Random Unitary Matrices with Structure

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June 2013

- We will consider several properties of a random unitary matrix $\mathcal{U}$.

- When $\mathcal{U}$ has uniform or Haar distribution, then these properties are very well understood.

- More general distributions yield very similar behavior, yet here there are few results.

- View generalizing as a form of universality.

- Many open questions.

Relationship to uncertainty principles

U - unitary matrix
P_1 and P_2 - coordinate projections with support sets S_1 and S_2.

Suppose there exists x such that support$(x) \subset S_1$ and support$(Ux) \subset S_2$. Then

$$\|P_2 U P_1 x\|_2 = \|P_2 U x\|_2 = \|Ux\|_2 = \|x\|_2,$$

so that $\|P_2 U P_1\| = 1$.

If no such x exists, then $\|P_2 U P_1\| < 1$.

Thus, coordinate projections (very simple matrices) allow us to address an uncertainty principle.

Free Probability

• Area of operator algebras initiated by Dan-Virgil Voiculescu and very influential in random matrix theory.

• Studies noncommutative random variables.

• *Freeness* plays the role for noncommutative random variables that independence plays for commutative random variables.

• If X and Y are free, then there is a straightforward way to determine the laws of $X + Y$ and XY from the individual laws of X and Y.

Definition

A *noncommutative probability space* is a pair $(\mathcal{A}, \phi)$ where $\mathcal{A}$ is a unital algebra (over $\mathbb{C}$) and ϕ is a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ satisfying $\phi(1_{\mathcal{A}}) = 1$.

Definition

Subalgebras $\{\mathcal{A}_i\}_{i \in \mathcal{I}}$ of $(\mathcal{A}, \phi)$ are *free* if for all $i_1, \ldots, i_n \in \mathcal{I}$, $a_{i_1} \in \mathcal{A}_{i_1}, \ldots, a_{i_n} \in \mathcal{A}_{i_n}$,

$$\phi(a_{i_1} \cdots a_{i_n}) = 0$$

whenever $i_j \neq i_{j+1}$, $i_1 \neq i_n$, $n \geq 2$ and $\phi(a_{i_j}) = 0$ for all $1 \leq j \leq n$.

Main (simplified) consequence: if a and b are elements of two *free* subalgebras, then the law of polynomials in a and b is determined soley by the individual laws of a and b.

Theorem (D.-V. Voiculescu, 1991)

Let $\mathcal{U}_N^{(1)}$ and $\mathcal{U}_N^{(2)}$ be independent, uniformly (Haar) distributed unitary matrices of size $N \times N$ and $\{A_N\}_{N=1}^{\infty}$ and $\{B_N\}_{N=1}^{\infty}$ sequences of (nonrandom) uniformly bounded self-adjoint matrices of size $N \times N$ with spectral measures converging to μ_A and μ_B. Then, as $N \to \infty$,

$$\mathcal{U}_N^{(1)} A_N \mathcal{U}_N^{(1)*} \quad \text{and} \quad \mathcal{U}_N^{(2)} B_N \mathcal{U}_N^{(2)*}$$

are asymptotically free.

Gives a limit law for $\mathcal{U}_N^{(1)} A_N \mathcal{U}_N^{(1)*} + \mathcal{U}_N^{(2)} B_N \mathcal{U}_N^{(2)*}$

and

$$A_N \mathcal{U}_N^{(1)} B_N \mathcal{U}_N^{(1)*} A_N$$

in terms of μ_A and μ_B.

Note: theorem only stated for Haar distributed unitaries.

Voiculescu's theorem addresses $A_N \mathcal{U}_N B_N \mathcal{U}_N^* A_N$.

Simplest case:

$$P_2 \mathcal{U}_N P_1 \mathcal{U}_N^* P_2,$$

where P_1 and P_2 are orthogonal projections with ranks proportional to N.

This is the most common example for free multiplicative convolution.

Uncertainty principle formulation is *simplest* instance of a fact of free probability

Theorem (G. Anderson and B.F., 2013)

Let $W_N^{(1)}$ and $W_N^{(2)}$ be independent and uniformly distributed on the set of signed permutation matrices, and let H_N be a general Hadamard matrix, i.e. unitary with $|H_N(j,k)| = 1/\sqrt{N}$ for all j, k and N.

