Flowers and Non-linear Constructions in Convex Geometry

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(Part 1: joint work with E. Milman and L. Rotem
Part 2: joint work with L. Rotem)

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Part 1

Flowers and Reciprocity

Joint work with Emanuel Milman and Liran Rotem

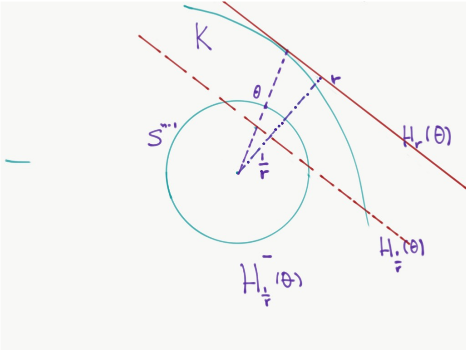
Indicatrix of the family of supporting functional

Let $K \in \mathcal{K}_\circ$ be the family of convex, closed sets, with $0 \in K$.

Let $K^\clubsuit$ be the indicatrix of the family of supporting functions $\{h_K(\theta)\}$, $\theta \in S^{n-1}$, i.e. the radial function

$$r_{K^\clubsuit}(\theta) = h_K(\theta) = \sup\{(\theta, x) \mid x \in K\}.$$

$K^\clubsuit$ is a star body, $K^\clubsuit \supseteq K$ and $= K$ iff $K = rB_2^n$ (the euclidean ball).



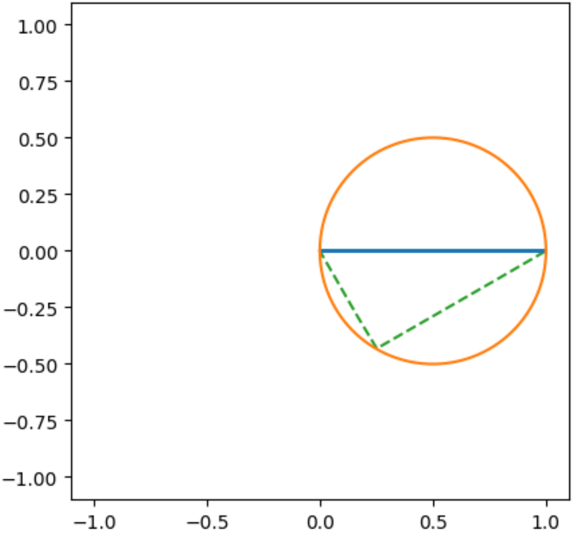
Note a few properties to start with:

1. $K^\clubsuit$ uniquely defines K;
2. $(\mathrm{Pr}_E K)^\clubsuit = K^\clubsuit \cap E$ for any subspace E, and $\clubsuit$ taken inside E;
3. For any K and $T \in \mathcal{K}_\circ$

$$(\mathrm{Conv}\, K \cup T)^\clubsuit = K^\clubsuit \cup T^\clubsuit.$$

Let $B_x := B\left(\frac{x}{2}, \frac{|x|}{2}\right)$ be the euclidean ball with $\frac{x}{2}$ its center and $\frac{|x|}{2}$ its radius, i.e. the interval $[0, x]$ is the diameter of B_x. For the interval $I = [0, x]$, $I^\clubsuit = B_x$ (Thales theorem).

