Isomorphic Steiner symmetrization of p-convex sets

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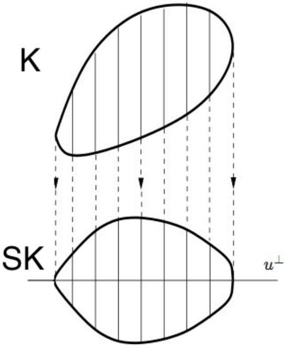
Notation

- $|K|$ will denote the volume of K.
- D_n- Euclidean ball with radius 1.
- S^{n-1}- Unit sphere.
- $Proj_E$- Orthogonal projection onto subspace E.
- $\overline{K}$ - will denote the convex hull of K. (non-standard notation).
- R_u - Reflection with respect to $u^\perp$.
- Let $h_K(x) = \sup_y \langle x, y \rangle$ denote the support function of K. Then, the mean width is defined to be

$$\omega(K) = 2M^*(K) = 2\int_{S^{n-1}} h_K(u) d\sigma(u).$$

Steiner symmetrization

Let $K \subset \mathbb{R}^n$ be a compact set, and let $u \in S^{n-1}$. Think of K as a family of line segments parallel to u. Translate each segment along u until it is symmetric with respect to $u^\perp$. The result $S_u K$, a set symmetric with respect to $u^\perp$, is called **Steiner symmetrization of K.**



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Properties

- $S_u K$ is symmetric with respect to $u^\perp$.
- $Vol(S_u K) = Vol(K)$ (Cavalieri's principle).
- $K \subseteq L$ implies $S_u K \subseteq S_u L$.
- $S_u(K + L) \supseteq S_u(K) + S_u(L)$ (Super-aditivity with respect to Minkowski sum).
- If K is convex, so is $S_u K$ (Brunn principle).
- Decreases surface area, diameter and outer radius.
- Increases inradius.
- If $u \perp v$ then $S_u S_v K$ is symmetric with respect to $u^\perp$ and $v^\perp$.
- Maps ellipsoids to ellipsoids. Any ellipsoid may be symmetrized to a ball with $n - 1$ symmetrizations.
- Euclidean ball - the only set invariant under any Steiner symmetrization.

Convergence

Theorem (Gross, 1917)

*Given a **convex** body K, there exists a sequence of directions $\{u_i\}$ such that the sequence*

$$S_{u_i} \dots S_{u_1} K$$

converges to a Euclidean ball with the same volume as K.

Many applications. For example, isoperimetric inequality.

Gross's result was extended by Mani:

Theorem (Mani, 1986)

Given a convex set K and sequence $\{u_i\}$ chosen uniformly at random, the sequence $S_{u_i} \dots S_{u_1} K$ converges to a ball with probability 1.

Convergence

Mani's result was recently extended by Volcic to compact sets:

Theorem (Volcic, 2012)

*Given a **compact** set K and sequence $\{u_i\}$ chosen uniformly at random, the sequence $S_{u_i} \dots S_{u_1} K$ converges to a ball with probability with respect to Hausdorff metric.*

And the more general case of measurable sets:

Theorem (Volcic, 2012)

*Given a **measurable** set K and sequence $\{u_i\}$ chosen uniformly at random, the sequence $S_{u_i} \dots S_{u_1} K$ converges to a ball with probability 1 with respect to the symmetric difference metric.*

Rate of convergence

- First result for convex set by Hadwiger - exponential rate of convergence $(Cn)^{n/2}$
- Bourgain, Lindenstrauss and Milman - $Cn \log n$ convergence up to universal constant.
- Klartag and Milman showed that one may approach the Euclidean ball isomorphically with at most $3n$ symmetrizations. That is,

Theorem (Klartag-Milman)

Let K be a convex set such that $|K| = |D_n|$. Then, there exist $3n$ Steiner symmetrizations that transform K into a set K' such that

$$cD_n \subseteq K' \subseteq CD_n.$$

- Klartag (2003) - polynomial in n and $\log \frac{1}{\varepsilon}$ - isometric result.

