# On products of $s$-nuclear operators, $s \in (0, 1]$

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# Nuclear operators

An operator $T : X \to Y$ is nuclear if it is of the form

$$Tx = \sum_{k=1}^{\infty} \langle x'_k, x \rangle y_k$$

for all $x \in X$, where $(x'_k) \subset X^*, (y_k) \subset Y$, $\sum_k ||x'_k|| \, ||y_k|| < \infty$.
We use the notation $N(X, Y)$

If $T$ is nuclear, then

$$T : X \to c_0 \to l_1 \to Y.$$

A. Grothendieck, Produits tensoriels topologiques et espases nucléaires, Mem. Amer. Math. Soc., Volume 16, 1955, 196 + 140.

Let U be a compact operator in $H$. Then U has the norm convergent expansion

$$U = \sum_{n=1}^{N} \mu_n(U)\,(f_n, \cdot)h_n,$$

where $(f_n)$, $(h_n)$ are ONS's, $\mu_1(U) \geq \mu_2(U) \geq \cdots > 0)$
The $\mu_n(U)$ are called the singular values of $U$.

Simon B., Trace ideals and their applications, London Math. Soc. Lecture Notes 35, Cambridge University Press, 1979.

# R. Schatten and J. von Neumann

- $$U \in S_p(H): \ \sum \mu_n^p(U) < \infty, \ \ p > 0.$$
  $$\sigma_p(U) = (\sum \mu_n^p(U))^{1/p}.$$
  $S_\infty^0(H)$ — all compact operators with the usual operator norm.
- $$S_p \circ S_q \subset S_r, \ 1/r = 1/p + 1/q;$$
  $p, q \in (0, \infty)$
  $$N(H) = S_1(H).$$

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# $s$-nuclear operators – Applications de puissance s.éme sommable

- An operator $T : X \to Y$ is $s$-nuclear $(0 < s \leq 1)$ if it is of the form
$$Tx = \sum_{k=1}^{\infty} \langle x'_k, x \rangle y_k$$
for all $x \in X$, where
$(x'_k) \subset X^*, (y_k) \subset Y$, $\sum_k ||x'_k||^s \, ||y_k||^s < \infty$. We use the notation $N_s(X, Y)$.
$\nu_s(T) := \inf(\sum_k ||x'_k||^s \, ||y_k||^s)^{1/s}$.

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$$N_p(H) = S_p(H), 0 < p \leq 1.$$

- R. Oloff, p-normierte Operatorenideale, Beiträge Anal. 4, 105-108 (1972).

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A natural question (due to Boris Mityagin):

- Is it true that a product of two nuclear operators in Banach spaces can be factored through a trace class (i.e., $S_1$-) operator in a Hilbert space?

- By using an example from

  Carleman T., Über die Fourierkoeffizienten einer stetigen Funktion, A. M., 41 (1918), 377-384.

  it was shown that
- The answer is negative.

O.I. Reinov, On products of nuclear operators, Func. Anal. and its Appl., **51**:4 (2017), 90-91.

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# $S_p$-factorization

**Definition**

*An operator $T : X \to Y$ can be factored through an operator from $S_p(H)$ (through $S_p$-operator), if there are operators $A \in L(X, H)$, $U \in S_p(H)$ and $B \in L(H, Y)$ such that $T = BUA$. We put*

$$\gamma_{S_p}(T) = \inf ||A||\, \sigma_p(U)\, ||B||.$$

# Factorization Theorem

**Theorem**

*Let $m \in \mathbb{N}$. If $X_1, X_2, \ldots, X_{m+1}$ are Banach spaces, $s_k \in (0,1]$ and $T_k \in N_{s_k}(X_k, X_{k+1})$ for $k = 1, 2, \ldots, m$, then the product*

$$T := T_m T_{m-1} \cdots T_1$$

*can be factored through an operator from $S_r(H)$, where*

$$1/r = 1/s_1 + 1/s_2 + \cdots + 1/s_m - (m+1)/2.$$

*Moreover,*

$$\gamma_{S_r}(T) \leq \prod_{k=1}^{m} \nu_{s_k}(T_k)$$

*(for $r = \infty$, we consider the class $S^0_\infty$).*

# Factorization Theorem: f.d. analogue

Finite dimensional analogue:

**Theorem**

*Under the above conditions, if the operator $T$ is of finite rank and $t \in (0, r]$, then*

$$\gamma_{S_t}(T) \leq (\dim T(X_1))^{1/t-1/r} \prod_{k=1}^{m} \nu_{s_k}(T_k).$$

*In particular, if all the operators $T_k$ are finite dimensional then*

$$\gamma_{S_t}(T) \leq (\min\ rank T)^{1/t-1/r} \prod_{k=1}^{m} \nu_{s_k}(T_k)$$

*(for $r = \infty$ we consider the class $S^0_\infty$).*

# Sharpness of f.d. Theorem

Sharpness of the previous theorem (with a proof):

**Theorem**

*There exists a constant $G > 0$ such that for every $n \in \mathbb{N}$ we can find an operator $A_n : l_1^n \to l_1^n$ with the following property:*
*If $m \in \mathbb{N}$, $s_k \in (0, 1]$ for $k = 1, 2, \dots, m$,*
$1/r = 1/s_1 + 1/s_2 + \cdots + 1/s_m - (m+1)/2$
*and $t \in (0, r]$,*
*then*

$$\gamma_{S_t}(A_n^m) \geq G n^{1/t - 1/r} \prod_{k=1}^{m} \nu_{s_k}(A_n).$$

# Proof

Fix $n \in \mathbb{N}$ and consider an unitary matrix

$$\left(n^{-1/2}\, e^{\frac{2\pi jl}{n} i}\right) \;\; (j, l = 1, 2, \ldots, n).$$

Let $A_n : l_1^n \to l_1^n$ -be the operator generated by this matrix. Clearly, if $s \in (0, 1]$, then

$$\nu_s(A_n) \leq n^{1/s - 1/2}.$$

On the other hand,

$$\Big(\sum_{\lambda} |\lambda|^p\Big)^{1/p} \leq \nu_s(A_n),$$

where $1/p = 1/s - 1/2$ and $(\lambda)$ is a system of all eigenvalues of $A_n$ (see 27.4.5 in

A. Pietsch, *Operator ideals*, 1980.)

Thus $\nu_s(A_n) = n^{1/s - 1/2}$.

# Proof continued

Consider $A_n^m$, where $m \in \mathbb{N}$, and suppose that

$$A_n^m = BUA,$$

where $A : l_1^n \to H$, $B : H \to l_1^n$, $U \in S_t(H)$ (if $t = r = \infty$, we consider the class $S_\infty^0$).
Consider a diagram

$$A_n^m B : \ H \xrightarrow{B} l_1^n \xrightarrow{A} H \xrightarrow{U} H \xrightarrow{B} l_1^n.$$

By Grothendieck theorem (see A. Pietsch, 22.4.4),

$$\sigma_2(AB) \leq c_G \, ||B|| \, ||A||$$

(here $c_G$ is a Grothendieck constant [A. Pietsch, 22.4.5]).
Therefore,

$$\sigma_q(UAB) \leq c_G ||B|| \, ||A|| \, \sigma_t(U),$$

where $1/q = 1/2 + 1/t$.

# Proof continued

Eigenvalues system of $A_n^m$ is $(\lambda^m)$ and coincides with the one of $UAB$. Consequently,

$$c_G||B||\,||A||\,\sigma_t(U) \geq \sigma_q(UAB) \geq (\sum_\lambda |\lambda^m|^q)^{1/q} = n^{1/q}.$$

But
$1/2 = 1/2 - 1/r + [(1/s_1 - 1/2) + (1/s_2 - 1/2) + \cdots + (1/s_m - 1/2) - 1/2] = -1/r + \sum_{k=1}^m (1/s_k - 1/2)$.
Therefore,

$$n^{1/q} = n^{1/2+1/t} = n^{1/t-1/r} \prod_{k=1}^m \nu_{s_k}(A_n).$$

Since $BUA$ is an arbitrary factorization of $BUA$ for $A_n^m$, one gets the desired inequality with a constant $G = 1/c_G$. ■

# Sharpness of the first theorem

Now, "summing" infinitely many finite rank operators, we obtain the sharpness of our first theorem:

**Theorem**

*Let* $m \in \mathbb{N}$, $s_k \in (0, 1]$ *for* $k = 1, 2, \ldots, m$ *and*

$$1/r = 1/s_1 + 1/s_2 + \cdots + 1/s_m - (m+1)/2.$$

*One can find the operators* $T_k \in N_{s_k}(X_k, X_{k+1})$ *in Banach spaces so that the product*

$$T := T_m T_{m-1} \cdots T_1$$

*can be factored through an operator from* $S_r(H)$, *but can not be factored through* $S_t$*-operator if* $t \in (0, r)$.

Thank you for your attention!