(In the next pictures the body K is blue and $K^\clubsuit$ is orange).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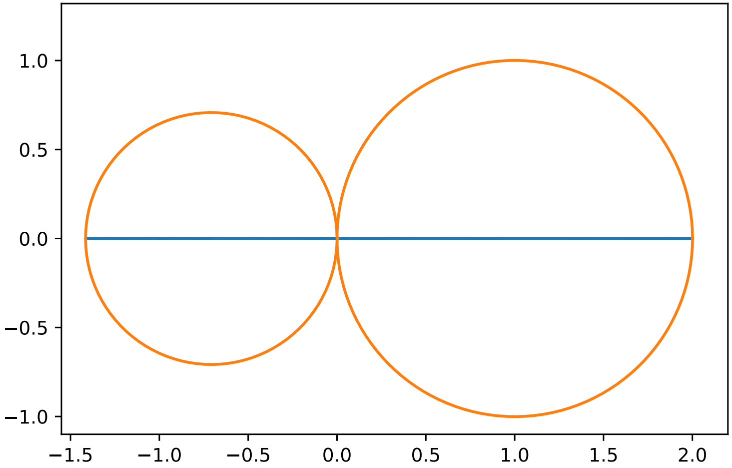
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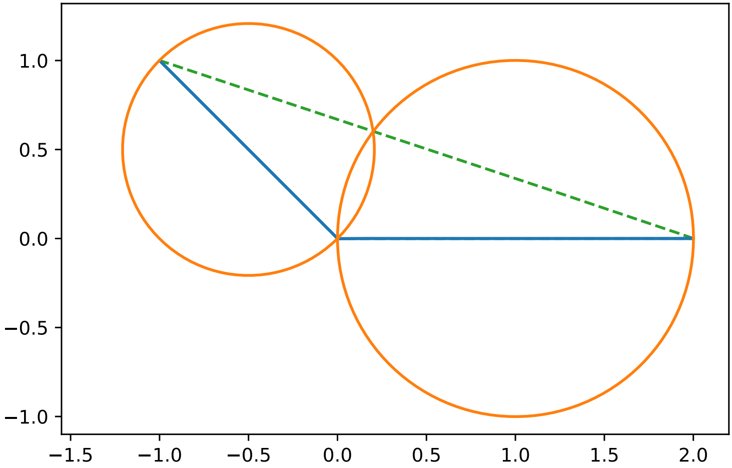
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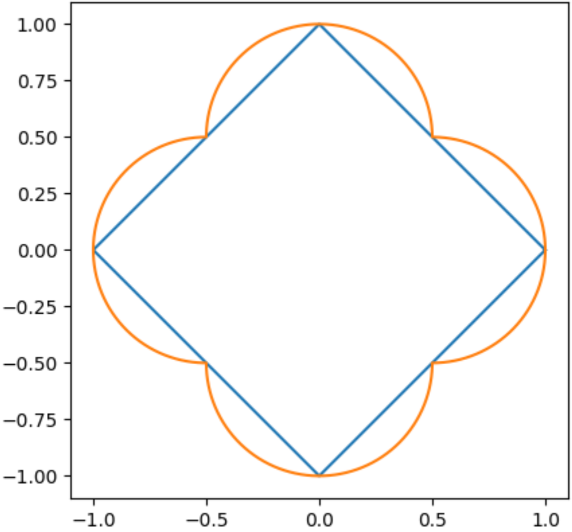
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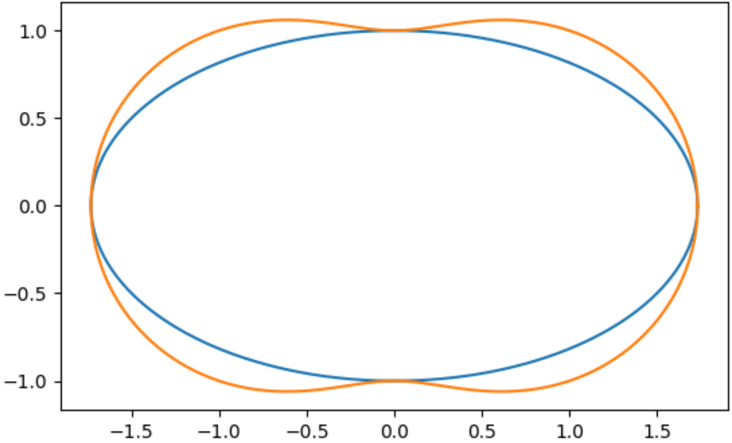
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Flowers

We call a flower

$$A = \bigcup_{\alpha} B_{x_\alpha}$$

a union of balls B_{x_α} (i.e. with diameters $[0, x_\alpha]$) which is a star body in $\mathbb{R}^n$.

Let $\mathcal{F}$ be the family of flowers in $\mathbb{R}^n$.

Fact 1a: Every indicatrix of $K \in \mathcal{K}_\circ$ is a flower

$$K^\clubsuit = \bigcup \{B_x \mid x \in \partial K\} \equiv \bigcup \{B_x \mid x \in K\}.$$

Write also for any A-star,

$$A^\clubsuit := \bigcup \{B_x \mid x \in A\} \equiv \bigcup \{B_{r_A(\theta)\theta} \mid \theta \in S^{n-1}\}.$$

Fact 1b: Every flower A is the indicatrix of some $K \in \mathcal{K}_\circ$: $\exists K$ s.t. $K^\clubsuit = A$.

Let A be a flower. Call

$$K = \{x \in A \mid B_x \subset A\} \text{— the core of } A.$$

Then

$$K \text{ is convex and } K^{\clubsuit} = A.$$

So $K = A^{-\clubsuit}$ (the inverse map).

In particular, if $A = \bigcup_{x \in \Lambda} B_x$ then

$$A^{-\clubsuit} = \operatorname{Conv} \Lambda.$$

Spherical inversion

We will also need a duality relation on a family of star bodies:

For A-star denote $\Phi(A)$ the star body s.t. $r_{\Phi(A)} = 1/r_A$ (considered by Moszyńska).

$\Phi(A)$ is "almost" a pointwise map:

Let $\mathcal{I} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ be $\mathcal{I}(x) = x/|x|^2$ (i.e. $\mathcal{I}$ is the spherical inversion).

Then $\partial\Phi(A) = \mathcal{I}(\partial A)$ (where ∂A is defined radially) (but $\mathcal{I}$ maps "interior" of A to the exterior of $\Phi(A)$ and vice versa).

Note: $\operatorname{Im}\Phi \equiv \overline{\operatorname{colm}\mathcal{I}}$, i.e. $\Phi(A) = \overline{\mathcal{I}(A)^c}$.

Map Φ and spherical inversion

Well-known facts on $\mathcal{I}$:

Fact 2. Let $A \subset \mathbb{R}^n$ be a sphere or a hyperplane.
Then $\mathcal{I}(A)$ is a hyperplane if $0 \in A$ and a sphere if $0 \notin A$.

So $\mathcal{I}(\partial B_x)$ is a hyperplane and $\Phi(B_x)$ a half-space containing 0.

