$, $\alpha \in \mathrm{A}$, *is direction of linearity for* $C_\alpha, \forall \alpha \in \mathrm{A} \setminus \mathrm{A}_0$,

*and for every* $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathrm{A}$ *the intersection* $\bigcap_{k=0}^{n} C_{\alpha_k}$ *non-empty* ($\neq \varnothing$), *then* $\bigcap_{\alpha \in \mathrm{A}} C_\alpha \neq \varnothing$.

## Remark 1

If all set $C_\alpha$, $\alpha \in \mathrm{A}$, are closed, and there exists $\mathrm{A}' \subset \mathrm{A}$ such that the intresection $\bigcap_{\alpha\in\mathrm{A}'} C_\alpha$ is bounded, then the condition (d) is fulfilled automatically because the common directions of recession simply do not exist.

# Set-theoretic, vector, geometric differences.

## And again definitions

For $C, S \subset \mathbb{R}^n$,

- *the set-theoretic difference* $C \setminus S := \{c \in C \colon c \notin S\}$;
- *the vector, or algebraic, difference* $C - S := \{c - s \colon c \in C, s \in S\}$;
- *the geometrical difference,* or Minkowski difference, $C \overset{*}{-} S := \{x \in \mathbb{R}^n \colon S + x \subset C\}$.

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# Problems

Let $\mathrm{A}$ and $\mathrm{B}$ are index sets. Let $\mathcal{C} := \{C_\alpha\}_{\alpha\in\mathrm{A}}$, and $\mathcal{S} := \{S_\beta\}_{\beta\in\mathrm{B}}$, are families of subsets in $\mathbb{R}^n$. Let

$$C := \bigcap_{\alpha\in\mathrm{A}} C_\alpha, \quad S := \bigcup_{\beta\in\mathrm{B}} S_\beta.$$

We investigate the following problems. What relations will be between $C$ and $S$, if for every sets of indexes $\{\alpha_0,\dots,\alpha_n\}\subset\mathrm{A}$, $\{\beta_0,\dots,\beta_n\}\subset\mathrm{B}$ *the intersection*

- $\bigcap_{k=0}^n \bigl(C_{\alpha_k} \overset{*}{-} S_{\beta_k}\bigr)$ *is non-empty set?*
- $\bigcap_{k=0}^n \bigl(C_{\alpha_k} - S_{\beta_k}\bigr)$ *is non-empty set?*
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## Covering Theorem

*Let all $C_\alpha$ are convex sets.*
*Suppose* $\operatorname{card} \mathrm{A} < \infty$ *and* $\operatorname{card}\{\beta \in \mathrm{B} \colon S_\beta \neq \varnothing\} < \infty$
*or* $\mathcal{C} = \{C_\alpha\}_{\alpha\in\mathrm{A}}$ *the condition* (d) *from Helly's Theorem is fulfilled, but with additional restrictions* $\mathrm{A}_0 = \varnothing$ *or* $\operatorname{card} \mathrm{B} < \infty$.
*Then following four statements are equivalent:*

- (T) *a translate of $C$ covers $S$;*
- (C) *for every $n+1$ members from $\mathcal{C}$ a translate of $S$ contains in the intersection of these $n+1$ sets;*
- (S) *for every $n+1$ members from $\mathcal{S}$ a translate of $C$ covers all these $n+1$ sets;*
- (CS) *for every $n+1$ indexes $\alpha_0,\dots,\alpha_n \in \mathrm{A}$ and $\beta_0,\dots,\beta_n \in \mathrm{B}$ the intersection $\bigcap_{k=0}^{n}\bigl(C_{\alpha_k} \stackrel{*}{-} S_{\beta_k}\bigr) \neq \varnothing$.*

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# Remarks

1. For $S = \{0\}$, $\mathcal{S} = \{S\}$, the implication (C)$\Rightarrow$(T) of this Theorem gives exactly Helly's Theorem, i. e. Theorem HR.
2. Even if $\mathcal{C}$ consists exactly of one element, then implication (S)$\Rightarrow$(T) of this Theorem generalizes Theorem VKE in the part "contains", where all $S_\beta$ are convex and closed (in our version the sets $S_\beta$ are arbitrary).
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## Intersection Theorem