Let $\{A_N\}_{N=1}^{\infty}$ and $\{B_N\}_{N=1}^{\infty}$ be as in the previous theorem

$$\left(\sigma(A_N) \to \mu_A \text{ and } \sigma(B_N) \to \mu_B\right)$$

and set

$$\mathcal{U}_N^{(j)} = W_N^{(j)} H_N W_N^{(j)*}, \quad j = 1, 2.$$

Then as $N \to \infty$,

$$\mathcal{U}_N^{(1)} A_N \mathcal{U}_N^{(1)*} \quad \text{and} \quad \mathcal{U}_N^{(2)} B_N \mathcal{U}_N^{(2)*}$$

are asymptotically free.

Thus, in this setting,

$$\mathcal{U} \quad \text{and} \quad WHW^*$$

behave the same, where

$$W \;\text{ is a random signed permutation}$$

$$H \;\text{ is a Hadamard matrix.}$$

Technical approach

For all N let $\left\{ U_N^{(i)} \right\}_{i \in I}$ be a set of independent copies of the random unitary constructed on the previous page.

We show that this sequence if *asymptotically liberating*:

for $i_1, \ldots, i_\ell \in I$ satisfying

$$\ell \geq 2, \ \ i_1 \neq i_2, \ \ \ldots, \ \ i_{\ell-1} \neq i_\ell, \ \ i_\ell \neq i_1, \tag{1}$$

there exists $c(i_1, \ldots, i_\ell)$ such that

$$\left| \mathbb{E}\mathrm{tr} \left(U_{i_1}^{(N)} A_1 U_{i_1}^{(N)*} \cdots U_{i_\ell}^{(N)} A_\ell U_{i_\ell}^{(N)*} \right) \right| \leq c(i_1, \ldots, i_\ell) \|A_1\| \cdots \|A_\ell\|$$

for all constant matrices $A_1, \ldots, A_\ell \in \mathbb{C}^{N \times N}$ *with trace zero.*

Let's return to

$$A := P_2 \mathcal{U}_N P_1 \mathcal{U}_N^* P_2,$$

where P_1 and P_2 are orthogonal projections with ranks pN and qN. Let

$$F(x) = \frac{1}{N} \sharp\{\lambda_i(A) \leq x\}.$$

Wachter showed in 1981 that when $F(x)$ converges almost surely to the distribution function with density

$$\begin{aligned} f_M(x) \quad := \quad & \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{2\pi x(1-x)} I_{[\lambda_-, \lambda_+]}(x) \\ & + (1 - \min(p,q))\delta_0(x) + (\max(p+q-1, 0))\delta_1(x), \end{aligned}$$

where

$$\lambda_\pm := p + q - 2pq \pm \sqrt{4pq(1-p)(1-q)}.$$

See B. Collins (2005) for extensive results.

Ensemble	Matrix Form	Matrix Name	Law
Gaussian	$X = X^*$	Wigner	Semicircle law
Laguerre	XX^*	Sample covariance	Marchenko-Pastur Law
Jacobi	$P_2 \mathcal{U} P_1 \mathcal{U}^* P_2$	MANOVA	(Kesten law)

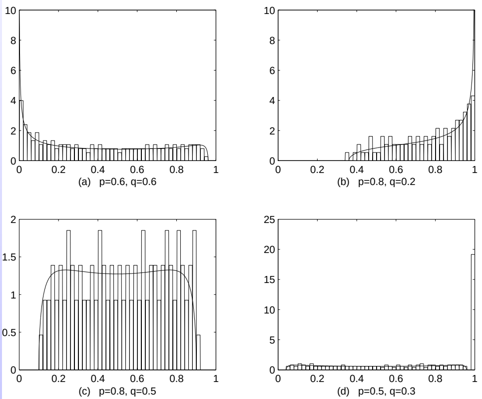

Figure: Plots for f_M for parameter pairs p, q

Toward Discrete Uncertainty Principles:

Let

$\{\mathbf{e}_j\}_{j=1}^n$ denote the standard Euclidean basis vectors in $\mathbb{C}^n$
$\{\mathbf{f}_j\}_{j=1}^n$ denote the Fourier basis vectors ($\mathbf{f}_j(k) = \frac{1}{\sqrt{n}} e^{-2\pi ijk/n}$)

and for $T, \Omega \subset \{1 \ldots n\}$, set

$$T = \{t_1, \ldots, t_k\} \text{ and } \Omega = \{\omega_1, \ldots, \omega_l\},$$

where

- $i \in T$ with probability $(1-p)$
- $i \in \Omega$ with probability $(1-q)$.