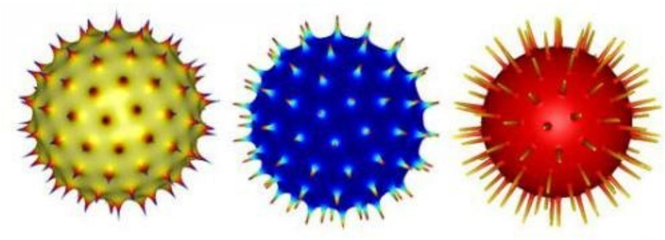
Fact

No results of this spirit exist for non-convex sets.

p-convex sets

Definition

Let $0 < p < 1$. A set K is called p-convex if for every λ and μ such that $\lambda^p + \mu^p = 1$ we have $\lambda K + \mu K \subseteq K$.



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Note: p-convex sets can differ greatly from convex sets. The Banach-Mazur distance of l_p to its convex hull is $n^{1/p-1}$.

Isomorphic result for p-convex sets

Theorem (S.)

Let K be a p-convex set for some $0 < p < 1$, such that $|K| = |D_n|$. Then, there exist $5n$ Steiner symmetrizations that transform the set K into a set K' such that $\alpha_p D_n \subseteq K' \subseteq \beta_p D_n$, where α_p, β_p are constants dependent only on p.

The proof is composed of several iterations. The plan:

1. First we show that $C_p n \log n$ symmetrizations are enough (following methods of Bourgain, Lindenstrauss and Milman).
2. Using Milman's quotient of subspace theorem (for p-convex sets) we get a bound of $C_p n$.
3. Iterating the above method we get the desired result of $5n$ symmetrizations.

Properties of p-convex sets

Fact

Steiner symmetrization preserves p-convexity.

Follows from super-additivity: For $\lambda^p + \mu^p = 1$,
$S_u(K) \supset S_u(\lambda K + \mu K) \supset \lambda S_u(K) + \mu S_u(K)$.

- Transforms projections to sections:
  $H \cap K \subset Proj_H(K) = S_H(K) \cap H$.
- Transforms (in some sense) sections to projections:
  $P_{H^\perp}(S_H(K)) \subset a_p(S_H(K) \cap H^\perp)$, whenever K is centrally-symmetric.
- Always contained in Minkowski symmetrization, i.e.

$$S_u K \subseteq \frac{1}{2}(\overline{K} + R_u \overline{K}).$$

First estimate ($C_p n \log n$) I

1. Apply n symmetrizations with respect to some orthonormal basis to obtain a centrally symmetric set.
2. Assume that $aD_n \subset K \subset bD_n$, for some $a, b > 0$. If $M^*(K) < \frac{b}{4}$, by a result of Klartag, there exist $5n$ Minkowski symmetrizations that transform $\overline{K}$ into a set K' such that $K' \subset \frac{b}{2} D_n$. Apply the corresponding Steiner symmetrizations and use the fact that they are contained in Minkowski symm.
3. If K contains an ellipsoid of volume $|2aD_n|$, symmetrize the ellipsoid to a ball with $n-1$ symmetrizations.
4. The above steps reduce the ratio $\frac{b}{a}$ at least by half each step. Thus, we may iterate steps 2 and 3 at most $\log \frac{b}{a}$ times.
5. Denote the new set (after the iterations) by Q. Denote by a_0, b_0 the improved quantities that satisfy $a_0 D_n \subseteq Q \subseteq b_0 D_n$.

First estimate $(C_p n \log n)$ II

6. Obviously, $M^*(Q) \geq \frac{1}{4} diam(Q)$. By Dvoretzky-Milman's theorem there exists a subspace E of dimension $h = [\delta n]$ such that

$$\frac{1}{4} M^*(Q) D_n \cap E \subset Proj_E(\overline{Q}) \subset 4M^*(Q) D_n \cap E,$$

   where $\delta > 0$ is some universal constant.