*Let all vector differences $C_\alpha - S_\beta$ are convex for all $\alpha \in \mathrm{A}, \beta \in \mathrm{B}$. Suppose* card $\mathrm{A}$ + card $\mathrm{B} < \infty$ *or*

(id) *each algebraic difference $C_\alpha - S_\beta$ is closed, for a finite subsets $\mathrm{A}_0 \subset \mathrm{A}, \mathrm{B}_0 \subset \mathrm{B}$ differences $C_\alpha - S_\beta$ are polyhedral for all $(\alpha, \beta) \in \mathrm{A}_0 \times \mathrm{B}_0$, and each common direction of recession for all $C_\alpha - S_\beta$, when $(\alpha, \beta) \in \mathrm{A} \times \mathrm{B}$, is direction of linearity for $C_\alpha - S_\beta$ $\forall (\alpha, \beta) \in (\mathrm{A} \times \mathrm{B}) \setminus (\mathrm{A}_0 \times \mathrm{B}_0)$*

*Then the following statements equivalent:*

(I) *there is a uniform vector $x \in \mathbb{R}^n$ such that for each index $\beta \in \mathrm{B}$ every translate $S_\beta + x$ meets all $C_\alpha$ from $\mathcal{C}$;*

(CSI) *for every $n+1$ indexes $\alpha_0, \dots, \alpha_n \in \mathrm{A}$ and $\beta_0, \dots, \beta_n \in \mathrm{B}$ the intersection $\bigcap_{k=0}^{n} (C_{\alpha_k} - S_{\beta_k}) \neq \varnothing$.*

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*Let all vector differences* $C_\alpha - S_\beta$ *are convex for all* $\alpha \in \mathrm{A}, \beta \in \mathrm{B}$. *Suppose* $\operatorname{card} \mathrm{A} + \operatorname{card} \mathrm{B} < \infty$ *or*

(id) *each algebraic difference* $C_\alpha - S_\beta$ *is closed, for a finite subsets* $\mathrm{A}_0 \subset \mathrm{A}$, $\mathrm{B}_0 \subset \mathrm{B}$ *differences* $C_\alpha - S_\beta$ *are polyhedral for all* $(\alpha, \beta) \in \mathrm{A}_0 \times \mathrm{B}_0$, *and each common direction of recession for all* $C_\alpha - S_\beta$, *when* $(\alpha, \beta) \in \mathrm{A} \times \mathrm{B}$, *is direction of linearity for* $C_\alpha - S_\beta$ $\forall (\alpha, \beta) \in (\mathrm{A} \times \mathrm{B}) \setminus (\mathrm{A}_0 \times \mathrm{B}_0)$

*Then the following statements equivalent:*

(I) *there is a uniform vector* $x \in \mathbb{R}^n$ *such that for each index* $\beta \in \mathrm{B}$ *every translate* $S_\beta + x$ *meets all* $C_\alpha$ *from* $\mathcal{C}$;

(CSI) *for every* $n+1$ *indexes* $\alpha_0, \dots, \alpha_n \in \mathrm{A}$ *and* $\beta_0, \dots, \beta_n \in \mathrm{B}$ *the intersection* $\bigcap_{k=0}^{n} (C_{\alpha_k} - S_{\beta_k}) \neq \varnothing$.

## Remark 2

If $\operatorname{card}\mathrm{B} = 1$ and $S_\beta = \{0\}$, then Intersection Theorem gives exactly Helly's Theorem, i. e. Theorem HR.