Therefore, for any flower A, $\Phi(A)$ is a convex body: the intersection of half-spaces containing 0, if

$$A = \bigcup_{x \in T} B_x \Rightarrow \Phi(A) = \bigcap_{x \in T} \Phi(B_x).$$

And another
Fact 3:

$$\Phi(K^{\clubsuit}) = K^{\circ}.$$

So

$$\mathcal{K}_{\circ} \overset{\clubsuit}{\to} \mathcal{F} \overset{\Phi}{\to} \mathcal{K}_{\circ} \qquad \text{(1-1 and onto maps)}$$
$$K \longrightarrow K^{\circ},$$

i.e.

$$\circ\, \Phi\, \clubsuit = \mathrm{Id} \quad \text{on} \quad \mathcal{K}_{\circ}.$$

We have the diagrams:

$$\mathcal{K}_\circ \xrightarrow{\clubsuit} \mathcal{F} \xrightarrow{\Phi} \mathcal{K}_\circ \xrightarrow{\clubsuit} \mathcal{F} \xrightarrow{\Phi} \mathcal{K}_\circ$$

$$\circ : \mathcal{K}_\circ \xrightarrow[\circ = \Phi\clubsuit]{} \mathcal{K}_\circ$$

$$\mathrm{Id} : \mathcal{K}_\circ \longrightarrow \mathcal{K}_\circ$$

$$\mathcal{F} \xrightarrow[* = \clubsuit\Phi]{} \mathcal{F}$$

$$* : A \longrightarrow A^*, \text{ 1–1, onto, order reversing}$$

i.e. $*$ is a duality on flowers $\mathcal{F}$:

$$\text{if} \quad A = K^\clubsuit \quad \text{then} \quad A^* = (K^\circ)^\clubsuit$$

Reciprocity

For function $f : S^{n-1} \to [0, \infty]$ define the Alexandrov body

$$A[f] = \{x \in \mathbb{R}^n \mid (x, \theta) \leq f(\theta),\ \forall\, \theta \in S^{n-1}\}.$$

Note that if $h_K(\theta)$ is a supporting function of $K \in \mathcal{K}_\circ$

$$A[h_K] = K.$$

We call $A[1/h_K] = K'$ a reciprocal body. Recall that the polar K° of K is

$$K^\circ = \{x \in \mathbb{R}^n \mid (x, y) \leq 1,\ \forall\, y \in K\}.$$

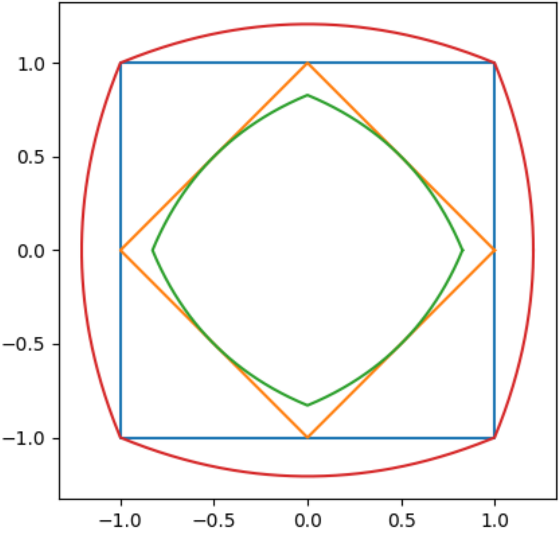
Easy: $K' \subseteq K^\circ$, $K'' \supseteq K$, and $'$ reverse order of embedding. It follows that $K''' = K'$, i.e.

Fact 5. K' is the duality on the image of this operation [i.e. on the family of reciprocal bodies].

Note:

$$K = K' \iff K = B_2^n$$
$$K^\circ = K' \iff K = B_2^n$$

In the next pictures: the started "chain" body K is in blue, K° – orange, K' – green and K'' – red.



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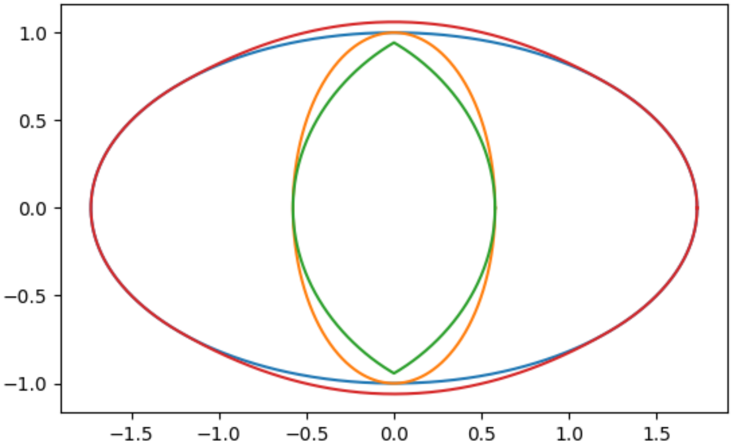
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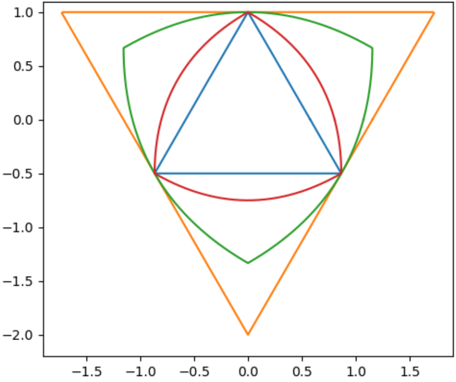
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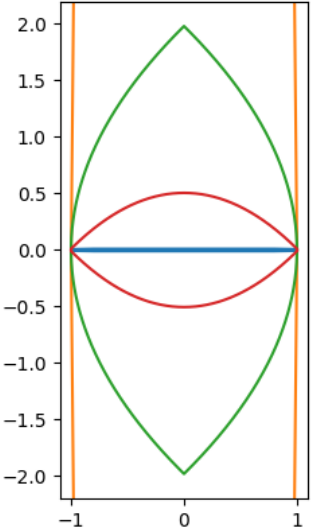
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Reciprocity

Some properties of the reciprocal operation $'$.

