Playing polygonal billiards with gaussian functions

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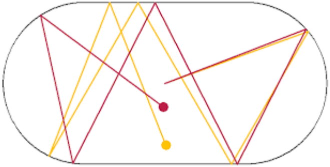
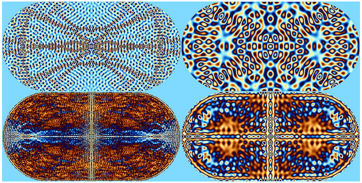
Quantum billiards

- The quantum billiard is given by the Schrödinger equation with Dirichlet boundary conditions.

$$(\Delta + \lambda)\psi_\lambda = 0, \quad \psi_\lambda|_D = 0$$

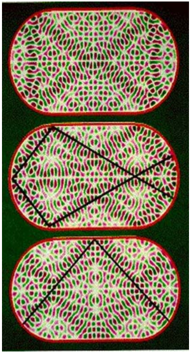
where $\lambda = \frac{2E}{\hbar^2}$





Classical dynamics $\leftrightarrow$ quantum dynamics?

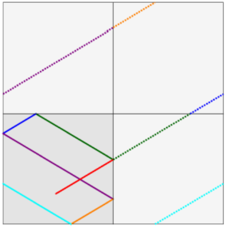
- How are the eigenfunctions distributed as $\lambda \to \infty$?



- In 2004 Bogomolny and Schmit conjectured that the eigenfunctions of the Laplacian on rational polygonal billiards ought to become localized along a finite number of vectors in momentum space ($\mathbb{S}^1$), as the eigenvalue (in other words, the energy) tends to infinity.
- Quantum limits = accumulation points of $\{dm_\lambda\}$

Rational polygons

- Simplest example: a square
- One may lift a billiard flow on a square to a geodesic flow on a torus



Rational polygons

- More interesting: take a triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{8}$ and $\frac{3\pi}{8}$
- We may unfold the billiard flow to the geodesic flow on a translation surface

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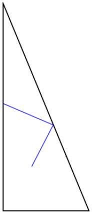

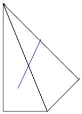

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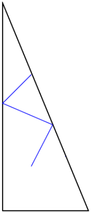

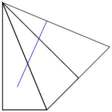

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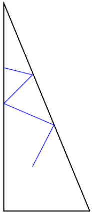

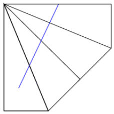

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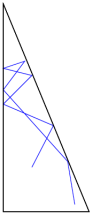
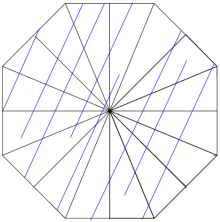
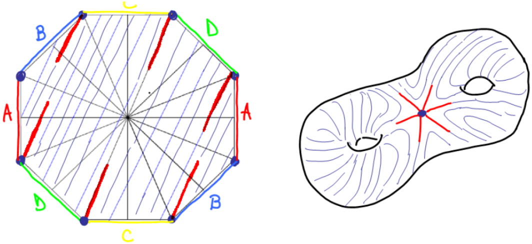




Translation surfaces and flat surfaces

- By gluing parallel edges we obtain a flat surface of genus 2 with a conical singularity of angle 6π



Translation surfaces and flat surfaces

- Let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a finite collection of polygons (not necessarily convex) in the euclidean plane
- A translation surface is the space obtained by edge identification
- That is, if $\{s_1, \dots, s_m\}$ is the collection of all edges in $\mathcal{P}$, then for any s_i there exists s_j parallel, of the same length and of opposite orientation
- For any rational polygon P there exists the corresponding finite translation surface Q which depends only on the dihedral group of P

Straight line on translation surfaces and flat surfaces

- Any straight line flow on Q can be locally identified as a straight line in the euclidean plane (as a flat surface)
- How far can it go? As long as it does not meet any singular point

Theorem (Zemljakov-Katok '75)

For any given time t there exists a direction η_0 so that the flow starting at x_0 at direction η_0 will not meet any singular point up to time t.

