Inequalities on mixed volumes and analogies with information theory.

Matthieu Fradelizi

Laboratoire d’Analyse et de Mathématiques Appliquées
Université Paris-Est Marne-la-Vallée

Asymptotic Geometric Analysis IV
Saint-Petersburg, 1-5 July 2019

Contents

Analogy between EPI and Brunn-Minkowski

Analogy between EPI and Brunn-Minkowski

1) Brunn-Minkowski inequality for compact sets.

Let A and B be two compact sets in $\mathbb{R}^n$. Then

$$\mathrm{vol}_n(A+B)^{\frac{1}{n}} \geq \mathrm{vol}_n(A)^{\frac{1}{n}} + \mathrm{vol}_n(B)^{\frac{1}{n}}.$$

Analogy between EPI and Brunn-Minkowski

1) Brunn-Minkowski inequality for compact sets.

Let A and B be two compact sets in $\mathbb{R}^n$. Then

$$\mathrm{vol}_n(A+B)^{\frac{1}{n}} \geq \mathrm{vol}_n(A)^{\frac{1}{n}} + \mathrm{vol}_n(B)^{\frac{1}{n}}.$$

2) Entropy Power Inequality.

Let X and Y be two independent random vectors in $\mathbb{R}^n$. Then

$$N(X+Y) \geq N(X) + N(Y),$$

where for $X \sim f$, $N(X) = \frac{1}{2\pi e} e^{\frac{2}{n}h(X)}$ and $h(X) = -\int f \log(f)$.

Analogy between EPI and Brunn-Minkowski

1) Brunn-Minkowski inequality for compact sets.
Let A and B be two compact sets in $\mathbb{R}^n$. Then

$$\mathrm{vol}_n(A+B)^{\frac{1}{n}} \geq \mathrm{vol}_n(A)^{\frac{1}{n}} + \mathrm{vol}_n(B)^{\frac{1}{n}}.$$

2) Entropy Power Inequality.
Let X and Y be two independent random vectors in $\mathbb{R}^n$. Then

$$N(X+Y) \geq N(X) + N(Y),$$

where for $X \sim f$, $N(X) = \frac{1}{2\pi e} e^{\frac{2}{n} h(X)}$ and $h(X) = -\int f \log(f)$.

3) Brunn-Minkowski inequality for matrices.
Let A and B be two non negative matrices in $\mathcal{M}_n(\mathbb{R})$. Then

$$\det(A+B)^{\frac{1}{n}} \geq \det(A)^{\frac{1}{n}} + \det(B)^{\frac{1}{n}}.$$

Because for $X \sim N(0,A)$ one has $N(X) = 2\pi \mathrm{e} \det(A)^{\frac{1}{n}}$.

Monotonicity of entropy and volume

Monotonicity of entropy and volume

Monotonicity of entropy: Artstein-Ball-Barthe-Naor (2004)
Let $X_1, \ldots, X_m, \ldots$ be i.i.d. random vectors then

$$m \mapsto h\left(\frac{X_1 + \cdots + X_m}{\sqrt{m}}\right) \quad \text{is increasing.}$$

Monotonicity of entropy and volume

Monotonicity of entropy: Artstein-Ball-Barthe-Naor (2004)
Let $X_1, \dots, X_m, \dots$ be i.i.d. random vectors then

$$m \mapsto h\left(\frac{X_1 + \cdots + X_m}{\sqrt{m}}\right) \quad \text{is increasing.}$$

Monotonicity of volume: Bobkov-Madiman-Wang conjecture (2011): For any compact set A in $\mathbb{R}^n$, let $A(m) = \frac{A+\cdots+A}{m}$. Then

$$\mathrm{vol}_n(A(m)) \leq \mathrm{vol}_n(A(m+1))?$$

Monotonicity of entropy and volume

Monotonicity of entropy: Artstein-Ball-Barthe-Naor (2004)
Let $X_1, \ldots, X_m, \ldots$ be i.i.d. random vectors then

$$m \mapsto h\left(\frac{X_1 + \cdots + X_m}{\sqrt{m}}\right) \quad \text{is increasing.}$$

Monotonicity of volume: Bobkov-Madiman-Wang conjecture (2011): For any compact set A in $\mathbb{R}^n$, let $A(m) = \frac{A+\cdots+A}{m}$. Then

$$\mathrm{vol}_n(A(m)) \leq \mathrm{vol}_n(A(m+1))?$$

Theorem

- *F., Madiman, Marsiglietti, Zvavitch (2016): yes for $n = 1$, no for $n \geq 12$.*

