B-property for the cross-polytope on subspaces

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based on joint work with

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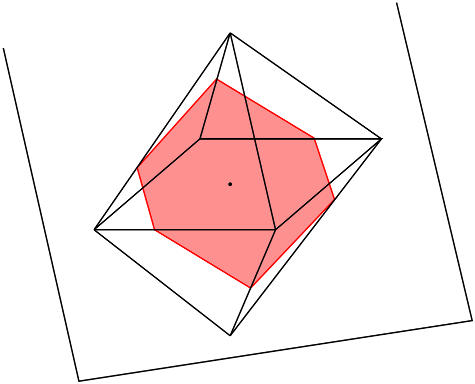
2 July 2019

Main result

<u>Theorem.</u> Let H be a subspace of $\mathbb{R}^n$. Then the function

$$F_H(t_1, \ldots, t_n) = \mathrm{vol}_H \left(\mathrm{diag}(e^{t_1}, \ldots, e^{t_n}) B_1^n \cap H\right)$$

is log-concave on $\mathbb{R}^n$.



Motivation 1

Theorem. $F_H(t) = \mathrm{vol}_H\left(\mathrm{diag}(e^{t_1}, \ldots, e^{t_n}) B_1^n \cap H\right)$ is log-concave.

Strong B-property: $t \mapsto \mu(\mathrm{diag}(e^{t_1}, \ldots, e^{t_n}) K)$ is log-concave.

E.g.

$\star$ $\frac{\mathrm{d}\mu(x)}{\mathrm{d}x} = \prod e^{-|x_i|^2}$, K convex, symmetric (C-EFM'04)

$\star$ $\frac{\mathrm{d}\mu(x)}{\mathrm{d}x} = \prod e^{-|x_i|^{p_i}}$, $p_i \in (0,1]$, K convex, symmetric

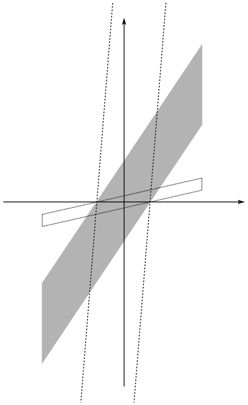
$\star$ $\frac{\mathrm{d}\mu(x)}{\mathrm{d}x} = e^{-(\sum |x_i|^2)^{p/2}}$, $p \in (0,1]$, K convex, symmetric (ENT'18)

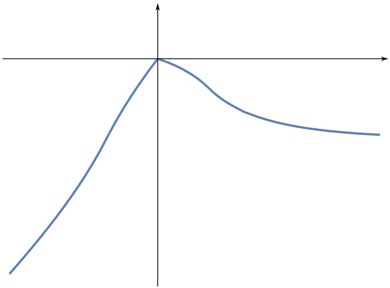
$\star$ $\mu = \mathrm{vol}_H$, $K = B_1^n$

Strong B-property – log-concave negative examples

- ⋆ $\mu =$ centred Gaussian measure, $K =$ box (C-ER'18)
- ⋆ $\mu =$ uniform on a parallelogram K,

  $f(t) = \log \mu(\mathrm{diag}(1, e^t)K)$ is not concave





Motivation 2

Conjecture (BLYZ'12). For every $n \geq 1$, every symmetric convex bodies K, L in $\mathbb{R}^n$ and $\lambda \in [0,1]$,

$$\mathrm{vol}_n(K^\lambda L^{1-\lambda}) \geq \mathrm{vol}_n(K)^\lambda \, \mathrm{vol}_n(L)^{1-\lambda}.$$

$K^\lambda L^{1-\lambda}$ =*geometric mean* (defined via support functions)

Fix $n \geq 1$. This conjecture is equivalent to (S'15'16)

Conjecture'. For every $N \geq n$, every n-dim subspace H of $\mathbb{R}^N$,

$$F_H(t_1, \ldots, t_N) = \mathrm{vol}_H \left(\mathrm{diag}(e^{t_1}, \ldots, e^{t_N}) B_\infty^N \cap H \right)$$

is log-concave on $\mathbb{R}^N$.

Proof

Theorem. $F_H(t) = \mathrm{vol}_H\left(\mathrm{diag}(e^{t_1}, \ldots, e^{t_n})B_1^n \cap H\right)$ is log-concave.

Key Lemma.

$$F_H(t) = c \cdot \exp\left(\sum t_i\right) \mathbb{E}\left[\frac{1}{\sqrt{\det\left(\sum e^{2t_j} Y_j v_j v_j^\top\right)}}\right],$$

Y_j are i.i.d. $Exp(1)$, v_j are vectors given by H.

To finish, is $\mathbb{E}\left[\frac{1}{\sqrt{\det\left(\sum e^{2t_j} Y_j v_j v_j^\top\right)}}\right]$ log-concave?

