Extensions of results on approximate John decomposition

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Starting point: Rudelson's theorem

A random vector y in $\mathbb{R}^d$ is called *isotropic*, if $\mathbb{E}\left[y \otimes y\right] = I$.

Rudelson's theorem '99

$y_1, \dots, y_k$: independent copies of an isotropic random vector y in $\mathbb{R}^d$ with $|y|^2 \leq \gamma$ almost surely.

$$k = \frac{c\gamma \ln d}{\varepsilon^2},$$

then

$$\mathbb{E}\left\|\frac{1}{k}\sum_{i=1}^{k} y_i \otimes y_i - I\right\| \leq \varepsilon.$$

Geometric motivation – *John's theorem*:

If B_2^d is the largest volume ellipsoid in a convex body L, then

$$\sum_{i=1}^{m} \alpha_i d u_i \otimes u_i = I \qquad \text{(Symm J)}$$

for some $\alpha_1, \dots, \alpha_m > 0$ with $\sum \alpha_i = 1$, where $u_1, \dots, u_m$ are *contact points* of $\mathrm{bd}\left(B_2^d\right)$ and $\mathrm{bd}\left(L\right)$.

Better bounds found by Batson, Spielman and Srivastava (2014).
Sharper form by Friedland and Youssef (2017) using Marcus, Spielman and Srivastava (2015).

BSS'14, MSS'15, FY'17

$u_1, \ldots, u_m$ are unit vectors in $\mathbb{R}^d$ that yield a John decomposition of I.
Then there is a multi-subset σ of $[m]$ of size $|\sigma| = \frac{cd}{\varepsilon^2}$ with
$\left\| \frac{1}{k} \sum_{i\in\sigma} du_i \otimes u_i - I \right\| < \varepsilon$.

Theorem 1: Extension of Rudelson's thm.

$\varepsilon \in (0,1)$. Let A be a symmetric positive semi-definite matrix.
$Q_1, \dots, Q_k$: independent random *positive semidefinite* matrices distributed according to not necessarily identical probability distributions with $\mathbb{E} Q_i = A$ for all $i \in [k]$.
Assume $\gamma \geq \|Q_i\|$ almost surely. $a := \|A\|$.

$$k := \left\lceil \frac{c\gamma(1+a)\ln d}{\varepsilon^2} \right\rceil,$$

Then

$$\mathbb{E}\left\| \frac{1}{k} \sum_{i \in [k]} Q_i - A \right\| \leq \varepsilon.$$

Example 1: Cannot remove $\ln d$

For any d, any $0 < \varepsilon < 1/4$, and any $0 < \gamma < d/4$ there are PSD matrices $Q_1, \dots, Q_m$ with $I \in \mathrm{conv}\{Q_i\}$, $\|Q_i\| < \gamma$ such that, for any multi-subset σ of $[m]$ with $|\sigma| = \frac{\gamma \log_2 d}{4\varepsilon}$, we have

$$\left\| \frac{1}{k} \sum_{i \in \sigma} Q_i - I \right\| \geq \varepsilon.$$

Non-symmetric matrices

General John's position: B_2^d replaced by K

$K = -K, L = -L \subset \mathbb{R}^d$ o-symmetric convex bodies. We say that K is in *John's position* in L if $K \subseteq L$ and for some scalars $\alpha_1, \ldots, \alpha_m > 0$ with $\sum_{i=1}^m \alpha_i = 1$, we have

$$\sum_{i=1}^m \alpha_i d u_i \otimes v_i = I \qquad \text{(NonSymm J)}$$

where (u_i, v_i) are *contact pairs* of $\mathrm{bd}\,(K)$ and $\mathrm{bd}\,(L)$, that is $u_i \in \mathrm{bd}\,(L) \cap \mathrm{bd}\,(K)$, $v_i \in \mathrm{bd}\,(L^\circ) \cap \mathrm{bd}\,(K^\circ)$ with $\langle u_i, v_i \rangle = 1$.

Gordon, Litvak, Meyer and Pajor (2004) /see also Giannopoulos – Perissinaki – Tsolomitis (2001) and Bastero – Romance (2002), and before these, V. Milman (unpublished)/:

M, GPT'01, BR'02, GLMP'04

Assume $K \subseteq L$ is in *maximum volume position*. Then K is in John's position in L.

Theorem 2: Stability of symmetric approx. John

Assume $B_2^d \subseteq K \subseteq (1+1/d^2)B_2^d$. Let K be in John's position in L. Then for any $\varepsilon \in (0,1)$ and for

$$k = \frac{cd \ln d}{\varepsilon^2}$$

there is a multiset $\sigma \subset [m]$ of size k such that

$$\left\| \frac{d}{k} \sum_{i\in\sigma} u_i \otimes v_i - I \right\| \leq \varepsilon.$$

Example 2: No approximation in the non-symmetric case

Fix d and $\varepsilon \in (0, 1/2)$ and $\delta \in \left(0, \sqrt{\frac{d}{4}}\right)$.

Then there is $K = -K \subseteq [-1, 1]^d$ whose largest volume ellipsoid is B_2^d, such that there are contact pairs of K and $[-1, 1]^d$ satisfying (NonSymm J) with the following property.

If M is any subset of the diads in (NonSymm J) such that some linear combination of members of M is at distance $< \varepsilon$ from I, then

$$|M| \geq \left(\frac{\delta}{4\varepsilon}\right)^2 d.$$

Example I for $\gamma = 1$

May assume, $d = 2^t$.

$$k = \frac{t}{4\varepsilon}.$$

1. No good approximation in ℓ_1^t.
2. ℓ_1^t embeds isometrically into ℓ_∞^d.
3. ℓ_∞^d = diagonal $d \times d$ matrices.

To show 1.: $e_1, \dots, e_t$: standard basis of ℓ_1^t, $e_{t+i} = -e_t$.

$$a := \frac{1}{4k}(1, 1, \dots, 1) \in \ell_1^t.$$

$$a \in \operatorname{conv}\left\{\frac{e_1}{2}, \dots, \frac{e_{2t}}{2}\right\}$$

But, for any $\sigma \subset [2t]$ of size k, we have

$$\left\| \frac{1}{k} \sum_{i \in \sigma} e_i - a \right\| \geq \frac{t}{4k} \geq \varepsilon.$$

Example 2

1. $\left\langle w_i^j, e_i \right\rangle = 1$ for all $j \in [d]$;
2. $|w_i^j - e_i| = \delta$ for all $j \in [d]$ and $w_i^j \in \{x : \ x_i = 1\}$;
3. $\{w_i^1, \ldots, w_i^d\}$: $(d-1)$-dim regular simplex centered at e_i.

Clearly, $I = \frac{1}{d} \sum\limits_{i,j=1}^{d} w_i^j \otimes e_i$.

$$K := \operatorname{conv}\left(B_2^d \cup \{\pm w_i^j\}_{i,j\in[d]}\right).$$

One can show that for each $i \in [d]$, we need at least

$$\left(\frac{\delta}{4\varepsilon}\right)^2$$

diads of the form $w_i^j \otimes e_i$ in a good approximation of I.