Theorems. $\forall K \in \mathcal{K}_\circ$

1. $(K^\clubsuit)^\circ = K'$ *[we write* $\circ \clubsuit = '$ *(in operator-type notation)];*
2. *Define by* $D(K)$ *the family of all convex bodies* $\{A\}$ *s.t.* $A' = K$*. Then:*
   - 2a. $\forall K$, $D(K)$ *is a closed convex subset of* $\mathcal{K}_\circ$*;*
   - 2b. *If* $D(K) \neq \emptyset$*, then* K' *is the maximal element in* $D(K)$*.*

!3. K *is reciprocal (i.e.* $\exists\, T \in \mathcal{K}_\circ$ *s.t.* $T' = K$*) iff* $K^\clubsuit$ *is convex.*

(So, reciprocal bodies are "more convex", both K and $K^\clubsuit$ are convex.)

Corollary (of 3). $(\mathrm{Pr}_E\, K)' = \mathrm{Pr}_E\, K'$ *for* K *reciprocal.*

Proof. As $K^\clubsuit$ is convex, then

$$(\mathrm{Pr}_E\, K)' = \left((\mathrm{Pr}_E\, K)^\clubsuit\right)^\circ = (K^\clubsuit \cap E)^\circ = \mathrm{Pr}_E (K^\clubsuit)^\circ = \mathrm{Pr}_E\, K'.$$

Relations between operations we have introduced

We discussed 4 operations on Convex/star-bodies

- $\circ$ — polarity $[\circ K \equiv K^\circ]$;
- $\clubsuit$ — taken indicatrix/flower $[\clubsuit K \equiv K^\clubsuit]$;
- Φ — duality for star-bodies/spherical inversion;
- $'$ — reciprocity $[{}'K \equiv K']$;

Let us see how they interplay.

Fact 6. On the class of convex bodies $\mathcal{K}_\circ$

(i) $\clubsuit = \Phi\circ$ $\quad (K^\clubsuit = \Phi(K^\circ))$ (correct also for K-star body);

(ii) $\clubsuit\circ = \Phi$ $\quad ((K^\circ)^\clubsuit = \Phi(K))$ (ONLY for convex K);

(iii) $\circ\clubsuit = {}'$ $\quad ((K^\clubsuit)^\circ = (K)')$;

(iv) $\Phi\clubsuit = \circ$ $\quad (K^\circ = \Phi(K^\clubsuit))$ (also for star-bodies).

As a consequence of Fact 6, let us show one direction in Theorem 3: if $K^{\clubsuit}$ is convex then K is reciprocal, i.e. $K'' = K$.

Indeed, by 6(ii). when $K^{\clubsuit}$ is convex

$$\clubsuit \circ \clubsuit = \Phi \clubsuit = \circ \qquad \text{(also by 6(iv))}.$$

Take $\circ$ from both parts:

$$\circ \clubsuit \circ \clubsuit K = K^{\circ\circ} = K$$

and by 6(iii) it follows $K'' = K$. $\square$

Fact 7. From 6(ii) follows that for $K \in \mathcal{K}_\circ$

$$\Phi K - \text{convex} \iff K^\circ \text{ is reciprocal.}$$

More remarkable properties of star-bodies called flowers

$$\mathcal{F} := \left\{ A = \bigcup_\alpha B_{x_\alpha} \right\} \text{ also, equivalently } = \left\{ \bigcup_\alpha \{ B_\alpha \mid 0 \in B_\alpha \} \right\}$$

where B_α are euclidean balls.

1. $\clubsuit$ and $\mathcal{F}$ are a preparational step for different dualities:

$$\Phi \clubsuit K = K^\circ \qquad \text{but } \circ \clubsuit K = K'.$$

2. Algebraic-geometric properties
   - (i) For $A, B \in \mathcal{F}$ also Minkowski sum $A + B \in \mathcal{F}$ (associative, commutative, monotone).
     Also, $\operatorname{Conv} A \in \mathcal{F}$ and $\operatorname{Conv} K^\clubsuit = (K'')^\clubsuit$.
   - (ii) $\forall$ subspace $E \hookrightarrow \mathbb{R}^n$, if $A \in \mathcal{F}$, then $A \cap E \in \mathcal{F}(E)$ and $\Pr_E A \in \mathcal{F}(E)$;
   - !(iii) If $A_i \in \mathcal{F}$ then also $\bigcup_i A_i \in \mathcal{F}$. Let $A_i = K_i^\clubsuit$ (for convex K_i). Then $A_1 \cap A_2 \in \mathcal{F}$ iff $K_1^\circ \cup K_2^\circ$ is convex;

Also, for a convex $K \in \mathcal{K}_\circ$

$$K^{\clubsuit,\clubsuit} = \bigcup_{\theta \in S^{n-1}} B_{h_K(\theta)\theta} \qquad \left[\text{recall } K^{\clubsuit} = \bigcup_{\theta \in S^{n-1}} B_{r_K(\theta)\theta}\right]$$

— · — · — · — · — ·

Let $K_i \in K_\circ$, $\lambda_i \geq 0$. Consider

$$P = \sum_i \lambda_i K_i \in \mathcal{K}_\circ.$$

By Minkowski theorem, $\mathrm{Vol}\, P$ is homogeneous polynomial in $\{\lambda_i\}$.

