On the minimizing rectifiable G chains of dimension 2 and of codimension 1 in finite dimensional normed spaces.

Ioann VASILYEV,
joint work with Thierry DE PAUW

UPEM and PDMI

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Some history.

Plateau's Problem formulation:
Find surface of least area with prescribed boundary.
The terms "surface", "area" and "boundary" should be explained.

In this problem one has to prove two things:

Today we shall concentrate ourselves only on the **existence**.

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- In this talk the ambient space X will be a finite dimensional normed space in which we consider the corresponding Hausdorff measure (see the appendix).
- The space of competitors will be the space of rectifiable chains $\mathcal{R}_m(X, G)$, where G is a complete normed Abelian group (see the appendix).
- The area is going to be the mass, the one associated to the Hausdorff measure (see the appendix).
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Why do we consider G minimizing chains?

- Historically, this started as a question of finding mathematical models for soap films on wire frames.
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Direct method of calculus of variations.

Let us briefly trace the main steps of the direct method of calculus of variations, since we shall utilize it.

- Choose a topology on the space of competitors (or "surfaces").
- Take a minimizing sequence.
- Extract a converging subsequence via a compactness argument.
- Show that the limiting object is a solution of the Plateau problem via a lower semicontinuity theorem.

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Throughout this talk we consider only the **flat norm** topology (see the appendix).

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Example: the Euclidean setting

In the Euclidean spaces the situation is nice: orthogonal projections do not increase the Hausdorff measure of rectifiable sets. From this one can derive the following fact.

Proposition

(Triangle inequality.) Let P be a polyhedral m-cycle (i.e. $\partial P = 0$) such that $P = \sum_{j=1}^{N} g_j[\sigma_j]$, where σ_j are non overlapping m-simplexes. Then

$$|g_1|\mathcal{H}^m(\sigma_1) \leqslant \sum_{j=2}^{N} |g_j|\mathcal{H}^m(\sigma_j).$$

From here using the strong and the weak approximation theorems one can derive the lower semicontinuity.

Remark

Alas, for instance in the space $X = (\mathbb{R}^3, \|\ldots\|_\infty)$ any linear projection $\pi : X \to W$ onto the hyperplane $W = \{x \in X : x + y + z = 0\}$ has Lipschitz constant $C > 1$.

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$(n-1)$-Hausdorff area contractions.

Defenition

Let $(X, \|\dots\|)$ be an n-dimensional normed space and let $W \in G(n-1, X)$. We say that a linear projection $\pi_W : X \to W$ is W-**good** if

$$\mathcal{H}^{n-1}_{\|\dots\|}(\pi_W(A)) \leqslant \mathcal{H}^{n-1}_{\|\dots\|}(A)$$

for any set $A \subset V$, where V is some $(n-1)$-dimensional linear subspace of X.

Theorem

(H.Busemann) In every finite dimensional normed space X each $W \in G(n-1, X)$ admits at least one W-good projection.

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Let $(X, \|\ldots\|)$ be a normed space with the unit ball B. The **Busemann–Hausdorff density** $\phi_{BH,m} : G(m,X) \to \mathbb{R}_+$ is defined for $W \in G(m,X)$ by

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Existence of good projections follows from another theorem, once again of Busemann.

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Lower semicontinuity ⇔ Triangle inequality

Theorem 1

(T.De Pauw, I.V.) Let $(X, \|\ldots\|)$ be an n-dimensional normed space, let G be a complete normed Abelian group and let m be an integer such that $1 \leqslant m \leqslant n-1$. Then the following are equivalent.

1. $\mathbb{M}$ *is lower semicontinuous with respect to flat convergence on* $\mathcal{P}_m(X, G)$.
2. *If* $P = \sum_{j=1}^{N} g_j[\sigma_j] \in \mathcal{P}_m(X, G)$, *where* σ_j *are non overlapping, is a cycle (which means that* $\partial P = 0$*) then*

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The existence theorem.