Construction of quasimodes on polygons

- **Goal:** construct approximate eigenfunctions (i.e. quasimodes) Ψ_λ of the Laplacian on P in the sense that

$$\frac{\|(\Delta + \lambda)\Psi_\lambda\|_{L^2(P)}}{\|\Psi_\lambda\|_{L^2(P)}} = O(\lambda^\delta)$$

  for some $\delta < \frac{1}{2}$

- **Main idea:** Take an initial state ψ_0 which is localized in position and momentum

  Average over the evolved state:

$$\Psi_\lambda = \int_{\mathbb{R}} H(t) e^{i\lambda t} U_t \psi_0 dt$$

  where $U_t = e^{it\Delta}$, $H \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} H = [-T, T]$ and T is a time-scale that depends on λ

Example: the plane

- Take an initial state

$$\phi_0(x) = \sqrt{\frac{\pi}{\hbar}}\gamma(\frac{x - x_0}{\hbar^{1/2}})e^{\frac{i\eta_0 \cdot x}{\hbar}}$$

  where $\hbar$ is a small parameter, $\eta_0 \in S^1$ and $\gamma(x) = \frac{1}{2\pi}e^{-|x|^2/2}$

- The state ϕ_0 is localized in position near x_0 on a sclae $\hbar^{1/2}$ and in momentum near $\eta_0/\hbar$ on a scale $\hbar^{-1/2}$
- Most of the mass of the evolved state $U_t\phi_0$ stays inside a ball which is evolved by the classical flow in direction η_0

Quasimodes on translation surfaces

- Recall that a translation surface Q looks locally like the euclidean plane (as long as we keep a "safety" distance from the conical singularities)
- Choose an initial state

$$\psi_0(x) = \chi\left(\frac{|x - x_0|}{\hbar^{1/2-\epsilon}}\right)\phi_0(x)$$

  where $\chi \in C_c^\infty(\mathbb{R}_+)$ is a suitable cutoff function
- Almost all of the mass of the Wigner distribution associated with ψ_0 lies inside the set $\Omega_0 = B(x_0, \hbar^{1/2-\epsilon}) \times B(\eta_0, \hbar^{1/2-\epsilon})$

Dynamical assumptions

- Given $T \leq \hbar^{3/4+\epsilon}$, there exists a direction $\eta_0 \in S^1$ such that $g_t\Omega_0$ does not self-intersect on the surface Q and avoids conical singularities
- We obtain the bound

$$\frac{\|(\Delta+\lambda)\Psi_\lambda\|_{L^2(Q)}}{\|\Psi_\lambda\|_{L^2(Q)}} = O(\frac{1}{T}) = O_\epsilon(\lambda^{3/8+\epsilon})$$

if we take $T = \hbar^{3/4+\epsilon}$ and recall $\hbar = \lambda^{-1/2}$

The main result for translation surfaces

Theorem

Let $\xi_0 \in \mathbb{S}$*. Then for any* $\epsilon > 0$ *there exists a continuous family of quasimodes* $\{\Psi_\lambda\}_{\lambda>0}$ *for the Laplacian on* Q *of spectral width* $O(\lambda^{3/8+\epsilon})$ *so that*

$$d\mu_{\Psi_\lambda}(\xi) \xrightarrow{w*} \delta(\xi - \xi_0)\ , \quad \text{as } \lambda \to \infty.$$

The main result for rational polygons

- Any rational polygon P may be unfolded to a translation surface Q under the action of the dihedral group D of P
- Given a quasimode Ψ_λ on Q, we may construct a quasimode on P by the method of images,

$$\Psi_\lambda^P(x) = \sum_{g \in D} \Psi_\lambda(gx).$$

Corollary

Let $\xi_0 \in \mathbb{S}$. Then for any $\epsilon > 0$ there exists a continuous family of quasimodes $\{\Psi_\lambda^P\}_{\lambda>0}$ for the Neumann Laplacian on P of spectral width $O(\lambda^{3/8+\epsilon})$ so that

$$d\mu_{\Psi_\lambda^P}(\xi) \xrightarrow{w*} \frac{1}{|D|} \sum_{g \in D} \delta(\xi - g\xi_0) \ , \quad \text{as } \lambda \to \infty,$$

where D is the dihedral group of P.