7. Choose an orthonormal basis in $E^{\perp}$ and apply $n - h$ Steiner symmetrizations with respect to this basis to obtain a new p-convex set Q'. As noted above, they transfer projections to sections so we get

$$\frac{1}{4} M^*(Q) D_n \cap E \subset \overline{Q'} \cap E \subset 4M^*(Q) D_n \cap E.$$

8. By a lemma of Kalton and Gordon, we get that $Q' \cap E$ is isomorphic to $D_n \cap E$ up to a constant C_p dependent on p only.

9. Recall that Q' contains a ball of radius a_0, while its section contains a ball of radius at least $C_p b_0 D_n$.

First estimate ($C_p n \log n$) III

10. Consider the maximal ellipsoid $\mathcal{E}$ inside the p-convex hull of $a_0 D_n$ and $C_p b_0 D_n \cap E$.
11. By our assumption $|\mathcal{E}| \leq |2a_0 D_n|$. A simple computation shows that $\frac{b_0}{a_0}$ must be bounded by some constant dependent only on p.
12. To sum it up, we have a bound of $Cn \log \frac{b}{a}$ symmetrizations.
13. John's theorem for $p - convex$ sets (Dilworth) guarantees the existence of an ellipsoid $\mathcal{E}$ such that

$$\mathcal{E} \subseteq Q' \subseteq n^{\frac{1}{p} - \frac{1}{2}} \mathcal{E}.$$

14. Symmetrizing $\mathcal{E}$ to the Euclidean ball we get that $\log \frac{b}{a} \leq C_p \log n$.

Note: Whenever we have a better bound for the Banach-Mazur d_K distance of our set to D_n, we automatically get a better result of $Cn \log d_K$.

Second estimate ($C_p n$ and $5n$)

We will now use Milman's quotient of subspace theorem for p-convex sets to improve the first estimate.

Theorem (Milman; Gordon-Kalton)

Let K be a p-convex set. Then, $\exists\ \gamma_p$, such that for every $0 < \lambda < 1$ there exist subspaces $F \subset E$ such that $\dim F \geq \lambda n$ and an ellipsoid $\mathcal{E}$ such that

$$\mathcal{E} \subset Proj_E(K) \cap F \subset \gamma_p \left(\frac{1}{1-\lambda} \log \frac{2}{1-\lambda} \right)^{\frac{2}{p}-1} \mathcal{E}.$$

Improving the estimate I

Main Idea:

- QS theorem guarantees the existence of large (λn) sections of projections isomorphic to Euclidean (up to some function of $f(\lambda)$).
- For a given λ, find such a section F of projection onto E with $\dim F = \lambda n$. Then, $\mathbb{R}^n = F \oplus F^{\perp}$.
- Transform the projection into section by applying less than $(1-\lambda)n$ Steiner symm in $E^{\perp}$.
- Using the first estimate, one may symmetrize the new set $K' \cap F$ using at most $Cn \log f(\lambda)$ symmetrizations.
- Additionally, we may symmetrize $K' \cap F^{\perp}$ (which is of dimension $(1-\lambda)n$) using $C_p(1-\lambda)n \log((1-\lambda)n)$ symmetrizations.
- Choosing the appropriate λ allows us to improve previous result.
- This procedure can be iterated.

The details I

- We show now that using the quotient of subspace theorem, we may improve the estimate to $C_p n \log\log n$.
- Choose $\lambda = 1 - \frac{1}{\log n}$ and apply QS theorem to obtain subspaces $F \subset E$ and an ellipsoid $\mathcal{E}$:

$$\mathcal{E} \subset Proj_E(K) \cap F \subset \gamma_p(\log n \cdot \log(2\log n))^{\frac{2}{p}-1}\mathcal{E}.$$

- As before, we may send projections to sections by symmetrizing using a basis in $E^{\perp}$:

$$\mathcal{E}' \subset K' \cap F \subset \gamma_p(\log n \cdot \log(2\log n))^{\frac{2}{p}-1}\mathcal{E}'$$