By the previous discussion, we want to study the eigenvalues of

$$U^* V V^* U = \underbrace{[\mathbf{e}_{t_1}, \ldots \mathbf{e}_{t_j}]^* [\mathbf{f}_{\omega_1}, \ldots, \mathbf{f}_{\omega_k}]}_{F_{\Omega T}} \underbrace{[\mathbf{f}_{\omega_1}, \ldots, \mathbf{f}_{\omega_k}]^* [\mathbf{e}_{t_1}, \ldots \mathbf{e}_{t_j}]}_{F^*_{\Omega T}}.$$

Key observation:

$\|F_{\Omega T}\| = 1$ if and only if there exists x with support contained in T such that $\hat{x}$ has support contained in Ω.

Using the previous theorem, we recover an earlier result.

Theorem (B.F., 2011)

The empirical eigenvalue distribution of $F_{\Omega_n T_n} F^*_{\Omega_n T_n}$ converges almost surely to f_M, reparametrized in terms of p and q:

$$f_{p,q}(x) = \frac{\sqrt{(x - r_-)(r_+ - x)}}{2\pi x(1-x)(1-\max(p,q))} \cdot I_{(r_-,r_+)}(x) + \frac{\max(0, 1-(p+q))}{1-\max(p,q)} \cdot \delta(x-1)$$

where

$$r_\pm = (\sqrt{p(1-q)} \pm \sqrt{q(1-p)})^2.$$

There is no support at 1 when $p + q < 1$.

Related work by: Tao, Meshulam, Donoho-Stark, Candès-Tao, Rudelson-Vershynin, Tropp.

Unanticipated instance of universality.

Let's return again to Voiculescu's theorem:

$\mathcal{U}_N^{(1)}$ and $\mathcal{U}_N^{(2)}$ are independent, uniformly (Haar) distributed unitary matrices of size $N \times N$.

$\{A_N\}_{N=1}^{\infty}$ and $\{B_N\}_{N=1}^{\infty}$ sequences of (nonrandom) uniformly bounded self-adjoint matrices of size $N \times N$ with spectral measures converging to μ_A and μ_B.

Then we can determine the law of

$$A_N \mathcal{U}_N^{(1)} B_N \mathcal{U}_N^{(1)*_N} A_N$$

in terms of μ_A and μ_B.

Other possibilities for $\mathcal{U}$?

Further possibility:

$$U \otimes U \text{ or } U^{\otimes k}.$$

If $U^{\otimes k}$ behaves in $\mathbb{C}^{n^k \times n^k}$ like $U \in \mathbb{C}^{n \times n}$, then the eigenvalue distributions of

$$P_2' U^{\otimes k} P_1' (U^{\otimes k})^* P_2' \text{ and } P_2 U P_1 U^* P_2$$

should be close when P_i' and P_i have normalized ranks p and q.

Consider

$$P_2 U^{\otimes k} P_1 (U^{\otimes k})^* P_2, \tag{2}$$

where U is uniformly distrubuted on $n \times n$ unitary matrices, rank $P_1 = pn^k$ and rank $P_2 = qn^k$.

Theorem (B.F. and R.R. Nadakuditi, 2013)

For $E \in [\lambda_-, \lambda_+]$, let $\mathcal{N}(E, \eta)$ denote the number of eigenvalues of (2) in $[E - \frac{\eta}{2}, E + \frac{\eta}{2}]$. Assume that $0 < c_0 < \frac{1}{16}$ and $k \leq c_0 \log n$. There exist absolute constants $C, \rho > 0$ such that for all $s > 0$ and $\alpha, \beta > 0$ satisfying $\alpha + \beta = \frac{1}{2} - c_0$, if

$$\eta := \frac{\sqrt{\rho} \log^{\frac{s}{2}+4} n}{n^{\beta}}, \tag{3}$$

then for all $\kappa > 0$

$$\mathbb{P}\left(\sup_{E \in [\lambda_- + \kappa, \lambda_+ - \kappa]} \left| \frac{\mathcal{N}(E,\eta)}{\eta n^k} - f_M(E) \right| > \frac{C}{n^{\alpha} \kappa^2} \right) \leq 2n^{k+2} e^{-\log^s n}. \tag{4}$$

Area	Results, Further Work
Classical random matrix theory	Universality for the Jacobi ensemble
Probability in high dimensions	Angles between subspaces
Free probability	Extend beyond Haar distributed unitaries
Combinatorics/discrete harmonic analysis	Uncertainty principles for finite Abelian groups

Universality $\longleftrightarrow$ Open Territory