## Corollary (intersection)

*Let $C \subset \mathbb{R}^n$ be a non-empty and $C - S_\beta$ are convex for all $\beta \in \mathrm{B}$, where $\operatorname{card}\mathrm{B} < \infty$ or all vector differences $C - S_\beta$ closed and at least one of them is bounded.*
*If for every $n+1$ indexes $\beta_0, \dots \beta_n$ a translate of $C$ intersects simultaneously all sets $S_{\beta_0}, \dots S_{\beta_n}$, then a translate of $C$ intersets all members of family $S$.*

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This Corollary generalizes and involves the Theorem VKE in the part "intersects".

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## Difference Theorem

*Let all differences $C_\alpha \setminus S_\beta$ are convex. Suppose* card $\mathrm{A}$ + card $\mathrm{B} < \infty$ *or*

(dd) *each difference $C_\alpha \setminus S_\beta$ is closed, for finite $\mathrm{A}_0 \subset \mathrm{A}$, $\mathrm{B}_0 \subset \mathrm{B}$ the differences $C_\alpha \setminus S_\beta$ are polyhedral $\forall(\alpha, \beta) \in \mathrm{A}_0 \times \mathrm{B}_0$, and each common direction of recession for all $C_\alpha \setminus S_\beta$ $\forall(\alpha, \beta) \in \mathrm{A} \times \mathrm{B}$ is direction of linearity for differences $C_\alpha \setminus S_\beta$ for all $(\alpha, \beta) \in (\mathrm{A} \times \mathrm{B}) \setminus (\mathrm{A}_0 \times \mathrm{B}_0)$.*

*The following statement are equivalent:*

(D) *the difference $C \setminus S$ is non-empty;*

(CSD) *for every $n+1$ indexes $\alpha_0, \dots, \alpha_n \in \mathrm{A}$, $\beta_1, \dots, \beta_n \in \mathrm{B}$ the intersection $\bigcap_{k=0}^{n} (C_{\alpha_k} \setminus S_{\beta_k}) \neq \varnothing$.*

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# Covering by a translate and the support function

## Support function

Let $a = (a_1, \ldots, a_n), s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ and $< a, s >:= \sum_{k=1}^n a_k s_k$ be the scalar product. Let $S \subset \mathbb{R}^n$. Denote by

$$H_S \colon \mathbb{R}^n \to [-\infty, +\infty], \quad H_S(a) := \sup_{s \in S} < a, s >, \quad a \in \mathbb{R}^n,$$

*the support function* of the set $S$.

## Theorem on support functions

*Let $C \subset \mathbb{R}^n$ be a convex bounded set, $\mathbb{S}$ be a family of sets from $\mathbb{R}^n$, and $\mathcal{S} := \bigcup_{S \in \mathbb{S}} S$. Suppose that $C$ is closed or $\mathcal{S}$ is open. Then the following four statements are equivalent.*

1. *A translate of $C$ covers the set $\mathcal{S}$.*
2. *For every $S_1, \ldots, S_{n+1} \in \mathbb{S}$ and for every closed semispaces $C_1, \ldots C_{n+1} \supset C$ there is a vector $x \in \mathbb{R}^n$ such that every translate $S_k + x$ contains in $C_k$ for all $k = 1, \ldots, n+1$.*
3. *For every $S_1, \ldots, S_{n+1} \in \mathbb{S}$ and for every vectors $a_1, \ldots, a_{n+1} \in \mathbb{R}^n$ and numbers $p_1, \ldots, p_{n+1} \geqslant 0$ the condition $\sum_{k=1}^{n+1} p_k a_k = 0$ implies inequality*

$$\sum^{n+1} p_k H_{S_k}(a_k) \leqslant \sum^{n+1} p_k H_C(a_k).$$

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## Theorem on support functions (continuation)

4 *For every* $S_1, \dots, S_{n+1} \in \mathcal{S}$ *and for every system of vectors*

$$\begin{cases} a_1 = (a_{11}, \dots, a_{1n}) \in \mathbb{R}^n, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ a_{n+1} = (a_{n+1,1}, \dots, a_{n+1,n}) \in \mathbb{R}^n \end{cases}$$

