Sharp Sobolev Inequalities via Projection Averages

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Isoperimetric inequality (IPI)

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If $K \in \mathcal{K}^n$ is n-dimensional, then

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- sets of finite perimeter (*De Giorgi, 1958*)
- euclidean inequality

Sobolev inequality (SI)

Theorem (Federer & Fleming and Maz'ya, 1960)

Let $n \geq 2$*. If* $f \in W^{1,1}(\mathbb{R}^n)$*, then*

$$\int_{\mathbb{R}^n} \|\nabla f\| dx \gtrsim \|f\|_{\frac{n}{n-1}}$$

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- $f \in BV(\mathbb{R}^n)$
- equivalent to isoperimetric inequality
- euclidean inequality

Petty projection inequality (PPI)

Theorem (Petty, 1971)

If $K \in \mathcal{K}^n$, then

$$V(K)^{n-1} V(\Pi^* K) \lesssim 1$$

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Notation:

$$K^* = \{y \in \mathbb{R}^n : x \cdot y \leq 1 \ \ \forall x \in K\}$$

$$h(\Pi K, u) = V_{n-1}(K|u^\perp)$$

$$\Pi^* K = (\Pi K)^*$$

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Affine Sobolev-Zhang inequality (ASZI)

Theorem (Zhang, J. Diff. Geom. 1999)

If $f \in C_c^1(\mathbb{R}^n)$, *then*

$$\left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} \|u \cdot \nabla f\| \, dx \right)^{-n} du \right)^{-\frac{1}{n}} \gtrsim \|f\|_{\frac{n}{n-1}}$$

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Theorem (Haberl & Schuster, 2019)

If $f \in W^{1,1}(\mathbb{R}^n)$*, then*

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- ASZI: average over 1–dimensional subspaces:

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Theorem (K., Schuster, 2019^+)

Let $1 \leq i \leq n$. If $f \in W^{1,1}(\mathbb{R}^n)$, then

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with equality if and only if

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Nested Sobolev–type Inequalities

A stronger result holds:

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- the only affine inequality is stronger than the euclidean inequalities in this family

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- $\mathcal{E}_{p,i} f := \left(\int_{Gr_{n,i}} \left(\int_{\mathbb{R}^n} \|\nabla f | F\|^p \, dx \right)^{-\frac{n}{p}} dF \right)^{-\frac{1}{n}}$

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$$\mathcal{E}_{p,n} f \gtrsim \mathcal{E}_{p,n-1} f \gtrsim \cdots \gtrsim \mathcal{E}_{p,2} f \gtrsim \mathcal{E}_{p,1} f \gtrsim \|f\|_{p^*}$$

where $p^* = \frac{np}{n-p}$.