Proof

$$\mathbb{E}\left[\frac{1}{\sqrt{\det\left(\sum e^{2t_j} Y_j v_j v_j^\top\right)}}\right] \quad \text{is log-concave?}$$

$$= \int_{(0,\infty)^n} \frac{1}{\sqrt{\det\left(\sum e^{2t_j} y_j v_j v_j^\top\right)}} e^{-\sum y_j} \mathrm{d}y$$

$$\overset{y_i = e^{s_i}}{=} \int_{\mathbb{R}^n} \frac{1}{\sqrt{\det\left(\sum e^{2t_j + s_j} v_j v_j^\top\right)}} e^{-\sum e^{s_j}} e^{\sum s_j} \mathrm{d}s$$

By Prékopa-Leindler, enough to show that the integrand is log-concave.

Is $(s, t) \mapsto \det\left(\sum e^{2t_j + s_j} v_j v_j^\top\right)$ log-convex?

Proof

Is $(s, t) \mapsto \det\left(\sum e^{2t_j + s_j} v_j v_j^\top\right)$ log-convex?

Fact. For $k \times k$ PSD matrices $A_1, \ldots, A_n$,

$$\det\left(\sum x_j A_j\right) = \sum_{j=(j_1,\ldots,j_k)} \underbrace{D(A_{j_1}, \ldots, A_{j_k})}_{\geq 0} x_{j_1} \ldots x_{j_k}.$$

Fact. For $a_j \geq 0$,

$$t \mapsto \sum a_j e^{t_j}$$

is log-convex.

Proof. Hölder: $\sum a_j e^{\lambda t_j + (1-\lambda) u_j} \leq \left(\sum a_j e^{t_j}\right)^\lambda \left(\sum a_j e^{u_j}\right)^{1-\lambda}$.

Proof of key lemma

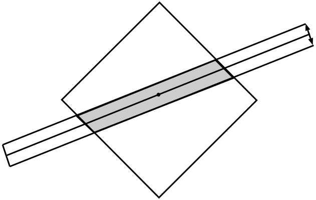
Key lemma. $\mathrm{vol}_H(B_1^n \cap H) = c \cdot \mathbb{E}\left[\frac{1}{\sqrt{\det\left(\sum Y_j v_j v_j^\top\right)}}\right].$

$H = \mathrm{span}(u_1, \dots, u_k)^\perp = \{x \in \mathbb{R}^n,\ \forall j \le k\ \langle x, u_j\rangle = 0\}$

$H(\varepsilon) = \{x \in \mathbb{R}^n,\ \forall j \le k\ |\langle x, u_j\rangle| \le \varepsilon/2\}$

$$vol_H(B_1^n \cap H) = c \cdot \lim_{\varepsilon \to 0} \varepsilon^{-k} \int_{H(\varepsilon)} e^{-\sum |x_i|} \mathrm{d}x$$

$$= c' \cdot \lim_{\varepsilon \to 0} \varepsilon^{-k} \mathbb{P}\left(X \in H(\varepsilon)\right)$$



$X = (X_1, \dots, X_n)$ i.i.d. $SymExp(1)$

Key fact. $X_i \overset{d}{=} \sqrt{2Y_i} G_i,\ Y_i \sim Exp(1),\ G_i \sim N(0,1)$

$\langle X, v\rangle = \sum X_i v_i = \sum G_i \sqrt{2Y_i} v_i = \langle G, \tilde{v}\rangle,\ \tilde{v}_i = \sqrt{2Y_i} v_i$

Proof of key lemma

$$vol_H(B_1^n \cap H) = c' \cdot \lim_{\varepsilon \to 0} \varepsilon^{-k} \mathbb{P}\left(X \in H(\varepsilon)\right)$$

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$$\begin{aligned}
\mathbb{P}\left(X \in H(\varepsilon)\right) &= \mathbb{P}\left(\forall j\ |\langle X, u_j\rangle| \le \varepsilon/2\right) \\
&= \mathbb{P}\left(\forall j\ |\langle G, \tilde{u}_j\rangle| \le \varepsilon/2\right) \\
&= \mathbb{E}_Y \mathbb{P}_G(Proj_{\tilde{U}} G \in \varepsilon K),
\end{aligned}$$

$$\tilde{U} = \mathrm{span}\{\tilde{u}_j\},\ K = \{x \in \tilde{U},\ \forall j \le k\ |\langle x, \tilde{u}_j\rangle| \le 1/2\}$$

Proof of key lemma

$$\begin{aligned}
vol_H(B_1^n \cap H) &= c' \cdot \lim_{\varepsilon \to 0} \varepsilon^{-k} \mathbb{E}_Y \mathbb{P}_G(Proj_{\tilde{U}} G \in \varepsilon K) \\
&= c'' \cdot \lim_{\varepsilon \to 0} \varepsilon^{-k} \mathbb{E}_Y \left[\varepsilon^k \operatorname{vol}(K) + o(\varepsilon^k) \right] \\
&= c'' \cdot \mathbb{E}_Y \left[\operatorname{vol}(K) \right] \\
&= c''' \cdot \mathbb{E}_Y \left[\frac{1}{\sqrt{\det\left(\sum Y_j v_j v_j^\top \right)}} \right],
\end{aligned}$$

where

$$\begin{bmatrix} \text{——} & u_1 & \text{——} \\ & \vdots & \\ \text{——} & u_k & \text{——} \end{bmatrix} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}. \qquad \square$$

Thanks!