Monotonicity of entropy and volume

Monotonicity of entropy: Artstein-Ball-Barthe-Naor (2004)
Let $X_1, \ldots, X_m, \ldots$ be i.i.d. random vectors then

$$m \mapsto h\left(\frac{X_1 + \cdots + X_m}{\sqrt{m}}\right) \quad \text{is increasing.}$$

Monotonicity of volume: Bobkov-Madiman-Wang conjecture (2011): For any compact set A in $\mathbb{R}^n$, let $A(m) = \frac{A+\cdots+A}{m}$. Then

$$\mathrm{vol}_n(A(m)) \leq \mathrm{vol}_n(A(m+1))?$$

Theorem

- *F., Madiman, Marsiglietti, Zvavitch (2016): yes for $n = 1$, no for $n \geq 12$.*
- *F., Lángi, Zvavitch (2019+): If A starshaped then yes for $m \geq n-1$.*

Monotonicity of entropy and volume

Monotonicity of entropy: Artstein-Ball-Barthe-Naor (2004)
Let $X_1, \ldots, X_m, \ldots$ be i.i.d. random vectors then

$$m \mapsto h\left(\frac{X_1 + \cdots + X_m}{\sqrt{m}}\right) \quad \text{is increasing.}$$

Monotonicity of volume: Bobkov-Madiman-Wang conjecture (2011): For any compact set A in $\mathbb{R}^n$, let $A(m) = \frac{A+\cdots+A}{m}$. Then

$$\mathrm{vol}_n(A(m)) \leq \mathrm{vol}_n(A(m+1))?$$

Theorem

- *F., Madiman, Marsiglietti, Zvavitch (2016): yes for $n = 1$, no for $n \geq 12$.*
- *F., Lángi, Zvavitch (2019+): If A starshaped then yes for $m \geq n - 1$.*

Corollary: If A starshaped then yes for $n \leq 3$.

Blachman-Stam inequality

Blachman-Stam inequality

1) Information. Let $Z \sim N(0, I)$ independent of X. The Fisher Information $I(X)$ may be defined by de Bruijn's identity,

$$I(X + \sqrt{t}Z) = 2\frac{d}{dt}h(X + \sqrt{t}Z).$$

Blachman-Stam inequality

1) Information. Let $Z \sim N(0, I)$ independent of X. The Fisher Information $I(X)$ may be defined by de Bruijn's identity,

$$I(X + \sqrt{t}Z) = 2\frac{d}{dt}h(X + \sqrt{t}Z).$$

Blachman-Stam inequality (1964): For X and Y independent

$$I(X + Y)^{-1} \geq I(X)^{-1} + I(Y)^{-1}.$$

Blachman-Stam inequality

1) Information. Let $Z \sim N(0, I)$ independent of X. The Fisher Information $I(X)$ may be defined by de Bruijn's identity,

$$I(X + \sqrt{t}Z) = 2\frac{d}{dt}h(X + \sqrt{t}Z).$$

Blachman-Stam inequality (1964): For X and Y independent

$$I(X+Y)^{-1} \geq I(X)^{-1} + I(Y)^{-1}.$$

2) Determinants. Bergstrøm's inequality. For A, B non-negative

$$\frac{\det(A+B)}{\det(A_1+B_1)} \geq \frac{\det(A)}{\det(A_1)} + \frac{\det(B)}{\det(B_1)}.$$

Blachman-Stam inequality

1) Information. Let $Z \sim N(0, I)$ independent of X. The Fisher Information $I(X)$ may be defined by de Bruijn's identity,

$$I(X + \sqrt{t}Z) = 2\frac{d}{dt}h(X + \sqrt{t}Z).$$

Blachman-Stam inequality (1964): For X and Y independent

$$I(X+Y)^{-1} \geq I(X)^{-1} + I(Y)^{-1}.$$

2) Determinants. Bergstrøm's inequality. For A, B non-negative

$$\frac{\det(A+B)}{\det(A_1+B_1)} \geq \frac{\det(A)}{\det(A_1)} + \frac{\det(B)}{\det(B_1)}.$$