*of a rank* $r > 0$ *there exists a nonzero minor*

$$\Delta = \begin{vmatrix} a_{k_1 j_1} & \cdots & a_{k_1 j_r} \\ \dots & \dots & \dots \\ a_{k_r j_1} & \cdots & a_{k_r j_r} \end{vmatrix}$$

*of* $r$*-th order such that*

## Theorem on support functions (continuation)

*for $k = 1, \dots, n+1$ the inequality*

$$\frac{1}{\Delta}\begin{vmatrix} a_{k_1 j_1} & \cdots & a_{k_1 j_r} & H_{S_{k_1}}(a_{k_1}) \\ \dots & \dots & \dots & \dots \\ a_{k_r j_1} & \cdots & a_{k_r j_r} & H_{S_{k_r}}(a_{k_r}) \\ a_{k j_1} & \cdots & a_{k j_r} & H_{S_k}(a_k) \end{vmatrix} \leqslant \frac{1}{\Delta}\begin{vmatrix} a_{k_1 j_1} & \cdots & a_{k_1 j_r} & H_C(a_{k_1}) \\ \dots & \dots & \dots & \dots \\ a_{k_r j_1} & \cdots & a_{k_r j_r} & H_C(a_{k_r}) \\ a_{k j_1} & \cdots & a_{k j_r} & H_C(a_k) \end{vmatrix}.$$

*is fulfilled.*

# Case $n = 2$, i. e. $\mathbb{R}^2 \leftrightarrow \mathbb{C}$

We adapt our results on the case of the complex plane. Let $S \subset \mathbb{C}$. Denote by

$$h_S \colon \mathbb{R} \to [-\infty, +\infty], \quad h_S(\theta) := \sup_{s \in S} \operatorname{Re} s e^{-i\theta}, \theta \in \mathbb{R},$$

*the support function* of the set $S \subset \mathbb{C}$. The function $h_S$ is $2\pi$-periodic.

## Covering Theorem for $\mathbb{C}$

*Let $C$ be a convex bounded set in $\mathbb{C}$ and $\mathbb{S}$ be a family of subsets $S \subset \mathbb{C}$, $\mathcal{S} = \bigcup_{S \in \mathbb{S}} S$. Suppose that $C$ is closed or $\mathcal{S}$ is open. Then the following four statements are equivalent.*

1. *A translate of $C$ covers the set $\mathcal{S}$.*
2. *For every $S_1, S_2, S_3 \in \mathbb{S}$ and for each closed triangle described around $C$ there is a point $z \in \mathbb{C}$ such that all three translates $S_1 + z, S_2 + z, S_3 + z$ contain in this triangle.*
3. *for every $S_1, S_2, S_3 \in \mathbb{S}$ and for every $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ and numbers $q_1, q_2, q_3 \geqslant 0$ the condition $q_1 e^{i\theta_1} + q_2 e^{i\theta_2} + q_3 e^{i\theta_3} = 0$ implies inequality*

$$q_1 h_{S_1}(\theta_1) + q_2 h_{S_2}(\theta_2) + q_3 h_{S_3}(\theta_3) \leqslant q_1 h_C(\theta_1) + q_2 h_C(\theta_2) + q_3 h_C(\theta_3).$$

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## Covering Theorem for $\mathbb{C}$ (continuation)

4. *For every $S_1, S_2, S_3 \in \mathcal{S}$ and for every numbers $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ following conditions are fulfilled.*
   (a) *If each diference of numbers $\theta_1, \theta_2, \theta_3$ is multiple to $\pi$, then for each pair $k, j \in \{1, 2, 3\}$ such that the difference $\theta_j - \theta_k$ is not multiple $2\pi$ the inequality $h_{S_1}(\theta_k) + h_{S_2}(\theta_j) \leqslant h_C(\theta_k) + h_C(\theta_j)$ is fulfilled.*
   (b) *If the difference $\theta_2 - \theta_1$ is not multiple $\pi$, then the inequality*

$$h_{S_1}(\theta_1)\frac{\sin(\theta_3-\theta_2)}{\sin(\theta_2-\theta_1)} + h_{S_3}(\theta_3) + h_{S_2}(\theta_2)\frac{\sin(\theta_1-\theta_3)}{\sin(\theta_2-\theta_1)}$$
$$\leqslant h_C(\theta_1)\frac{\sin(\theta_3-\theta_2)}{\sin(\theta_2-\theta_1)} + h_C(\theta_3) + h_C(\theta_2)\frac{\sin(\theta_1-\theta_3)}{\sin(\theta_2-\theta_1)}$$