3. Also Vol $P^\clubsuit$ is a homogeneous polynomial in $\{\lambda_i\}$ with coefficients which we will call $\clubsuit$-mixed volumes of $\{K_i\}$. For these numbers **all** corresponding relations are *elliptic* (not *hyperbolic*) and exactly the same kind as in the "dual mixed volume" theory of Lutwak. Say, Brunn-Minkowski type $\clubsuit$-inequality is for A and B in $\mathcal{K}_\circ$

$$|(A+B)^\clubsuit|^{1/n} \leq |A^\clubsuit|^{1/n} + |B^\clubsuit|^{1/n},$$

and elliptic type $\clubsuit$-Alexandrov-Fenchel inequality

$$V_\clubsuit(A_1, A_2, \dots, A_n)^2 \leq V_\clubsuit(A_1, A_1, A_3, \dots, A_n) \cdot V_\clubsuit(A_2, A_2, A_3 \cdots A_n)$$

where $A_i \in \mathcal{K}_\circ$ and

$$V_\clubsuit(A_1, \dots, A_n) = |B_2^n| \int_{S^{n-1}} h_{A_1}(\theta) \cdot \ldots \cdot h_{A_n}(\theta) d\mu(\theta),$$

$h_{A_i}(\theta)$ is the supporting functional of A_i.

Kubota formulas for ♣-mixed volumes

Let $W_{\clubsuit,i}(K) = V_\clubsuit(\underbrace{K,\dots,K}_{(n-i)\text{-times}},\underbrace{B_2^n,\dots,B_2^n}_{i\text{-times}})$.

Then for every $1 \le i \le n$

$$W_{\clubsuit,n-i}(K) = \frac{\omega_n}{\omega_i}\int_{G_{n,i}} \left|(\mathrm{Proj}_E\, K)^\clubsuit\right| d\mu(E)$$

(ω_i is the volume of the euclidean ball B_2^i). Also

$$\left(\frac{|K|}{\omega_n}\right)^{1/n} \le \left(\frac{W_1(K)}{\omega_n}\right)^{1/n-1} \le \cdots \le \frac{W_{n-1}(K)}{\omega_n} = \frac{W_{\clubsuit,n-1}(K)}{\omega_n}$$

$$\le \left(\frac{W_{\clubsuit,n-2}(K)}{\omega_n}\right)^{1/2} \le \cdots \le \left(\frac{W_{\clubsuit,1}(K)}{\omega_n}\right)^{1/n-1} \le \left(\frac{|K^\clubsuit|}{\omega_n}\right)^{1/n}.$$

New summations on $\mathcal{K}_\circ$ and $\mathcal{F}$

Summation on flowers implies strange summations on $\mathcal{K}_\circ$ and also another one on $\mathcal{F}$.

Let $A, B \in \mathcal{F}$. Then $A + B \in \mathcal{F}$.

Let $K, T \in \mathcal{K}_\circ$ s.t. $A = K^\clubsuit$ and $B = T^\clubsuit$.

Let $C := A + B = P^\clubsuit$, $P \in \mathcal{K}_\circ$.

Define $K \underset{\clubsuit}{\oplus} T = P$ (the "club" sum). This sum is commutative, associative, monotone and $\{0\}$ is its unit element. However !
$K \underset{\clubsuit}{\oplus} K \supset 2K$ but not in general $=$.

Consider now the subset $\mathcal{R} \hookrightarrow \mathcal{K}_\circ$ of reciprocal bodies. Then, for $T, K \in \mathcal{R}$, $T^\clubsuit, K^\clubsuit$ are convex and $T^\clubsuit + K^\clubsuit$ is also convex.

This means that $P = K \underset{\clubsuit}{\oplus} T$ is reciprocal.

So,there is a summation on $\mathcal{R}$!

Note, Minkowski sum does not preserve reciprocity.

Also, in this case

$$K \underset{\clubsuit}{\oplus} K = 2K.$$

(Sum is 1-homogeneous.)

More on reciprocal bodies $\mathcal{R}$

We add: If K and $T \in \mathcal{R}$, then

$$K \cap T \in \mathcal{R}.$$

Also note: if $K = -K$, $K \in \mathcal{R}$, then $\exists\, r$, $R > 0$, s.t.

$$B(0, r) \subseteq K \subseteq B(0, R) \qquad \text{and} \quad R/r \leq 2.$$

However, for non-origin-symmetric bodies, the smallest R/r may be any large even in dim 2.

Example. Let $\mathcal{E}$ be an ellipsoid (in $\mathbb{R}^2$) and 0 is a focus of $\mathcal{E}$. Then

(i) $\mathcal{E}^{\clubsuit} =$ euclidean ball B, $0 \in B$.

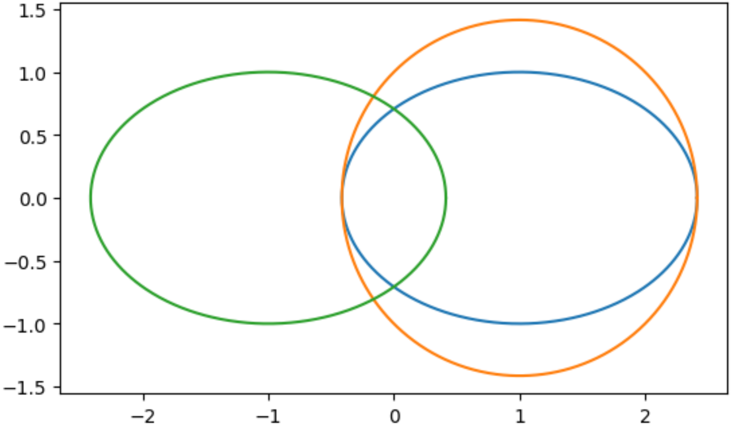
(ii) $\mathcal{E}$ is reciprocal and $\mathcal{E}'$ is an ellipsoid.