3) Convex. Dembo-Cover-Thomas (1991) + Milman conjecture: For K convex let $J(K) = \mathrm{vol}_n(K)/\partial(K) = I(K)^{-1}$. For any convex A, B

$$J(A+B) \geq J(A) + J(B)?$$

Blachman-Stam inequality

1) Information. Let $Z \sim N(0, I)$ independent of X. The Fisher Information $I(X)$ may be defined by de Bruijn's identity,

$$I(X + \sqrt{t}Z) = 2\frac{d}{dt}h(X + \sqrt{t}Z).$$

Blachman-Stam inequality (1964): For X and Y independent

$$I(X + Y)^{-1} \geq I(X)^{-1} + I(Y)^{-1}.$$

2) Determinants. Bergstrøm's inequality. For A, B non-negative

$$\frac{\det(A + B)}{\det(A_1 + B_1)} \geq \frac{\det(A)}{\det(A_1)} + \frac{\det(B)}{\det(B_1)}.$$

3) Convex. Dembo-Cover-Thomas (1991) + Milman conjecture: For K convex let $J(K) = \mathrm{vol}_n(K)/\partial(K) = I(K)^{-1}$. For any convex A, B

$$J(A + B) \geq J(A) + J(B)?$$

Giannopoulos-Hartzoulaki-Paouris (2003): Yes for $B = B_2^n$.

Blachman-Stam inequality

1) Information. Let $Z \sim N(0, I)$ independent of X. The Fisher Information $I(X)$ may be defined by de Bruijn's identity,

$$I(X + \sqrt{t}Z) = 2\frac{d}{dt}h(X + \sqrt{t}Z).$$

Blachman-Stam inequality (1964): For X and Y independent

$$I(X+Y)^{-1} \geq I(X)^{-1} + I(Y)^{-1}.$$

2) Determinants. Bergstrøm's inequality. For A, B non-negative

$$\frac{\det(A+B)}{\det(A_1+B_1)} \geq \frac{\det(A)}{\det(A_1)} + \frac{\det(B)}{\det(B_1)}.$$

3) Convex. Dembo-Cover-Thomas (1991) + Milman conjecture: For K convex let $J(K) = \mathrm{vol}_n(K)/\partial(K) = I(K)^{-1}$. For any convex A, B

$$J(A+B) \geq J(A) + J(B)?$$

Giannopoulos-Hartzoulaki-Paouris (2003): Yes for $B = B_2^n$.
F.-Giannopoulos-Meyer (2003): Yes iff $n \leq 2$.

Projections

Projections

Recall $J(A) = \frac{|A|}{\partial(A)}$.

Projections

Recall $J(A) = \frac{|A|}{\partial(A)}$. For which A: $J(A + [0, u]) \geq J(A)$? (P)

Projections

Recall $J(A) = \frac{|A|}{\partial(A)}$. For which A: $J(A + [0, u]) \geq J(A)$? (P)

F.-Giannopoulos-Meyer 2003: (P) holds for any A iff $n \leq 2$.

Projections

Recall $J(A) = \frac{|A|}{\partial(A)}$. For which A: $J(A+[0,u]) \geq J(A)$? (P)

F.-Giannopoulos-Meyer 2003: (P) holds for any A iff $n \leq 2$.

Artstein-Avidan-Florentin-Ostrover 2014: $(P) \Leftrightarrow J(A) \leq J(P_{u^\perp}A)$

Projections

Recall $J(A) = \frac{|A|}{\partial(A)}$. For which A: $J(A + [0, u]) \geq J(A)$? (P)
F.-Giannopoulos-Meyer 2003: (P) holds for any A iff $n \leq 2$.
Artstein-Avidan-Florentin-Ostrover 2014: $(P) \Leftrightarrow J(A) \leq J(P_{u^\perp} A)$

Proposition (F.-Madiman-Zvavitch 2019+)

Let K be a convex body. Let $\mathcal{L}(K) = \{TK; T \text{ linear}\}$. TFAE
(i) $J(A) \leq J(P_{u^\perp} A)$, for any $A \in \mathcal{L}(K)$.

Projections

Recall $J(A) = \frac{|A|}{\partial(A)}$. For which A: $J(A + [0, u]) \geq J(A)$? (P)
F.-Giannopoulos-Meyer 2003: (P) holds for any A iff $n \leq 2$.
Artstein-Avidan-Florentin-Ostrover 2014: $(P) \Leftrightarrow J(A) \leq J(P_{u^\perp}A)$

Proposition (F.-Madiman-Zvavitch 2019+)

Let K be a convex body. Let $\mathcal{L}(K) = \{TK; T \text{ linear}\}$. TFAE
(i) $J(A) \leq J(P_{u^\perp}A)$, for any $A \in \mathcal{L}(K)$.
(ii) $|A||P_{[u,v]^\perp}A|\sqrt{1-(u \cdot v)^2} \leq |P_{u^\perp}A||P_{v^\perp}A|$, for any $A \in \mathcal{L}(K)$.