Theorem 2

(T.De Pauw, I.V.) Let $(X, \|\ldots\|)$ be an n-dimensional normed space and let G be a locally compact complete normed Abelian group. Then the following Plateau problem admits a solution:

$$\begin{cases} \text{minimize } \mathbb{M}(T) \\ T \in \mathcal{R}_{n-1}(X, G) \quad \text{such that } \partial T = B, \end{cases}$$

where $B \in \mathcal{R}_{n-2}(X, G)$ has a compact support and satisfies $\partial B = 0$. Moreover, among the solutions, there is at least one, say T_0 such that $spt(T_0)$ is compact.

Proof.

We use the direct method. The lower semicontinuity is the corollary from the previous slide. The compactness requires a more involved argument. □

Remark

Our result gives a (partial) answer to a question posed by Ambrosio & Schmidt.

The existence theorem.

Theorem 2

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Proof.

We use the direct method. The lower semicontinuity is the corollary from the previous slide. The compactness requires a more involved argument. □

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The lack of projections and the Burago–Ivanov theorem.

Remark

For $m \neq \{1, n-1\}$ it is possible to construct examples of m-volume densities which admit convex extension to $\Lambda_2 X$, but fail to admit good projections in general.

Despite of the lack of area minimizing projections, the 2 dimensional area is convex.

Theorem

(D.Burago, S.Ivanov) In every finite-dimensional normed space X, the two-dimensional Busemann–Hausdorff area density admits a convex extension to $\Lambda_2 X$.

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(H.Busemann) Has the m-dimensional Busemann–Hausdorff area $\phi_{BH,m}$ a convex extension to $\Lambda_m X$ for a general codimension m?

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Density contractors.

Convexity itself is not sufficient for our goals, so we define another object that we call density contractors, which will play the role of good projections in dimension 2.

Defenition

We call a **density contractor** of $W \in G(m, X)$ any Borel probability measure μ on $\mathrm{Hom}_m(X, X)$ satisfying

- μ is supported in $\mathrm{Hom}(X, W)$;
- If $V \in G(m, X)$ and $A \subseteq V$ is Borel, then

$$\int_{\mathrm{Hom}_m(X,X)} \mathcal{H}^m_{\|\dots\|}(\pi(A))d\mu(\pi) \leq \mathcal{H}^m_{\|\dots\|}(A),$$

with equality when $V = W$.

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Good projections are particular case of density contractors: $\mu = \delta_{\pi_W}$.

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Existence of density contractors.

Using Burago–Ivanov's theorem, we prove that in dimension 2 these contractors exist.

Theorem 3

(T.De Pauw, I.V.) For each $W \in G(2, X)$ there exists at least one density contractor.

Remark

In the two dimensional case the contractors that we construct are of the following form: $\mu = \sum \lambda_i \lambda_j \delta_{\pi_{i,j}}$.

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The triangle inequality.

Based on our theorem on the density contractors, we prove the triangle inequality.

Theorem 4

(T.De Pauw, I.V.) Let $(X, \|\ldots\|)$ be an n-dimensional Banach space with the unit ball $B = \{x \in X : \|x\| \leqslant 1\}$. Let $P \in \mathcal{P}_2(X, G)$ be a 2-cycle (i.e. $\partial P = 0$) such that $P = \sum_{j=1}^{N} g_j[\sigma_j]$, where σ_j are non overlapping 2-simplexes. Then

$$|g_1|\mathcal{H}^2_{\|\ldots\|}(\sigma_1) \leqslant \sum_{j=2}^{N} |g_j|\mathcal{H}^2_{\|\ldots\|}(\sigma_j).$$

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Approximation of the norm $\|\ldots\|$ by polyhedral norms + density contractors. □

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The lower semicontinuity a the existence theorem.

Using our triangle inequality, we prove that the mass is lower semicontinuous.

Corollary

Let $(X, \|\ldots\|)$ be an n-dimensional normed space and let G be a complete normed Abelian group. Then the mass $\mathbb{M}$ is lower semicontinuous with respect to flat convergence on $\mathcal{R}_2(X, G)$.

Proposition

Let $(X, \|\ldots\|)$ be an n-dimensional normed space and let G be a locally compact complete Abelian group. Then the following Plateau problem admits a solution:

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