(iii) Let B be a euclidean ball, $0 \in B$. Then B is a flower (of some ellipsoid) and B° is reciprocal. If 0 is not the center B then B is not reciprocal.

Fact 8.

(i) If K and T are reciprocal then $(K^\circ + T^\circ)^\circ$ is also reciprocal.

(ii) If K and T are star-bodies, such that $\Phi(K)$ and $\Phi(T)$ are convex, then $\Phi(K + T)$ is also convex.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Convexity property and arithmetic-harmonic means inequality for operations $\circ, *, \clubsuit, \Phi, \prime$.

Below K, T are in $\mathcal{K}_\circ$ and $A, B \in \mathcal{F}$:

$$\circ: \left(\frac{K+T}{2}\right)^\circ \subseteq \frac{K^\circ + T^\circ}{2} \text{ and } \frac{K+T}{2} \supseteq \left(\frac{K^\circ + T^\circ}{2}\right)^\circ, \text{ Firey}$$

The same convexity and arithmetic-harmonic means ineqalities are correct for:

$*$ (for A and B); Φ (for K and T; and also for A and B)

$\prime$: for K and T reciprocal and flower summation $\underset{\clubsuit}{\oplus}$

$\clubsuit$: convexity property is correct for K and T.

Proof of the Characterization Theorem

Lemma

Let K be any convex body $0 \in K$. Consider the subset

$$\mathrm{Inn}_S\, K := T = \bigcup \{ B(x, |x|) \; and \; B(x, |x|) \subset K \} = \bigcup_{\alpha} \{ B(\alpha) \subset K | 0 \in B(\alpha) \}$$

(spherical inner hull)*(so any such ball passes through* 0 *and is in* K*). Then* T *is a* convex *subset of* K*. Moreover,* T *is the largest* $A \subset K$ *s.t.* $\Phi(A)$ *is convex.*

(Surprising! But that said – easy.)

Note that $T = \Phi \circ \circ \Phi\, K := \Phi\, \mathrm{Conv}\, \Phi K$.

(Formal checking: $\Phi\, \partial B(x, |x|)$ is a hyperplane outside $\Phi\, K$.)

Actually T is the maximal convex subset of K s.t.

$$\Phi T \quad \text{is convex.}$$

Using this lemma let us prove Theorem 3.

Proof.

We want to show that $K'' = K \Rightarrow K^{\clubsuit}$ convex. This means $K'' := \circ\clubsuit \circ \clubsuit K = K$.

$$\begin{gathered}(\text{act by } \clubsuit) \Rightarrow \clubsuit \circ \clubsuit \circ \clubsuit K = \clubsuit K \\ (\text{use } \clubsuit = \Phi \circ)\ \clubsuit \circ \Phi \circ \circ \Phi \circ K = \clubsuit K \\ \clubsuit \circ [\Phi \circ \circ \Phi] \circ K = \clubsuit K\end{gathered}$$

and $\clubsuit \circ = \Phi$ on convex sets, but, by the lemma, $\Phi \circ \circ \Phi(\circ K)$ convex,

$$\Phi\Phi \circ \circ \Phi \circ K = \clubsuit K \Rightarrow \circ \circ\ \Phi \circ K = \clubsuit K$$

which means $\mathrm{Conv}(\clubsuit K) = \clubsuit K$ (recall $\Phi \circ = \clubsuit$). □

The above proof is not intuitive.

Let us see some intuition behind on one example.

Let $A \in D(T)$, i.e. $A' = T$. Also $T' \in D(T)$. Recall T' is a maximal set in $D(T) : A \subset T'$. If $A \neq T'$, then it is not reciprocal (because otherwise $A = A'' = T'$).

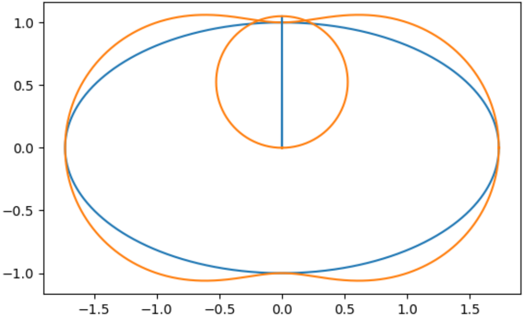
So, if $K^\clubsuit$ is not convex we would like to find another body K_1 s.t. $K \subsetneq K_1$ but $\operatorname{Conv} K_1^\clubsuit = \operatorname{Conv} K^\clubsuit$ and then $K_1' = K'$, i.e. K is not reciprocal.

Example: Our K is an ellipsoid E and $K_1 = \operatorname{Conv}(E \cup I)$, I is a special interval (see picture).

We use Fact 2:

$$[\operatorname{Conv}(K \cup T)]^\clubsuit = K^\clubsuit \cup T^\clubsuit$$

This fact and example demonstrate how lack of convexity of $K^\clubsuit$ is used to prove that K is not reciprocal.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Proof of the Lemma. Let $B_i = B_i(x_i, |x_i|) \subseteq K$, $i = 1, 2$.