Projections

Recall $J(A) = \frac{|A|}{\partial(A)}$. For which A: $J(A+[0,u]) \geq J(A)$? (P)

F.-Giannopoulos-Meyer 2003: (P) holds for any A iff $n \leq 2$.

Artstein-Avidan-Florentin-Ostrover 2014: $(P) \Leftrightarrow J(A) \leq J(P_{u^\perp}A)$

Proposition (F.-Madiman-Zvavitch 2019+)

Let K be a convex body. Let $\mathcal{L}(K) = \{TK; T \text{ linear}\}$. TFAE

(i) $J(A) \leq J(P_{u^\perp}A)$, for any $A \in \mathcal{L}(K)$.

(ii) $|A||P_{[u,v]^\perp}A|\sqrt{1-(u\cdot v)^2} \leq |P_{u^\perp}A||P_{v^\perp}A|$, for any $A \in \mathcal{L}(K)$.

(iii) $|A|V(A[n-2], Z_1, Z_2) \leq \frac{n}{n-1}V(A[n-1], Z_1)V(A[n-1], Z_2)$, for any $A \in \mathcal{L}(K)$ and any Z_1, Z_2 zonoids.

Projections

Recall $J(A)=\frac{|A|}{\partial(A)}$. For which A: $J(A+[0,u])\geq J(A)$? (P)
F.-Giannopoulos-Meyer 2003: (P) holds for any A iff $n\leq 2$.
Artstein-Avidan-Florentin-Ostrover 2014: $(P)\Leftrightarrow J(A)\leq J(P_{u^\perp}A)$

Proposition (F.-Madiman-Zvavitch 2019+)

Let K be a convex body. Let $\mathcal{L}(K)=\{TK; T \text{ linear}\}$. TFAE
(i) $J(A)\leq J(P_{u^\perp}A)$, for any $A\in\mathcal{L}(K)$.
(ii) $|A||P_{[u,v]^\perp}A|\sqrt{1-(u\cdot v)^2}\leq |P_{u^\perp}A||P_{v^\perp}A|$, for any $A\in\mathcal{L}(K)$.
(iii) $|A|V(A[n-2],Z_1,Z_2)\leq \frac{n}{n-1}V(A[n-1],Z_1)V(A[n-1],Z_2)$, for any $A\in\mathcal{L}(K)$ and any Z_1,Z_2 zonoids.
(iv) $|A+s[0,u]+t[0,v]|$ has only real roots, for any $A\in\mathcal{L}(K)$.
Moreover (P) holds for $K=B_2^n$, $B_2^n+[0,u]$, B_∞^n.

Projections

Recall $J(A) = \frac{|A|}{\partial(A)}$. For which A: $J(A + [0, u]) \geq J(A)$? (P)
F.-Giannopoulos-Meyer 2003: (P) holds for any A iff $n \leq 2$.
Artstein-Avidan-Florentin-Ostrover 2014: $(P) \Leftrightarrow J(A) \leq J(P_{u^\perp} A)$

Proposition (F.-Madiman-Zvavitch 2019+)

Let K be a convex body. Let $\mathcal{L}(K) = \{TK; T \text{ linear}\}$. TFAE
(i) $J(A) \leq J(P_{u^\perp} A)$, for any $A \in \mathcal{L}(K)$.
(ii) $|A||P_{[u,v]^\perp} A|\sqrt{1 - (u \cdot v)^2} \leq |P_{u^\perp} A||P_{v^\perp} A|$, for any $A \in \mathcal{L}(K)$.
(iii) $|A|V(A[n-2], Z_1, Z_2) \leq \frac{n}{n-1} V(A[n-1], Z_1)V(A[n-1], Z_2)$, for any $A \in \mathcal{L}(K)$ and any Z_1, Z_2 zonoids.
(iv) $|A + s[0, u] + t[0, v]|$ has only real roots, for any $A \in \mathcal{L}(K)$.
Moreover (P) holds for $K = B_2^n$, $B_2^n + [0, u]$, B_∞^n.

Soprounov-Zvavitch 2016: (P) holds for $K = \Delta^n$.

Projections

Recall $J(A) = \frac{|A|}{\partial(A)}$. For which A: $J(A+[0,u]) \geq J(A)$? (P)

F.-Giannopoulos-Meyer 2003: (P) holds for any A iff $n \leq 2$.