   *is fulfilled.*

## Covering Theorem for $\mathbb{C}$ (continuation)

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$$h_{S_1}(\theta_1)\frac{\sin(\theta_3-\theta_2)}{\sin(\theta_2-\theta_1)} + h_{S_3}(\theta_3) + h_{S_2}(\theta_2)\frac{\sin(\theta_1-\theta_3)}{\sin(\theta_2-\theta_1)}$$
$$\leqslant h_C(\theta_1)\frac{\sin(\theta_3-\theta_2)}{\sin(\theta_2-\theta_1)} + h_C(\theta_3) + h_C(\theta_2)\frac{\sin(\theta_1-\theta_3)}{\sin(\theta_2-\theta_1)}$$

*is fulfilled.*

## Covering Theorem for $\mathbb{C}$ (continuation)

4 *For every $S_1, S_2, S_3 \in \mathcal{S}$ and for every numbers $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ following conditions are fulfilled.*

(a) *If each diference of numbers $\theta_1, \theta_2, \theta_3$ is multiple to $\pi$, then for each pair $k, j \in \{1, 2, 3\}$ such that the difference $\theta_j - \theta_k$ is not multiple $2\pi$ the inequality $h_{S_1}(\theta_k) + h_{S_2}(\theta_j) \leqslant h_C(\theta_k) + h_C(\theta_j)$ is fulfilled.*

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### Theorem on support functions (for unbounded convex sets)

*Let $n \in \mathbb{N}$, $C$ be a unbounded convex closed set in $\mathbb{R}^n$, and $\mathcal{S}$ be a family of subsets in $\mathbb{R}^n$, and $S$ be the union all members from $\mathcal{S}$. Suppose* card $\mathcal{S} < \infty$, *and the set $C$ is polyhedral or each direction of recession for $C$ is direction of linearity for $C$.*
*Then the statements 1–4 from Theorem on support function are equivalent.*

### Covering Theorem for $\mathbb{C}$ (for unbounded convex sets)

*Let $C$ be a unbounded convex closed polygon in $\mathbb{C}$, and $\mathcal{S}$ be a family of subsets in $\mathbb{C}$, and $S$ be the union all members from $\mathcal{S}$. Suppose* card $\mathcal{S} < \infty$. *Then the statements 1–4 from Covering Theorem for $\mathbb{C}$ are equivalent.*

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# Definitions for unbounded sets