Let $a_i \in B_i$. We should show that $\forall\, \lambda$, $0 < \lambda < 1$, $\exists$ a ball $B \subseteq K$ from our family of balls and $\lambda a_i + (1 - \lambda) a_2 \in B$.

We will prove that $\forall\, z \in \mathrm{Conv}(B_1, B_2) := A$, $\exists$ such a ball $B \subset A(\subseteq K)$ and $z \in B$.

$$\text{Set} \qquad A = \bigcup_{\lambda \in [0,1]} \left\{ (1 - \lambda) B_1 + \lambda B_2 \right\}$$
$$= \bigcup_{\lambda} B\big((1 - \lambda) x_1 + \lambda x_2, (1 - \lambda)|x_1| + \lambda |x_2| \big)$$

Then $\exists\, \lambda$ and $z \in B((1 - \lambda) x_1 + \lambda x_2, (1 - \lambda)|x_1| + \lambda |x_2|) = B^1$, and $0 \in B^1$ ball. Then $\exists$ a ball $\widetilde{B}$ inside this ball $B^1(\subseteq K)$ s.t. $0 \in \partial \widetilde{B}$, $z \in \widetilde{B}$.

Part 2

Applications of the Language of Flowers for Non-linear Constructions in Convex Geometry

Joint work with Liran Rotem

We will now use flowers to construct different functions of convex bodies.

Actually, we will discuss the power function.

Consider a flower $F = \bigcup B_x$; let $x = r_\theta \theta$, $\theta \in S^{n-1}$, $r_\theta \geq 0$.
Let $K = F^{-\clubsuit}$ (i.e. $K^{\clubsuit} = F$). We call representation $F = \bigcup B_x$ is canonical if $x \in \partial K \ \forall x$ (∂K is a radial boundary: $\lambda x \in K$ for $\lambda < 1$ and $\lambda x \notin K$ for $\lambda > 1$).

Then $\forall \theta \ \exists ! x = r_\theta \theta$ in the set $\{B_x\}$.

Let $f(t) \geq 0$ for $t \geq 0$, $f(0) = 0$.

Define $f(F) := \bigcup B_{f(r_\theta)\theta}$ is a flower.
Note $\Phi(\bigcup B_{f(r_\theta)\theta}) = A[1/f(r_K)]$.
Then for $K = F^{-\clubsuit}$ (the core of F) define

$$f(K) = f(F)^{-\clubsuit}, \text{i.e. } f(K) = (A[1/f(r_K)])^\circ.$$

This is a *naïve* definition. (However, it may also be useful for new geometric inequalities.)

The problem: If f_i, $i = 1, 2$, are two such functions, then typically

$$(f_1 \circ f_2)(K) \neq f_1(f_2(K)).$$

We should correct it to build K^λ, $0 \leq \lambda \leq 1$, which satisfy the semigroup property.

This is possible:

Theorem (Milman–Rotem). *There are maps $F \mapsto F^{\lambda}$ on the class of flowers satisfy:*

1. *If $F_1 \subseteq F_2$ then $F_1^{\lambda} \subseteq F_2^{\lambda}$.*
2. $(cF)^{\lambda} = c^{\lambda} F^{\lambda}$.
3. $(F^{\lambda})^{\mu} = F^{\lambda\mu}$.

And for the convex bodies from $\mathcal{K}_{\circ}$ we have built a power map s.t. the above 3 conditions are satisfied.

Recently, we (jointly with Rotem) constructed maps with much stronger properties. (The construction of the theorem above corresponds to the so-called *h*-power case, we considered earlier.)

Actually, there is a function "power" defined on all convex bodies s.t. $K = -K$ and satisfies all properties of the power function t^{α}, but for $|\alpha| \leq 1$.

Theorem (Milman–Rotem).

There is a map $K \to K^\alpha$, $0 < \alpha < 1$, *such that*

1. $\forall\, 0 < \alpha < 1$, $K \subset T \Rightarrow K^\alpha \subseteq T^\alpha$;
2. $\forall\, 0 < \alpha < 1$, $\forall\, K\, \forall\, \lambda > 0 \Rightarrow (\lambda K)^\alpha = \lambda^\alpha K^\alpha$;
3. $\forall\, 0 < \alpha, \beta < 1$, $\forall\, K$,
$$(K^\alpha)^\beta = K^{\alpha\beta};$$
4. $(K^\alpha)^\circ = (K^\circ)^\alpha$.
5. $\forall$ *ellipsoids* $\mathcal{E}$, $\forall\, 0 < \alpha < 1$, $\mathcal{E}^\alpha$ *agrees with its natural definition.*

Because the interpretation of K° is "K^{-1}", we define the power function K^α for any $-1 \leq \alpha \leq 1$.

Moreover, we are also able to construct a "geometric mean" for any two convex bodies containing 0 in the interior, and actually also "weighted" geometric means which are connected to the powers.

Weighted geometric means $G_\lambda(K, T)$

Define for numbers $a, b > 0$,

$$a \#_\lambda b = a^{1-\lambda} \cdot b^\lambda.$$

Check $a \#_\mu (a \#_\lambda b) = a \#_{\lambda\mu} b$.

For an ellipsoid E we define a positive definite operator u, s.t. $h_E(x) = \sqrt{(u_E x, x)}$, and we define E^λ by $u_{E^\lambda} = (u_E)^\lambda$.

Define the λ-geometric mean of two positive definite matrices X and Y (introduced by Pusz–Woronowich (1975))

$$X \#_\lambda Y = X^{1/2}(X^{-1/2} Y X^{-1/2})^\lambda X^{1/2}.$$

Note, if $XY = YX$ then this is $X^{1-\lambda} Y^\lambda$.