Artstein-Avidan-Florentin-Ostrover 2014: $(P) \Leftrightarrow J(A) \leq J(P_{u^\perp}A)$

Proposition (F.-Madiman-Zvavitch 2019+)

Let K be a convex body. Let $\mathcal{L}(K) = \{TK; T \text{ linear}\}$. TFAE

(i) $J(A) \leq J(P_{u^\perp}A)$, *for any* $A \in \mathcal{L}(K)$.

(ii) $|A||P_{[u,v]^\perp}A|\sqrt{1-(u\cdot v)^2} \leq |P_{u^\perp}A||P_{v^\perp}A|$, *for any* $A \in \mathcal{L}(K)$.

(iii) $|A|V(A[n-2], Z_1, Z_2) \leq \frac{n}{n-1}V(A[n-1], Z_1)V(A[n-1], Z_2)$, *for any* $A \in \mathcal{L}(K)$ *and any* Z_1, Z_2 *zonoids.*

(iv) $|A + s[0,u] + t[0,v]|$ *has only real roots, for any* $A \in \mathcal{L}(K)$.

Moreover (P) *holds for* $K = B_2^n$, $B_2^n + [0,u]$, B_∞^n.

Soprounov-Zvavitch 2016: (P) holds for $K = \Delta^n$.

See also Brazitikos-Giannopoulos-Liakopoulos 2018, Giannopoulos-Koldobsky-Valettas 2018, Alonso-Gutiérrez-Artstein-Avidan-González-Merino-Jiménez-Villa 2019

Steiner polynomial

Steiner polynomial

Marcus' question: The Steiner polynomial

$$P_Z(t) = |Z + tB_2^n|$$

of a zonoid Z has only real roots?

Steiner polynomial

Marcus' question: The Steiner polynomial

$$P_Z(t) = |Z + tB_2^n|$$

of a zonoid Z has only real roots?
Notice that this holds for any Z in dimension 2.

Steiner polynomial

Marcus' question: The Steiner polynomial

$$P_Z(t) = |Z + tB_2^n|$$

of a zonoid Z has only real roots?
Notice that this holds for any Z in dimension 2.
Victor Katsnelson (2009): for n large enough some P_Z have non real roots when $Z = B_2^{n-1} \times \{0\}$.

Steiner polynomial

Marcus' question: The Steiner polynomial

$$P_Z(t) = |Z + tB_2^n|$$

of a zonoid Z has only real roots?
Notice that this holds for any Z in dimension 2.
Victor Katsnelson (2009): for n large enough some P_Z have non real roots when $Z = B_2^{n-1} \times \{0\}$.

Proposition (F.-Madiman-Zvavitch 2019+)

For $n \geq 3$ odd, P_Z has non real roots when $Z = B_2^{n-1} \times \{0\}$.
For $n \geq 4$ even, P_Z has non real roots when $Z = B_2^{n-2} \times \{0\}^2$.

3 bodies inequalities

3 bodies inequalities

1) Information. Let X, Y, Z be indep. r.v. Madiman (2008):

$$h(X+Y+Z)+h(X) \leq h(X+Y)+h(X+Z).$$

3 bodies inequalities

1) Information. Let X, Y, Z be indep. r.v. Madiman (2008):

$$h(X+Y+Z) + h(X) \leq h(X+Y) + h(X+Z).$$

2) Determinants. Madiman (2008). Let $A, B, C \geq 0$. Then

$$\det(A+B+C)\det(A) \leq \det(A+B)\det(A+C).$$

3 bodies inequalities

1) Information. Let X, Y, Z be indep. r.v. Madiman (2008):

$$h(X+Y+Z)+h(X) \leq h(X+Y)+h(X+Z).$$

2) Determinants. Madiman (2008). Let $A, B, C \geq 0$. Then

$$\det(A+B+C)\det(A) \leq \det(A+B)\det(A+C).$$

3) Convexity. Bobkov-Madiman (2012): Let A, B, C be convex

$$|A+B+C||A| \leq 3^n |A+B||A+C|.$$

3 bodies inequalities

1) Information. Let X, Y, Z be indep. r.v. Madiman (2008):

$$h(X+Y+Z)+h(X) \leq h(X+Y)+h(X+Z).$$

2) Determinants. Madiman (2008). Let $A, B, C \geq 0$. Then

$$\det(A+B+C)\det(A) \leq \det(A+B)\det(A+C).$$

3) Convexity. Bobkov-Madiman (2012): Let A, B, C be convex

$$|A+B+C||A| \leq 3^n|A+B||A+C|.$$

We define $c(A,B,C) = \frac{|A+B+C||A|}{|A+B||A+C|}$ and $c_n = \sup_{A,B,C\subset\mathbb{R}^n} c(A,B,C)$.