Let $C \subset \mathbb{C}$. Let's define as $B_C(\theta) := h_C(\theta) + h_C(\theta + \pi)$ *breadth* of $B$ in direction $\theta$, and $b_C := \inf_\theta B_C(\theta)$ a *thickness* of $C$.
If a vector $e^{i\theta}$ is a direction of recession (resp. linearity) for $C$, then we name as also $\theta$.
For convex $C$, we set
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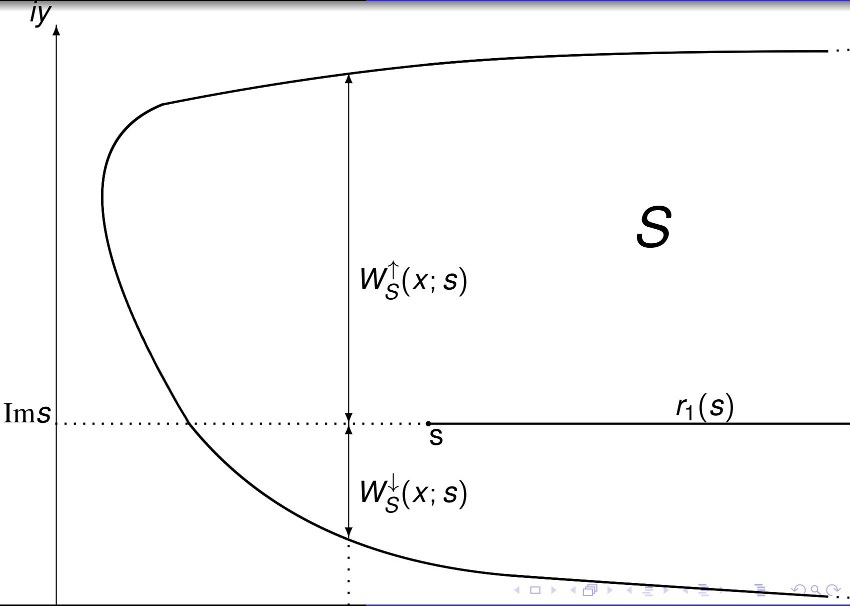
Suppose that a convex set $S \subset \mathbb{C}$ is bounded or has only one direction of recession (for determinancy of $\theta = 0$) to within summand, multiple to $2\pi$, and also contains a ray $r_1(s) := s + t$ from the beginning at $s \in S$. Let's define the *cut-off upper and lower width* of the convex set $S$ concerning point $c$ by the direction $\theta = 0$:

$$\begin{cases} W_S^{\uparrow}(x;s) := \sup\{\operatorname{Im} z - \operatorname{Im} s \colon z \in S, \operatorname{Im} z \geqslant \operatorname{Im} s, \operatorname{Re} z = x\}, & x \in \mathbb{R}, \\ W_S^{\downarrow}(x;s) := \sup\{\operatorname{Im} s - \operatorname{Im} z \colon z \in S, \operatorname{Im} z \leqslant \operatorname{Im} s, \operatorname{Re} z = x\}, & x \in \mathbb{R}. \end{cases}$$


$iy$
$S$
$W^{\uparrow}_{S}(x;s)$
$\mathrm{Im}\, s$
$r_1(s)$
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## Theorem

*Let $C \subset \mathbb{C}$ be a unbounded convex set, $S \subset \mathbb{C}$.*

- *If $C$ has two directions of recession $\theta_1, \theta_2 \in \mathbb{R}$ and $\theta_1 - \theta_2$ isn't multiple $\pi$, and $S$ is bounded, then a translate of $C$ covers $S$.*
- *If $0 < \theta_2 - \theta_1 \leqslant \pi$ and the arc $\smile(\theta_1, \theta_2) := \{e^{i\theta} : \theta_1 < \theta < \theta_2\}$, contains in $0^+C$, and $S$ is convex set such that an arc $\smile(\theta_1', \theta_2') \supset 0^+S$, where $\theta_1 < \theta_1' < \theta_2' < \theta_2$, then a translate of $C$ covers $S$.*
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## Theorem (continuation)

- *If closed set $C$ has only one direction of recession $\theta = 0$ to within summand, multiple to $2\pi$, then a translate of $C$ covers $S$ iff $S$ is bounded or has only one direction of recession $\theta = 0$ to within summand, multiple to $2\pi$, and in both cases there are $s \in S, x_S \in \mathbb{R}$ such that inequalities*

$$\begin{cases} W_S^{\uparrow}(x; s) \leqslant W_C^{\uparrow}(x + x_S; c), & x \in \mathbb{R}, \\ W_S^{\downarrow}(x; s) \leqslant W_C^{\downarrow}(x + x_S; c) & x \in \mathbb{R}. \end{cases}$$

*is fulfilled.*

## Remark 4

The case of arbitrary unbounded convex set $C \subset \mathbb{R}^n$, $n \geqslant 3$, is much more complicated. For this case it is necessary to use new geometrical characteristics. Here these questions aren't discussed as they require the considerable additional preparation.

# Thank you for your attention!