For ellipsoids E_1, E_2 we set $G_\lambda(E_1, E_2) = E_3$ if $u_{E_3} = u_{E_1} \#_\lambda u_{E_2}$.

Note that

$$G_\mu(E_1, G_\lambda(E_1, E_2)) = G_{\lambda\mu}(E_1, E_2).$$

Theorem (Milman–Rotem).

There is a family of maps $G_\lambda(K,T)$, $0 \le \lambda \le 1$, defined on any pair K and T of centrally-symmetric convex bodies which satisfies the following properties:

1. $G_\lambda(K,K) = K$.
2. *If* $K_1 \subseteq T_1$ *and* $K_2 \subseteq T_2$ *then* $G_\lambda(K_1,K_2) \subseteq G_\lambda(T_1,T_2)$.
3. $G_\lambda(\alpha K, \beta T) = \alpha^{1-\lambda}\beta^\lambda G_\lambda(K,T)$ *for all* $\alpha, \beta > 0$.
4. G_λ *is a continuous function of* K, T *and* λ *with respect to the Hausdorff metric on* $\mathcal{K}_s^n$.
5. G_λ *satisfies the harmonic mean – geometric mean – arithmetic mean inequality*

$$((1-\lambda)K^\circ + \lambda T^\circ)^\circ \subseteq G_\lambda(K,T) \subseteq (1-\lambda)K + \lambda T.$$

6. $G_\lambda(K, T)^\circ = G_\lambda(K^\circ, T^\circ)$.
7. $G_\lambda(uK, uT) = u(G_\lambda(K, T))$ *for all invertible linear maps* u.
8. $G_\lambda(K, G_\mu(K, T)) = G_{\lambda\mu}(K, T)$.
9. $G_\lambda(B_2^n, K) = K^\lambda$, *where the power* K^λ *was defined in the theorem above.*
10. *For ellipsoids* E *and* F *the mean* $G_\lambda(E, F)$ *agrees with the one defined above.*

This convex body $G_\lambda(K, T)$ *we call* λ*-geometric mean of* K *and* T.

Return to a general construction $f(K)$.

Consider a function $f(\theta, r) \geq 0$ for $\theta \in S^{n-1}$, $r \geq 0$.

Let $F = \bigcup B_{r(\theta)\theta}$. Then

$$f(F) := \bigcup_{\theta} B_{f(\theta, r(\theta))\theta}; \quad \text{Note } \Phi(f(F)) = A[1/f(\theta, r(\theta))].$$

Similarly, for $K = F^{-\clubsuit}$, define

$$f(K) := f(F)^{-\clubsuit} = f(K^{\clubsuit})^{-\clubsuit}.$$

Then $f(K) = (A[1/f(\theta, r(\theta))])^{\circ}$
We use this to define *composition* of two convex bodies T and K.

First, for flowers

$$F_1 = \bigcup B_{r_1(\theta)\theta} \quad \text{and} \quad F_2 = \bigcup B_{r_2(\theta)\theta} \quad \text{(in canonical presentation)},$$

define $F_1 \circ F_2 := \rho_{F_1}(F_2)$ where ρ_{F_1} is a 1-homogeneous function built by the radial function $\rho_1(\theta)$ of F_1, i.e. $\rho_{F_1}(\theta, r) = \rho_1(\theta) \cdot r$.

This means

$$F_1 \circ F_2 = \bigcup B_{\rho_1(\theta) \cdot r_2(\theta)\theta}.$$

Now let $F_1 = T^\clubsuit$ and $F_2 = K^\clubsuit$.

Then $r_2(\theta) = r_K(\theta)$, the radial function of K, and $\rho_1(\theta) = h_T(\theta)$. So we define

$$T \circ K := \left(\bigcup B_{h_T(\theta) \cdot r_K(\theta)\theta}\right)^{-\clubsuit}.$$

Note $T \circ T^\circ = B_2^n$ and $(T \circ K)^\circ = A\left[\frac{1}{h_T \cdot r_K}\right]$.

This may also be seen as

$$T \circ K = h_T(K) \equiv [h_T(K^{\clubsuit})]^{-\clubsuit}$$

We have, connected with T, another function r_T, the radial function of T, and we may define a different composition

$$T \odot K := r_T(K) = [r_T(K^{\clubsuit})]^{-\clubsuit}.$$

This is

$$\Big[\bigcup B_{r_T(\theta)\cdot r_K(\theta)\theta}\Big]^{-\clubsuit}.$$

So $T \odot K$ is a commutative "product".

If $T = B_2^n$ then both compositions preserve K, i.e. the identical map on $\mathcal{K}_\circ$.

Problem. Find bodies T s.t. the Brunn-Minkowski type inequality

$$|T \circ (K_1 + K_2)|^{1/n} \geq |T \circ K_1|^{1/n} + |T \circ K_2|^{1/n}$$

is correct (for any of 2 compositions h or r)?

Let us rewrite $T \odot K$ in an explicit form.

Define $T \cdot K$ to be the star body with the radial function

$$r_{T \cdot K} = r_T(\theta) \cdot r_K(\theta).$$

Then $T \odot K = \mathrm{Conv}(T \cdot K)$.

In the same notation we may use $A \cdot B$ for flowers.

Then

$$T \circ K = \mathrm{Conv}(T^{\clubsuit} \cdot K),$$

because the radial function of $T^{\clubsuit}$ is $h_T(\theta)$.