3 bodies inequalities

1) Information. Let X, Y, Z be indep. r.v. Madiman (2008):

$$h(X+Y+Z) + h(X) \leq h(X+Y) + h(X+Z).$$

2) Determinants. Madiman (2008). Let $A, B, C \geq 0$. Then

$$\det(A+B+C)\det(A) \leq \det(A+B)\det(A+C).$$

3) Convexity. Bobkov-Madiman (2012): Let A, B, C be convex

$$|A+B+C||A| \leq 3^n|A+B||A+C|.$$

We define $c(A,B,C) = \frac{|A+B+C||A|}{|A+B||A+C|}$ and $c_n = \sup_{A,B,C\subset\mathbb{R}^n} c(A,B,C)$.
Courtade's question (2016): $c(B_2^n, B, C) \leq 1$, for any B, C?

3 bodies inequalities

1) Information. Let X, Y, Z be indep. r.v. Madiman (2008):

$$h(X+Y+Z)+h(X) \le h(X+Y)+h(X+Z).$$

2) Determinants. Madiman (2008). Let $A, B, C \ge 0$. Then

$$\det(A+B+C)\det(A) \le \det(A+B)\det(A+C).$$

3) Convexity. Bobkov-Madiman (2012): Let A, B, C be convex

$$|A+B+C||A| \le 3^n |A+B||A+C|.$$

We define $c(A,B,C) = \frac{|A+B+C||A|}{|A+B||A+C|}$ and $c_n = \sup_{A,B,C\subset\mathbb{R}^n} c(A,B,C)$.
Courtade's question (2016): $c(B_2^n, B, C) \le 1$, for any B, C?
Notice that $c(A, [0,u], [0,v]) \le 1$ iff $I(A) \le I(P_{u^\perp} A)$.

3 bodies inequalities

1) Information. Let X, Y, Z be indep. r.v. Madiman (2008):

$$h(X+Y+Z)+h(X) \leq h(X+Y)+h(X+Z).$$

2) Determinants. Madiman (2008). Let $A, B, C \geq 0$. Then

$$\det(A+B+C)\det(A) \leq \det(A+B)\det(A+C).$$

3) Convexity. Bobkov-Madiman (2012): Let A, B, C be convex

$$|A+B+C||A| \leq 3^n|A+B||A+C|.$$

We define $c(A,B,C)=\frac{|A+B+C||A|}{|A+B||A+C|}$ and $c_n=\sup_{A,B,C\subset\mathbb{R}^n} c(A,B,C)$.
Courtade's question (2016): $c(B_2^n, B, C) \leq 1$, for any B, C?
Notice that $c(A,[0,u],[0,v]) \leq 1$ iff $I(A) \leq I(P_{u^\perp}A)$.
F.-Madiman-Zvavitch (2019+):

1. $c_2 = 1$, $c_3 = 4/3$ and $c(3/2)^{n/4} \leq c_n \leq 2^{n-2}$.

3 bodies inequalities

1) Information. Let X, Y, Z be indep. r.v. Madiman (2008):

$$h(X+Y+Z)+h(X) \leq h(X+Y)+h(X+Z).$$

2) Determinants. Madiman (2008). Let $A, B, C \geq 0$. Then

$$\det(A+B+C)\det(A) \leq \det(A+B)\det(A+C).$$

3) Convexity. Bobkov-Madiman (2012): Let A, B, C be convex

$$|A+B+C||A| \leq 3^n|A+B||A+C|.$$

We define $c(A,B,C) = \frac{|A+B+C||A|}{|A+B||A+C|}$ and $c_n = \sup_{A,B,C\subset\mathbb{R}^n} c(A,B,C)$.
Courtade's question (2016): $c(B_2^n, B, C) \leq 1$, for any B, C?
Notice that $c(A,[0,u],[0,v]) \leq 1$ iff $I(A) \leq I(P_{u^\perp}A)$.
F.-Madiman-Zvavitch (2019+):

1. $c_2 = 1$, $c_3 = 4/3$ and $c(3/2)^{n/4} \leq c_n \leq 2^{n-2}$.
2. $$c(A,B,C) \leq 1 \quad \text{if} \begin{cases} A = B_2^n \text{ and } B \text{ is a zonoid} \\ A = \Delta^n \text{ and } n = 2,3,4 \end{cases}$$

End

Thank you