- Symmetrize the ellipsoid $\mathcal{E}'$ to Euclidean ball:

$$\lambda_1 D_n \cap F \subset K'' \cap F \subset \gamma_p(\log n \cdot \log(2\log n))^{\frac{2}{p}-1}\lambda_1 D_n \cap F$$

The details II

- By the first estimate $K'' \cap F$ can be symmetrized with $C_p n \log \log n$ symmetrizations to obtain $\tilde{K}$:

$$\delta \alpha_p D_n \cap F \subset \tilde{K} \cap F \subset \delta \beta_n D_n \cap F.$$

- And the same for $\tilde{K} \cap F^{\perp}$:

$$\mu \alpha_p D_n \cap F^{\perp} \subset K_1 \cap F^{\perp} \subset \mu \beta_n D_n \cap F^{\perp},$$

  where K_1 is the result of the symmetrization process.
- Combining the above we get that K_1 is isomorphic to D_n with some new constants α'_p, β'_p.
- Applying the first estimate we finish the proof with additional $Cn \log \frac{\beta'_p}{\alpha'_p}$ symmetrizations.
- In total, we have performed at most $C_p n \log \log n$ symmetrizations.

Best bound

- Actually there is no need to continue iterating the above method.
- One may assume that the optimal bound is $n\theta(n)$ for some perhaps unbounded function $\theta(n)$.
- Applying exactly the same procedure gives an a posteriori , which implies a result of $C_p n$ symmetrizations.
- Adding one more iteration of QS theorem allows us to replace the constant C_p with 5.
- However, the cost of such estimate is that the isomorphism constants α_p, β_p might be worse.

- Assume that there exists a monotone function $\theta(n)$ such that for each p-convex set we need at least $n\theta(n)$ symm. to approach a Euclidean ball up to constants α_p, β_p.
- Choose $\lambda = 1 - \frac{1}{\theta(n)}$ and apply QS theorem to obtain subspaces $F \subset E$ and an ellipsoid $\mathcal{E}$:

$$\mathcal{E} \subset Proj_E(K) \cap F \subset \gamma_p(\theta(n) \log 2\theta(n))^{\frac{2}{p}-1}\mathcal{E}.$$

- Choose a basis in $E^\perp$ and apply $\frac{n}{\theta(n)}$ symm. in $E^\perp$ to obtain the following:

$$\mathcal{E}' \subset K' \cap F \subset \gamma_p(\theta(n) \log 2\theta(n))^{\frac{2}{p}-1}\mathcal{E}'$$

- Symmetrize $\mathcal{E}'$ to a ball. Then, there exists λ_1 s.t.

$$\lambda_1 D_n \cap F \subset K'' \cap F \subset \gamma_p(\theta(n) \log 2\theta(n))^{\frac{2}{p}-1}\lambda_1 D_n \cap F$$

- By the first estimate, there exist $C_p n \log(\gamma_p \theta(n))$ Steiner symm (in F) applied to K'' result in $\tilde{K}$ s.t.

$$\delta \alpha_p D_n \cap F \subset \tilde{K} \cap F \subset \delta \beta_n D_n \cap F.$$

- Additionally, there exist $\frac{n}{\theta(n)} \theta(\frac{n}{\theta(n)}) \leq n$ symm. (in $F^{\perp}$) applied to $\tilde{K}$ result in K_1 s.t.

$$\mu \alpha_p D_n \cap F^{\perp} \subset K_1 \cap F^{\perp} \subset \mu \beta_n D_n \cap F^{\perp}.$$

- Combining the above we get that K_1 is isomorphic to D_n with some new constants α'_p, β'_p.
- Applying the first estimate to K_1 we obtain a new set K_2 as desired, after $C_p n \log \frac{\beta'_p}{\alpha'_p} + C_p n \log(\gamma_p \theta(n)) + n$ symmetrizations. This contradicts the fact that $\theta(n)$ is unbounded.